P_1–nonconforming polyhedral finite elements in high dimensions

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Abstract
We consider the lowest–degree nonconforming finite element methods for the approximation of elliptic problems in high dimensions. The P_1–nonconforming polyhedral finite element is introduced for any high dimension. Our finite element is simple and cheap as it is based on the triangulation of domains into polytopes, which are combinatorially equivalent to d–dimensional cube, rather than the triangulation of domains into simplices. Our nonconforming element is nonparametric, and on each polytope it contains only linear polynomials, but it is sufficient to give optimal order convergence for second–order elliptic problems.

1 Introduction

We are interested in the lowest–degree nonconforming finite element methods for the approximation of elliptic problems in high dimensions. Efficient numerical methods to approximate solutions of partial differential equations in high dimensions are very demanding. For instance, in computational finance, efficient numerical methods are necessary to approximate high dimensional basket options (see [2, 21, 19] and the references therein). Also, in the approximation of the Einstein equations of general relativity, one needs to work on high dimensional dynamics modeling (see [1, 9, 22], and the references therein). For possible applications in fluid mechanics in high dimensions ≥ 4, see [13, 10, 11, 12, 24] and so on for the uniqueness, existence and regularity results on the solution of Navier–Stokes equations. However, practical application areas in fluid mechanics are hardly found.

In high dimensions it is much simpler to to adopt cubic type of elements rather than simplicial elements. In our paper we develop finite elements based on the tri-

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angulation of domains into polytopes, which are combinatorially equivalent to d–dimensional cube. In order to have lowest degree conforming finite elements on d–cubes, one needs to have multilinear polynomial spaces whose dimensions are at least 2^d. Hence to reduce the dimension of approximation polynomial space, we develop nonconforming elements which are nonparametric, but on each polytope it contains only P_1 polynomials which is sufficient to give optimal order convergence for second–order elliptic problems.

To present most effectively the idea of developing the nonconforming polyhedral finite elements, which are nonparametric, we briefly review the nonconforming elements of lowest degrees from parametric elements to nonparametric elements, and from rotated bilinear elements to P_1–nonconforming quadrilateral elements. By this brief review it will be very natural to expose our idea to develop the final nonconforming polyhedral elements in high dimensions.

In this section we present our model problem, and then some notations and preliminaries are given.

1.1 The model problem

Let $\Omega \in \mathbb{R}^d$ be a simply–connected polyhedral domain with boundary Γ. Consider the second–order elliptic problem:

$$-\nabla\cdot(\mathbf{A}(\mathbf{x})\nabla u) + cu = f, \quad \Omega, \tag{1a}$$
$$u = 0, \quad \Gamma, \tag{1b}$$

where the uniformly positive–definite matrix–valued function $\mathbf{A}$ and the nonnegative function $c > 0$ are assumed to be sufficiently smooth. The weak formulation of (1) is to find $u \in H_0^1(\Omega)$ fulfilling

$$a(u,v) = \ell(v)\ \forall v \in H_0^1(\Omega), \tag{2}$$

where the bilinear form $a(\cdot,\cdot) : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ and the linear form $\ell : H_0^1(\Omega) \to \mathbb{R}$ are given by

$$a(u,v) = (\mathbf{A}\nabla u, \nabla v) + (cu, v), \tag{3a}$$
$$\ell(v) = (f, v), \tag{3b}$$

for all $u, v \in H_0^1(\Omega)$.

1.2 Notations and preliminary results

For be a domain $S \in \mathbb{R}^d$, we adopt standard notations for Sobolev spaces with the inner products and norms

$$
\begin{aligned}
&L^2(S) = \{f : S \to \mathbb{R} \mid \int_S |f(\mathbf{x})|^2\,dx < \infty\},\\
&\qquad (f,g)_S = \int_S f(x)g(x)\,dx;\ \|f\|_{0,S} = \sqrt{(f,f)};\\
&H^1(S) = \{f \in L^2(S) \mid \|\nabla f(x)\|_{0,S} < \infty\},\\
&\qquad\qquad (f,g)_{H^1(S)} = (f,g)_S + (\nabla f, \nabla g)_S;\ \|f\|_{1,S} = \sqrt{(f,f)_{H^1(S)}};\\
&H^1_0(S) = \{f \in H^1(S) \mid f|_{\partial S} = 0\};\\
&H^k(S) = \{f \in L^2(S) \mid \|\partial^\alpha f(\mathbf{x})\|_{0,S} < \infty\ \forall |\alpha| \le k\},\\
&\qquad (f,g)_{H^k(S)} = \sum_{|\alpha|\le k} (\partial^\alpha f, \partial^\alpha g)_S;\ \|f\|_{k,S} = \sqrt{(f,f)_{H^k(S)}}.
\end{aligned}
$$

Here, and in what follows, if $S = \Omega$ the subindex Ω may be dropped as well as the subindex 0.

Denote by $\operatorname{conv} S$ the interior of the convex hull of S, which is an open set. The 0– and 1–faces of d–polyhedral domain S are the vertices and edges of S, respectively. In particular, the $(d-1)$–faces of S will be called the "facets" of d–dimensional polyhedral domain, and by μ_j we designate the barycenter of facet F_j's.

The organization of the paper is as follows. In Section 2, the lowest–degree parametric and nonparametric nonconforming quadrilateral elements for two and three dimensions are briefly reviewed. In Section 3, we introduce the nonparametric P_1–NC polyhedral finite element space in $\mathbb{R}^d$ for any $d \ge 2$. Here, and in what follows, P_1 means "piecewise linear" and NC means "nonconforming."

2 The parametric and nonparametric P_1–simplicial and quadrilateral nonconforming finite elements

In this section we review the simplicial and quadrilateral NC (nonconforming) finite element spaces in two and three dimensions.

2.1 The parametric simplicial and rectangular NC elements in two and three dimensions

The NC elements for elliptic and Stokes equations in two and three dimensions have been well known since the work of Crouzeix and Raviart [7] was published.

Denote the reference element as follows:

$$\widehat{K} = \begin{cases} \widehat{\Delta}^d = d\text{–simplex, } \textit{i.e.,}\ \mathrm{conv}\{\mathbf{0}, \widehat{\mathbf{e}}_1, \cdots, \widehat{\mathbf{e}}_d\}, \\ \widehat{Q}^d = d\text{–cube, } \textit{i.e.,}\ (-1,1)^d. \end{cases} \tag{4}$$

1. The lowest–degree simplicial Crouzeix-Raviart element (1973) [7]:
   a. $\widehat{K} = \widehat{\Delta}^d$, $d = 2,3$;
   b. $\widehat{P}_{\widehat{K}} = P_1(\widehat{K}) = \mathrm{Span}\{1, \widehat{x}_1, \cdots, \widehat{x}_d\}$;
   c. $\widehat{\Sigma}_{\widehat{K}} = \{\widehat{\varphi}(\widehat{\xi}_j), \quad \widehat{\xi}_j$ barycenter of facets, $j = 1, \cdots, d+1$, $\forall \widehat{\varphi} \in \widehat{P}(\widehat{K})\}$.

   All odd–degree simplicial NC elements were introduced for the Stokes problems in [7].

   Remark 1. It is straightforward to define the simplicial NC elements on d–simplicial triangulation in any high dimension. However, for high dimension it is not easy to see how the d–simplices are packed in the domain. Thus the development of d–cubical elements is beneficial in this regard.

2. The Han rectangular element (1984) [15]:
   a. $\widehat{K} = \widehat{Q}^2$;
   b. $\widehat{P}_{\widehat{K}} = P_1(\widehat{K}) \oplus \mathrm{Span}\{\widehat{x}_1^2 - \frac{5}{3}\widehat{x}_1^4, \widehat{x}_2^2 - \frac{5}{3}\widehat{x}_2^4\}$;
   c. $\widehat{\Sigma}_{\widehat{K}} = \{\widehat{\varphi}(\widehat{\xi}_j), \widehat{\xi}_j, j = 1, \cdots, 4,$ midpoints of facets; $\int_{\widehat{Q}^2} \widehat{\varphi}\ \forall \widehat{\varphi} \in \widehat{P}_{\widehat{K}}\}$.
3. The Rannacher–Turek rotated Q_1 element (1992, [20], also Z. Chen [5]):
   a. $\widehat{K} = \widehat{Q}^d, d = 2,3$;
   b. $\widehat{P}_{\widehat{K}} = P_1(\widehat{K}) \oplus \mathrm{Span}\{\widehat{x}_1^2 - \widehat{x}_d^2, \widehat{x}_{d-1}^2 - \widehat{x}_d^2\}$;
   c. $\widehat{\Sigma}^{(m)}_{\widehat{K}} = \{\widehat{\varphi}(\widehat{\xi}_j), \quad \widehat{\xi}_j, j = 1, \cdots, 2d,$ barycenters of facets $\widehat{F}_j$, $\forall \widehat{\varphi} \in \widehat{P}_{\widehat{K}}\}$;
   $\widehat{\Sigma}^{(i)}_{\widehat{K}} = \{\frac{1}{|\widehat{F}_j|} \int_{\widehat{F}_j} \widehat{\varphi} d\sigma, \quad \widehat{F}_j, j = 1, \cdot, 2d,$ are facets, $\forall \widehat{\varphi} \in \widehat{P}_{\widehat{K}}\}$.

   Remark 2. The two DOFs generate two different NC elements, and for general quadrilateral meshes the NC element with the DOFs $\widehat{\Sigma}^{(i)}_{\widehat{K}}$ gives optimal convergence rates while that with the DOFs $\widehat{\Sigma}^{(m)}_{\widehat{K}}$ leads to suboptimal convergence rates.

4. The DSSY element(DOUGLAS-SANTOS-Sheen-YE, 1999) [8]: For $\ell = 1,2$, define

$$\theta_\ell(t) = \begin{cases} t^2, & \ell = 0; \\ t^2 - \frac{5}{3}t^4, & \ell = 1; \\ t^2 - \frac{25}{6}t^4 + \frac{7}{2}t^6, & \ell = 2. \end{cases}$$

a. $\widehat{K} = \widehat{Q}^d, d = 2,3;$
b. $\widehat{P}_{\widehat{K}} = P_1(\widehat{K}) \oplus \mathrm{Span}\{\theta_\ell(\widehat{x}_1) - \theta_\ell(\widehat{x}_d), \theta_\ell(\widehat{x}_{d-1}) - \theta_\ell(\widehat{x}_d)\};$
c. $\widehat{\Sigma}_{\widehat{K}}^{(m)} = \{\widehat{\varphi}(\widehat{\xi}_j), \widehat{\xi}_j$ barycenters of facets, $j = 1,\cdots,2d,\ \forall\widehat{\varphi} \in \widehat{P}_{\widehat{K}}\}$
$\widehat{\Sigma}_{\widehat{K}}^{(i)} = \{\frac{1}{|\widehat{F}_j|}\int_{\widehat{F}_j}\widehat{\varphi}d\sigma, \widehat{F}_j, j = 1,\cdots,2d,$ are facets, $\forall\widehat{\varphi} \in \widehat{P}_{\widehat{K}}\}.$

Remark 3. The benefit of the DSSY element is the Mean Value Property

$$\widehat{\varphi}(\widehat{\xi}_j) = \frac{1}{|\widehat{F}_j|}\int_{\widehat{F}_j}\widehat{\varphi}d\sigma\ \forall\widehat{\varphi} \in \widehat{P}_{\widehat{K}} \tag{5}$$

holds if $\ell = 1,2$. Thus, for $\ell = 1,2$, the two DOFs $\widehat{\Sigma}_{\widehat{K}}^{(m)}$ and $\widehat{\Sigma}_{\widehat{K}}^{(i)}$ generate an identical NC elements with optimal convergence rates. The case of $\ell = 0$ reduces to the Rannacher–Turek rotated Q_1 element.

5. For truly quadrilateral triangulations, $P_1(\widehat{K})$ for the Rannacher–Turek element and the DSSY element should be modified such that $P_1(\widehat{K})$ is replaced by $Q_1(\widehat{K})$ in the reference elements with an additional DOF $\int_{\widehat{Q}^2}\widehat{\varphi}(\widehat{x}_1,\widehat{x}_2)\widehat{x}_1\widehat{x}_2 d\widehat{x}_1 d\widehat{x}_2$ (Cai–Douglas–Santos–Sheen–Ye, CALCOLO, 2000) [4].

Let $(\mathscr{T}_h)_{0<h<1}$ denote a family of quasiregular triangulations of Ω into simplices or quadrilaterals K_j's where $\mathrm{diam}(K_j) \le h\ \forall K_j \in \mathscr{T}_h$. If K is a d–simplex, or a parallelogram or a parallelepiped, there is a unique (up to rotation in the order of the vertices) affine map $F_K : \widehat{K} \to K$. Set

$$\mathscr{N}C_K = \{v : v = \widehat{v}\circ F_K^{-1},\ \widehat{v} \in \widehat{P}_{\widehat{K}}\}.$$

The global (*parametric*) NC element space is defined as follows:

$$\mathscr{N}C^h = \big\{v \in L^2(\Omega) \mid v|_K \in \mathscr{N}C_K\ \forall K \in \mathscr{T}_h; \langle[[v]]_F, 1\rangle_F = 0 \\ \forall \text{ interior facets } F \in \mathscr{T}_h\big\},$$

and

$$\mathscr{N}C_0^h = \left\{v \in \mathscr{N}C^h \mid \langle v_F, 1\rangle_F = 0\ \forall \text{ boundary facets } F \in \mathscr{T}_h\right\},$$

where $[[v]]_F$ denotes the jump across the facets $F = \partial K \cap \partial K'$ for all $K, K' \in \mathscr{T}_h$.

The (parametric) NC Galerkin method for (2) is to find $u_h \in \mathscr{N}C_0^h$ such that

$$a_h(u_h, v_h) = \ell(v_h)\ \forall v_h \in \mathscr{N}C_0^h, \tag{6}$$

where

$$a_h(u,v) = \sum_{K\in\mathscr{T}_h}(\mathbf{A}\nabla u, \nabla v)_K + (cu, v)\ \forall u, v \in \mathscr{N}C_0^h + H_0^1(\Omega).$$

2.2 The nonparametric NC quadrilateral and hexahedral elements

Recall that finite elements need to contain at least the P_1 space in order to have a full approximation property for the second–order elliptic problems due to the Bramble–Hilbert lemma.

In this subsection the nonparametric DSSY-type nonconforming quadrilateral elements will be briefly reviewed. Then the P_1–NC quadrilateral elements will be reviewed, which are essentially nonparametric, but which are the lowest degrees–of–freedom elements as they contain only P_1 spaces on each quadrilateral or hexahedron.

2.2.1 The nonparametric DSSY-type nonconforming quadrilateral elements

It was questionable if, for truly quadrilateral triangulations, any 4–DOF DSSY–type nonconforming element can be defined or not. A DSSY–type element needs to fulfill the Mean Value Property (5) such that $\widehat{\Sigma}_{\widehat{K}}^{(m)}$ and $\widehat{\Sigma}_{\widehat{K}}^{(i)}$ generate an identical NC elements. It turns out that we may not have such a finite element in the class of parametric finite elements. Instead, it is possible to define such DSSY–type element in the class of nonparametric finite elements. Indeed, a class of nonparametric DSSY nonconforming quadrilateral elements [16] were developed with 4 DOFs fulfilling the Mean Value Property (5).

Such nonparametric DSSY nonconforming hexahedral elements in three dimensions with 6 DOFs fulfilling three–dimensional Mean Value Property will appear elsewhere [23].

2.2.2 The P_1–NC quadrilateral element

For general convex quadrilateral triangulation ($d = 2$ or $d = 3$), it is possible to define a *nonparametric* P_1–NC quadrilateral element (see Park (PhD Thesis, 2002) and Park–Sheen (SINUM, 2003) [17, 18]).

1. The nonparametric P_1–NC quadrilateral ($d = 2$) or hexahedral ($d = 3$) element.
   a. K, any convex quadrilateral or hexahedron;
   b. $P_K = P_1(K)$;
   c. $\Sigma_K = \{\varphi(\mu_j), j = 1,\cdots,d+1, \forall\varphi \in P_K\}$, where μ_j is any barycenter of the two opposite facets $F_{j,\pm}$ for $j = 1,\cdots,d$, and μ_{d+1} is any other barycenter of facets $F_{j,\pm}, j = 1,\cdots,d$.
2. **Lemma 1.** *[17, 18]. If $u \in P_1(K)$, then $u(\mu_{1,-}) + u(\mu_{1,+}) = \cdots = u(\mu_{d,-}) + u(\mu_{d,+})$. Conversely, if $u_{j,\pm}$ are given values at $\mu_{j,\pm}$, for $1 \le j \le d$, satisfying $u_{1,-} + u_{1,+} = \cdots = u_{d,-} + u_{d,+}$, then there exists a unique function $u \in P_1(K)$ such that $u(\mu_{j,\pm}) = u_{j,\pm}$, $1 \le j \le d$.*

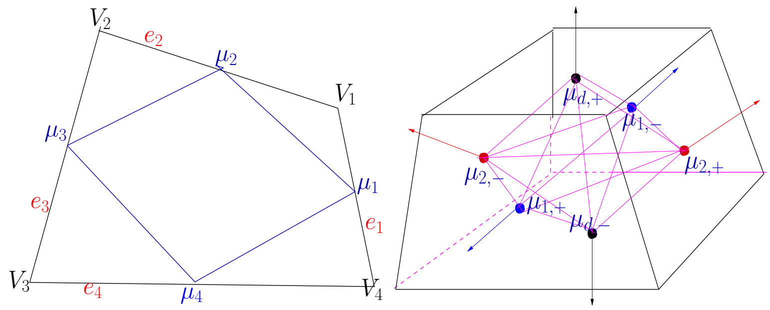

Fig. 1 Left. For $j = 1,\cdots,4$, μ_j denotes the midpoint of edge e_j of any quadrilateral $\text{conv}\{V_1,V_2,V_3,V_4\}$. Then $\text{conv}\{\mu_1,\mu_2,\mu_3,\mu_4\}$ is a parallelogram. **Right.** For $j = 1,2,3, \iota = \pm$, $\mu_{j,\iota}$ denotes the barycenter of face $f_{j,\iota}$ of any hexahedron. Then $\text{conv}\{\mu_{j,\iota}, j = 1,2,3, \iota = \pm\}$ forms an octahedron, which is a dual of the hexahedron.

It is shown in [17, 18] that the above elements are unisolvent and optimal error estimates hold for the second–order elliptic problems (2).

3 The P_1–nonconforming polyhedral finite element

We now extend the P_1–NC quadrilateral or hexahedral element to any dimension $d \geq 2$.

The notion of polytope is the generalization of quadrilateral to higher dimension, introduced by Coexter [6]. See also [3, 14]. The stream of developing the P_1–NC polyhedral finite element basis functions is a follow–up of that given in [17, 18].

For polyhedral set, polytope, parallelotope, and so on, we adopt the following definitions. Here, we just modify to have those sets to be open sets instead of closed sets.

Definition 1. [14, p.26] A set $K \subset \mathbb{R}^d$ is called a *polyhedral* set provided K is the intersection of a finite family of open half spaces of $\mathbb{R}^d$.

Definition 2. [14, p.17, p.31] Let K be a convex subset of $\mathbb{R}^d$. A point $x \in \overline{K}$ is an extreme point of K provided $y,z \in \overline{K}, 0 < \lambda < 1$, and $x = \lambda y + (1-\lambda)z$ imply $x = y = z$. The set of all extreme points of K is denoted by $\text{ext}\,K$. An open convex set $K \subset \mathbb{R}^d$ is a *polytope* provided $\text{ext}\,K$ is a finite set. For a polytope of dimension d, we use d–polytope. We use k–face if the face is of dimension k. A subset $F \subset \overline{K}$ is called a *face* of a polytope K if either $F = \emptyset$ or $F = K$, or if there exists a supporting hyperplane H of K such that $F = \overline{K} \cap H$. The set of all faces of K is denoted by $\mathscr{F}(K)$. The 0– and 1–faces of d–polytope K are the vertices and edges of K, respectively. In particular, the $(d-1)$–faces of d–polytope K will be called the *facets* of K. For a polytope (or polyhedral set) K, $\text{ext}\,K$ consists of all vertices of K.

The following proposition is a well–known result from the above definitions.

Proposition 1. *A set $K \subset \mathbb{R}^d$ is a polytope if and only if K is a bounded polyhedral set.*

Definition 3. [3] We say $\sum_{j=1}^{d} \lambda_j \mathbf{x}_j$ is a *convex combination* of $\mathbf{x}_j \in \mathbb{R}^d, j = 1,\cdots,d$, denoted by

$$\sum_{j=1}^{d}{}^{c} \lambda_j \mathbf{x}_j. \tag{7}$$

if $\sum_{j=1}^{d} \lambda_j = 1$ and $\lambda_j \geq 0\, \forall j$. The vectors $\mathbf{x}_j \in \mathbb{R}^d, j = 1,\cdots,d$, are said to be *affinely independent* if

$$\sum_{j=1}^{d} \lambda_j \mathbf{x}_j = \mathbf{0} \quad \text{with } \sum_{j=1}^{d} \lambda_j = 0 \text{ implies } \lambda_j = 0\, \forall j.$$

For affinely independent vectors $\mathbf{x}_j, j = 1,\cdots,k$, a *$k$–parallelotope* K is a bounded polytope which can be represented by

$$\mathbf{x} = \mathbf{a} + \sum_{j=1}^{k} \lambda_j \mathbf{x}_j,\, 0 \leq \lambda_j \leq 1\, \forall j. \tag{8}$$

In the meanwhile a bounded k–polytope can be represented by

$$\mathbf{x} = \sum_{j=1}^{2k}{}^{c} \lambda_j \mathbf{x}_j.$$

with suitable $\mathbf{x}_j \in \operatorname{ext} K, j = 1,\cdots,2k$, if it is combinatorially equivalent to a k–cube. Two polytopes are said to be *combinatorially equivalent* if there is a one–to–one correspondence between the set of all faces of P and that of all faces of Q with incidence–relation preserved.

If a k–polytope is combinatorially equivalent to k–cube, $(-1,1)^k$, K is assumed to have $2d$ boundaries which are flat $(d-1)$–faces combinatorially equivalent to the $(d-1)$–dimensional cube $(-1,1)^{d-1}$. In particular, denote by $(F_{j,-}, F_{j,+}), j = 1,\cdots,d$, the pairs of opposite $(d-1)$-faces. For each vertex V_j, there are d edges which meet at the vertex. For $j = 1,\cdots,2d$, denote by $\mu_{j,\pm}$ the barycenter of facet $F_{j,\pm}$.

The convex hull of the barycenters of facets of d–polytope K forms the dual of K, and their diagonals intersect at one point and are bisected by this point. Indeed, we have the following lemma.

Lemma 2. *Let $K \in \mathbb{R}^d$ be a d–polytope which is combinatorially equivalent to the d–dimensional cube $(-1,1)^d$, with 2^d vertices: $V_j, j = 1,\cdots,2^d$. Assume that K has d pairs of opposite boundaries $F_{j,\pm}, j = 1,\cdots,d$, which are flat $(d-1)$–faces combinatorially equivalent to the $(d-1)$–dimensional cube $(-1,1)^{d-1}$. Let*

$\{\mu_{j,+},\mu_{j,-}, j=1,\cdots,d\}$ *be the barycenters of boundaries of* $F_{j,\pm}$.
Then $\mathrm{conv}\{\mu_{j,+},\mu_{j,-}, j=1,\cdots,d\}$ *forms a d–polytope, which is the dual of* K, *and the midpoint of* $\mu_{j,+}$ *and* $\mu_{j,-}$ *coincides for* $j=1,\cdots,d$.

Proof. For $j=1,\cdots,d$, and $\iota=\pm$, let $V_k^{(j,\iota)}, k=1,\cdots,2^{d-1}$, denote the vertices of $F_{j,\pm}$. Then notice that

$$\frac{1}{2}\left[\sum_{k=1}^{2^{d-1}} V_k^{(j,+)} + \sum_{k=1}^{2^{d-1}} V_k^{(j,-)}\right] = \frac{1}{2}\sum_{k=1}^{2^d} V_k$$

which implies that the midpoint of $\mu_{j,+}$ and $\mu_{j,-}$ coincides for every $j=1,\cdots,d$. This proves the lemma.

The Lemma 2 enables to generalize the P_1–NC quadrilateral or hexahedral element to any $d\ge 2$ dimension.

From now on, *we assume that a k–polytope is combinatorially equivalent to a k–cube, for* $0<k\le d$. We are ready to generalize the P_1–NC quadrilateral element to any high dimension as follows.

Definition 4. Define the d–dimensional P_1–NC polyhedral element as follows:

(i) K, d–polytope;
(ii) $P_K = P_1(K)$;
(iii) $\Sigma_K = \{\varphi(\mu_j), j=1,\cdots,d+1, \forall\varphi\in P_K\}$, where μ_j is any barycenter of the two opposite facets $F_{j,\pm}$ for $j=1,\cdots,d$, and μ_{d+1} is any other barycenter of facets $F_{j,\pm}, j=1,\cdots,d$.

Now, we have the following lemma.

Lemma 3. *If* $u\in P_1(K)$, *then the following* $d-1$ *constraints hold:* $u(\mu_{1,-})+u(\mu_{1,+})=\cdots=u(\mu_{j,-})+u(\mu_{j,+})=\cdots=u(\mu_{d,-})+u(\mu_{d,+})$. *Conversely, if* $u_{j,\pm}$ *are given values at* $\mu_{j,\pm}$, *for* $1\le j\le d$, *satisfying* $u_{1,-}+u_{1,+}=\cdots=u_{j,-}+u_{j,+}=\cdots=u_{d,-}+u_{d,+}$, *then there exists a unique function* $u\in P_1(K)$ *such that* $u(\mu_{j,\pm})=u_{j,\pm}$, $1\le j\le d$.

Proof. Due to Lemma 2, we have $\mu_{j,-}+\mu_{j,+}=2\mathbf{c}$, $\forall j=1,\cdots,d$, and the linearity of ϕ implies $\phi(\mu_{j,-})+\phi(\mu_{j,+})=2\phi(\mathbf{c})$, $\forall j=1,\cdots,d$.

To show the converse suppose that $u_{j,\pm}$ are given values at $\mu_{j,\pm}$, for $1\le j\le d$, satisfying $u_{1,-}+u_{1,+}=\cdots=u_{j,-}+u_{j,+}=\cdots=u_{d,-}+u_{d,+}$, Without loss of generality, we may assume that $\mu_j=\mu_{j,-}$ is chosen from the pair of barycenters $\mu_{j,-}$ and $\mu_{j,+}$ for all $j=1,\cdots,d$. Since $\mathrm{conv}\{\mathbf{c},\mu_j,\ j=1,\cdots,d\}$ forms a d–simplex, any function $\phi\in P_1(\mathrm{conv}\{\mathbf{c},\mu_j,\ j=1,\cdots,d\})$ is uniquely determined by the $d+1$ values at $\mathbf{c},\mu_j,\ j=1,\cdots,d$. From the constraint and Lemma 2, the value at $\mathbf{c}$ can be determined by any additional value at any $\mu_{j_0,+}$. This shows the claim of the converse holds.

Owing to Lemma 2 and Lemma 3, the following unisolvency holds.

Theorem 1. *The d–dimensional P_1–NC polyhedral element defined in Definition 4 is unisolvent.*

3.1 Global P_1–NC polyhedral finite element spaces

Let $(\mathscr{T}_h)_{0<h<1}$ denote a family of quasiregular triangulations of Ω into d–polytopes K_j's where $\operatorname{diam}(K_j) \le h \, \forall K_j \in \mathscr{T}_h$ with all their k–faces are combinatorially equivalent to k–cube for all $k \le d-1$. Set

$$\mathscr{N}C_K = P_1(K) \, \forall K \in \mathscr{T}_h.$$

The above Lemma 3 enables to define the d–dimensional P_1–NC polyhedral element spaces, which are nonparametric. Indeed, the global P_1–NC polyhedral finite element spaces are defined as follows:

$$\mathscr{N}C^h = \big\{ v \in L^2(\Omega) \mid v|_K \in \mathscr{N}C_K \, \forall K \in \mathscr{T}_h; \, \langle [[v]]_F, 1 \rangle_F = 0 \\ \forall \text{ interior } (d-1)\text{–faces (or facets) } F \in \mathscr{T}_h \big\},$$

and

$$\mathscr{N}C_0^h = \Big\{ v \in \mathscr{N}C^h \mid \big\langle v_f, 1 \big\rangle_F = 0 \, \forall \text{ boundary facets } F \in \mathscr{T}_h \Big\},$$

where $[[v]]_F$ denotes the jump across the facets $F = \partial K \cap \partial K'$ for all $K, K' \in \mathscr{T}_h$.

3.2 Basis and its dimension

Following the idea in [17, 18] for two and three dimensions, denote by $\mathscr{M}_h$ the set of all barycenters of facets in $\mathscr{T}_h$. Let $\{V_j \in \mathscr{T}_h, j = 1, \cdots, N_V^i\}$ be the set of all interior vertices in $\mathscr{T}_h$. Then for $j = 1, \cdots, N_V^i$, let $K_l^{(j)}, l = 1, \cdots, N_j$ form the set of all d–polytopes in $\mathscr{T}_h$ which share the common vertex V_j. Denote by $\mathscr{M}(V_j)$ the set of all barycenters of the facets of those $K_l^{(j)}, l = 1, \cdots, N_j$ sharing the common vertex V_j. Now, define $\phi_j \in \mathscr{N}C_0^h$ by

$$\phi_j(\mu) = \begin{cases} 1, & \mu \in \mathscr{M}(V_j), \\ 0, & \mu \in \mathscr{M}_h \setminus \mathscr{M}(V_j). \end{cases}$$

Then the following theorem holds (see [17, 18] for two and three dimensions):

Theorem 2. $\phi_j, j = 1, \cdots, N_V^i$ *are linearly independent. Moreover, we have*

$$\mathscr{N}C_0^h = \operatorname{Span}\{\phi_j, j = 1, \cdots, N_V^i\}; \quad \dim(\mathscr{N}C_0^h) = N_V^i.$$

3.3 Local and global Interpolation operators

Let K be a d–polytope combinatorially equivalent to $[-1,1]^d$ with facets and barycenters F_j and μ_j, respectively, for $j=1,\cdots,2d$. Denote by $V_k^{(j)}, k=1,\cdots,2^{d-1}$, the vertices of $F_j, j=1,\cdots,2d$. Then the interpolation operator $\mathscr{I}_K : C^0(\overline{K}) \to P_1(K)$ is defined as follows: if $u \in C^0(\overline{K})$, due to Lemma 3 one can define $\mathscr{I}_K u \in P_1(K)$ such that

$$(\mathscr{I}_K u)(\mu_j) = \frac{1}{2^{d-1}} \sum_{k=1}^{2^{d-1}} u(V_k^{(j)}),\ j=1,\cdots,2d.$$

The global interpolation operator $\mathscr{I}_h : C^0(\overline{\Omega}) \to \mathscr{N}C_h$ is then defined element by element such that

$$\mathscr{I}_h\,|_K = \mathscr{I}_K\ \forall K \in \mathscr{T}_h.$$

Since linear polynomials remain unchanged by $\mathscr{I}_h$, the Bramble–Hilbert lemma (which holds for high dimensional spaces) leads to the following estimate:

$$\|\mathscr{I}_h - u\| + h|\mathscr{I}_h - u|_{1,h} \le Ch^2|u|_2\ \forall u \in H^2(\Omega), \tag{9}$$

where $|\cdot|_{1,h}$ designates the broken semi-norm defined by $|v|_{1,h} = \sqrt{\sum_{K\in\mathscr{T}_h} \|\nabla v\|_{0,K}^2}$ for all $v \in H^1(\Omega) + \mathscr{N}C_h$.

3.4 The P_1–NC polyhedral Galerkin methods

Then the NC Galerkin method for (2) is to find $u_h \in \mathscr{N}C_0^h$ such that

$$a_h(u_h, v_h) = \ell(v_h)\ \forall v_h \in \mathscr{N}C_0^h, \tag{10}$$

where

$$a_h(u,v) = \sum_{K\in\mathscr{T}_h} (\mathbf{A}\nabla u, \nabla v)_K + (cu, v)\ \forall u,v \in \mathscr{N}C_0^h + H_0^1(\Omega),$$

and $\ell : \mathscr{N}C_0^h + H_0^1(\Omega) \to \mathbb{R}$ is as in (3).

Theorem 3. *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_h \in \mathscr{N}C_0^h$ be the solutions of* (2) *and* (10)*, respectively. Then the following optimal error estimates hold for the second–order elliptic problems:*

$$\|u_h - u\|_{1,h} \le Ch|u|_2, \tag{11a}$$

$$\|u_h - u\|_0 \le Ch^2|u|_2. \tag{11b}$$

Proof. The theorem follows from the usual argument by using the second Strang lemma and the interpolation estimate (9).

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References

[1] Baumgarte, T.W., Shapiro, S.L.: Numerical relativity: solving Einstein's equations on the computer. Cambridge University Press (2010)
[2] Broadie, M., Glasserman, P., et al.: A stochastic mesh method for pricing high-dimensional american options. Journal of Computational Finance **7**, 35–72 (2004)
[3] Brondsted, A.: An introduction to convex polytopes, vol. 90. Springer Science & Business Media (2012)
[4] Cai, Z., Douglas, Jr., J., Santos, J.E., Sheen, D., Ye, X.: Nonconforming quadrilateral finite elements: A correction. Calcolo **37**(4), 253–254 (2000)
[5] Chen, Z.: Projection finite element methods for semiconductor device equations. Computers Math. Applic. **25**, 81–88 (1993)
[6] Coxeter, H.: Regular Polytopes. Dover Books on Mathematics. Dover Publications (2012)
[7] Crouzeix, M., Raviart, P.: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. R.A.I.R.O.– Math. Model. Anal. Numer. **7**(R-3), 33–75 (1973)
[8] Douglas, Jr., J., Santos, J.E., Sheen, D., Ye, X.: Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems. ESAIM–Math. Model. Numer. Anal. **33**(4), 747–770 (1999)
[9] El Naschie, M.: Deriving the essential features of the standard model from the general theory of relativity. Chaos, Solitons & Fractals **24**(4), 941–946 (2005)
[10] Frehse, J., Růžička, M.: On the regularity of the stationary Navier-Stokes equations. Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV. Ser. **21**(1), 63–95 (1994)
[11] Frehse, J., Růžička, M.: Regularity for the stationary Navier–Stokes equations in bounded domains. Arch. Ration. Mech. Anal. **128**(4), 361–380 (1994)
[12] Frehse, J., Růžička, M.: Regular solutions to the steady Navier-Stokes equations. In: A. Sequeira (ed.) Navier-Stokes equations and related nonlinear problems. Proceedings of the 3rd international conference, held May 21–27, 1994 in Funchal, Madeira, Portugal., pp. 131–139. Plenum Press, Funchal (1995)
[13] Frehse, J., Růžička, M.: Existence of regular solutions to the steady Navier-Stokes equations in bounded six-dimensional domains. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze **23**(4), 701–719 (1996)

[14] Grünbaum, B., Klee, V., Perles, M.A., Shephard, G.C.: Convex polytopes. Springer (1967)
[15] Han, H.: Nonconforming elements in the mixed finite element method. J. Comp. Math. **2**, 223–233 (1984)
[16] Jeon, Y., Nam, H., Sheen, D., Shim, K.: A class of nonparametric DSSY nonconforming quadrilateral elements. ESAIM–Math. Model. Numer. Anal. **47**(6), 1783–1796 (2013)
[17] Park, C.: A study on locking phenomena in finite element methods. Ph.D. thesis, Department of Mathematics, Seoul National University, Korea (2002). Available at http://www.nasc.snu.ac.kr/cpark/papers/phdthesis.ps.gz
[18] Park, C., Sheen, D.: P_1-nonconforming quadrilateral finite element methods for second-order elliptic problems. SIAM J. Numer. Anal. **41**(2), 624–640 (2003)
[19] Pettersson, U., Larsson, E., Marcusson, G., Persson, J.: Improved radial basis function methods for multi-dimensional option pricing. Journal of Computational and Applied Mathematics **222**(1), 82–93 (2008)
[20] Rannacher, R., Turek, S.: Simple nonconforming quadrilateral Stokes element. Numer. Methods Partial Differential Equations **8**, 97–111 (1992)
[21] Reisinger, C., Wittum, G.: Efficient hierarchical approximation of high-dimensional option pricing problems. SIAM Journal on Scientific Computing **29**(1), 440–458 (2007)
[22] Shapiro, S.L., Teukolsky, S.A.: Black holes, star clusters, and naked singularities: numerical solution of Einstein's equations. Phil. Trans. R. Soc. Lond. A **340**(1658), 365–390 (1992)
[23] Sheen, D., Shim, K.: A class of nonparametric DSSY nonconforming hexahedral elements. In preparation
[24] Struwe, M.: Regular solutions of the stationary Navier-Stokes equations on $\mathbb{R}^5$. Mathematische Annalen **302**(1), 719–741 (1995)