The extended Prandtl closure model applied to the two-dimensional turbulent classical far wake

Ashleigh J. Hutchinson

Abstract Prandtl's mixing length closure model has been used extensively in turbulent wake flows. Although the simplicity of this model is advantageous, it contains mathematical and physical limitations. In particular, this model results in a poor estimation of the flow on the center-line and near the wake boundary. Prandtl constructed an improved model, which will be referred to as the extended mixing length model, in an attempt to address many of the limitations of the original model. In this work, the extended Prandtl model is considered. A similarity solution that leaves both the governing equation for the stream-wise mean velocity deficit and the conserved quantity invariant is obtained. The governing partial differential equation is reduced to an ordinary differential equation. The ordinary differential equation, which must be solved subject to appropriate boundary conditions and the conserved quantity, cannot be solved analytically and thus a double-shooting method is developed to obtain the stream-wise mean velocity deficit. A plot of the mean velocity deficit is then produced.

Key words: Extended Prandtl's mixing length, turbulent classical wake, conserved quantity, mean velocity deficit

1 Introduction

In turbulent flows, the time averaged Navier-Stokes equation is used to solve for the mean flow variables. Unknown Reynolds stress terms arise, resulting in an incomplete system of equations. In order to complete the system of equations, a closure model is needed. Many closure models have been proposed. Algebraic closure mod-

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els are the simplest type. In algebraic closure models, the Reynolds stresses which are co-variances, are related to a single mean velocity gradient by a turbulent viscosity function [1]. The effective viscosity is expressed as the sum of the kinematic viscosity, which is an intrinsic property of the fluid, and the turbulent or eddy viscosity which is not a characteristic of the fluid [2, 3]. In most algebraic closure models, the kinematic viscosity is neglected as it is negligible when compared to the turbulent viscosity.

Prandtl's mixing length closure model [4] falls under the class of algebraic closure models. In Prandtl's original model [4], the concept of a mixing length is introduced. The Reynolds stresses are written in terms of the square of this mixing length, and in terms of the square of the mean velocity gradient perpendicular to the axis of the wake. This model has been successfully applied to the turbulent classical far wake and other free shear flows [5]. Prandtl's mixing length model is convenient in that it is fairly easy to implement mathematically.

Prandtl's mixing length model has various limitations. When applied to turbulent wake flows, the predicted width of the wake is underestimated [4, 6]. Another failing of Prandtl's mixing length model is that the mixing length cannot be derived from the model and its form must be independently imposed. Prandtl assumed that the mixing length is proportional to the width of the wake. These limitations have been addressed [7] by modifying Prandtl's model by including the kinematic viscosity. Prandtl neglected the kinematic viscosity in his analysis and it is shown that by including the kinematic viscosity, the mixing length can be derived using a systematic method [7]. It is also shown that when the kinematic viscosity is included, the predicted width of the wake lies outside of the predicted width when the kinematic viscosity is neglected.

Prandtl realised the limitations of his closure model and put forth a new extended version. In this model, the kinematic viscosity is still neglected. Instead, two mixing lengths are introduced and the turbulent viscosity is considered as a function of both the first and second derivatives of the mean velocity deficit perpendicular to the axis of the wake [8]. This new form increases the mathematical complexity of the model. However, this model suffers from the same issue as the original model in that the two mixing lengths have to be independently specified. This issue can again be addressed by including the kinematic viscosity. However, including the kinematic viscosity further complicates the model and the numerical method and so is excluded in the current paper. Prandtl's hypothesis that each mixing length is proportional to the width of the wake is used to specify the form of each mixing length.

The aim of this work is to obtain an expression for the mean flow variables when the extended Prandtl model is applied to the two-dimensional turbulent classical far wake. A similarity solution, that leaves both the conserved quantity and the governing equation for the stream-wise mean velocity deficit invariant, is obtained. The partial differential equation is reduced to an ordinary differential equation which cannot be solved analytically. As an initial study, the kinematic viscosity is not included which simplifies the numerical method significantly.

This paper is presented as follows. In Section 2, the derivation of the governing equations, boundary conditions, and conserved quantity for the two-dimensional turbulent classical far wake is provided. The extended Prandtl closure model is used to complete the system of equations. In Section 3, similarity solutions are considered. Each mixing length is assumed to be proportional to the width of the wake. In Section 3, a numerical method is developed to solve the reduced ordinary differential equation. The similarity velocity profile is then plotted. Conclusions and further work are given in Section 4.

2 Mathematical model for the two-dimensional turbulent classical far wake

In this section a brief review of the derivation of the momentum equation for the two-dimensional turbulent classical wake far downstream of a stationary slender object is provided. A more in-depth derivation can be obtained from [9, 7]. The mean velocity profile for the two-dimensional turbulent classical wake is illustrated in Figure 1. A Cartesian coordinate system is used with the origin positioned at the trailing edge of the slender object. A laminar incompressible fluid with constant velocity $(U,0)$ flows past the stationary slender object. Downstream of the object, a wake is formed. For large Reynolds number flows, the wake that forms is turbulent. The turbulent wake region merges smoothly with the laminar mainstream flow.

In this work, the components of the mean velocity deficit, $\overline{v}_x$ and $\overline{v}_y$, in the wake region are considered. The work conducted in [7] is expanded upon by considering an effective viscosity of the form $E = E\left(x, y, \frac{\partial \overline{v}_x}{\partial y}, \frac{\partial^2 \overline{v}_x}{\partial y^2}\right)$ so that the extended Prandtl model can be investigated.

In algebraic closure models, the effective viscosity is expressed as the sum of the kinematic viscosity, ν, and the turbulent or eddy viscosity, ν_T [1]:

$$E = \frac{\mu + \mu_T}{\rho} = \nu + \nu_T. \tag{2.1}$$

For Prandtl's extended model, the turbulent viscosity is of the form

$$\nu_T = \nu_T\left(x, y, \frac{\partial \overline{v}_x}{\partial y}, \frac{\partial^2 \overline{v}_x}{\partial y^2}\right). \tag{2.2}$$

The Reynolds number for the mean flow is defined as

$$Re = \frac{UL}{E_C}. \tag{2.3}$$

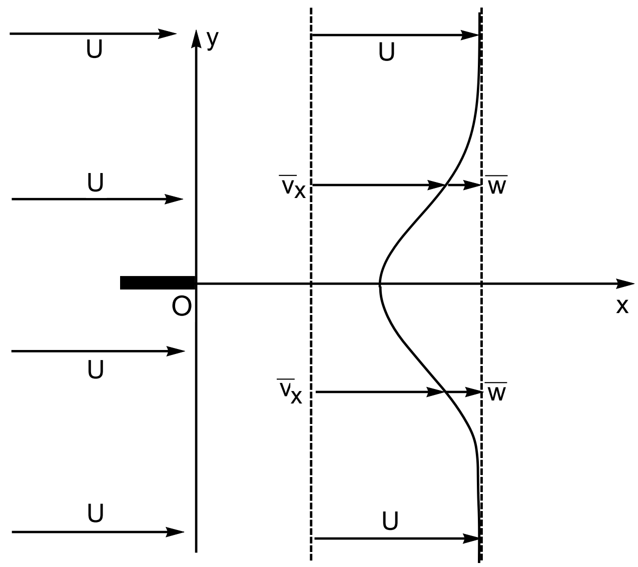

Fig. 1 The two-dimensional turbulent classical far wake behind a thin symmetric planar body aligned with a uniform flow. The mean velocity in the x–direction, $\overline{v}_x$, and the mean velocity deficit, $\overline{w}$, are shown.

Here, L is the length downstream over which the reduction in velocity is not negligible, and $E_C = \nu + \nu_{TC}$ is the characteristic effective viscosity where ν is the viscosity, and ν_{TC} is the characteristic turbulent viscosity.

Boundary layer theory is used to describe wake flows. Dimensionless variables for the x and y coordinates, the mean velocity components $\overline{v}_x$ and $\overline{v}_y$, and the mean fluid pressure $\overline{p}$ are now introduced [10]:

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta} = y\frac{\sqrt{Re}}{L},$$

$$\overline{v}_x{}^* = \frac{\overline{v}_x}{U}, \quad \overline{v}_y{}^* = \overline{v}_y\frac{\sqrt{Re}}{U}, \quad \overline{p}^* = \frac{\overline{p}}{\rho U^2}, \quad E^* = \frac{E}{E_C}. \tag{2.4}$$

The dimensionless effective viscosity is

$$E^* = \frac{\nu}{\nu + \nu_{TC}} + \frac{\nu_{TC}}{\nu + \nu_{TC}}\nu_T^*, \tag{2.5}$$

where $\nu_T^* = \nu_T/\nu_{TC}$ is the dimensionless turbulent viscosity. The dimensionless mean velocities are given by

$$\overline{v}_x{}^*(x^*, y^*) = 1 - \overline{w}^*(x^*, y^*), \quad \overline{v}_y{}^*(x^*, y^*) = 0 + \overline{v}_y{}^*(x^*, y^*), \tag{2.6}$$

where $\overline{w}^*(x^*, y^*)$ is the dimensionless mean velocity deficit in the x–direction. In order to derive the appropriate approximation of the momentum equation for the far wake flow, these variables are substituted into the momentum equation, the boundary layer approximation is implemented, and terms which are products and powers of the velocity deficits or their derivatives are neglected. The y– component of the momentum equation simply gives $\dfrac{d\overline{p}^*}{dy^*} = 0$, and from mainstream matching, $\dfrac{d\overline{p}^*}{dx^*} = 0$. Thus, there is no external pressure gradient. The momentum equation in the x– direction reduces to

$$\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left[E\left(x, y, \frac{\partial \overline{w}}{\partial y}, \frac{\partial^2 \overline{w}}{\partial y^2}\right)\frac{\partial \overline{w}}{\partial y}\right], \tag{2.7}$$

where the star notation has been suppressed for convenience. The momentum equation must be solved subject to

$$\overline{w}(x, \pm y_b) = 0, \qquad \frac{\partial \overline{w}}{\partial y}(x, \pm y_b) = 0, \tag{2.8}$$

$$\frac{\partial \overline{w}}{\partial y}(x, 0) = 0, \tag{2.9}$$

where the boundary $y = \pm y_b(x)$ is left unspecified. For a wake that extends to infinity in the y–direction, $y_b(x) = \infty$. For the purpose of obtaining numerical solutions, $y_b(x)$ can be considered to be the effective half width of the wake. The first conditions, (2.8), state that the turbulent wake flow merges smoothly with the mainstream flow. The second condition, (2.9), expresses the condition that the mean velocity deficit is a maximum on the center-line.

In order to derive the conserved quantity, Equation (2.7) is integrated with respect to y over the width of the wake. The boundary conditions, (2.8), are imposed. This results in the condition

$$2\int_0^{y_b(x)} \overline{w}\,dy = D, \tag{2.10}$$

where D is the drag.

Prandtl's extended mixing length model states that the effective viscosity is of the form

$$E(x, \overline{w}_y, \overline{w}_{yy}) = \nu + l_1^2(x)\left[(\overline{w}_y)^2 + l_2^2(x)(\overline{w}_{yy})^2\right]^{1/2}, \tag{2.11}$$

where l_1 and l_2 are known as the mixing lengths. In order to express this in the form

$$E^* = \frac{\nu}{\nu + \nu_{TC}} + \frac{\nu_{TC}}{\nu + \nu_{TC}}\nu_T^*, \tag{2.12}$$

dimensionless mixing lengths corresponding to l_1 and l_2 need to be defined. The mixing lengths are chosen to scale with the boundary layer thickness, δ. In other words,

$$l_1^* = \frac{l_1}{\delta}, \quad l_2^* = \frac{l_2}{\delta}. \tag{2.13}$$

In terms of the dimensionless variables, $\nu_{TC}\nu_T^*$ is given by

$$\nu_{TC}\nu_T^* = U\delta (l_1^*)^2 \left[\left(\overline{w}^*_{y^*}\right)^2 + (l_2^*)^2 \left(\overline{w}^*_{y^*y^*}\right)^2 \right]^{1/2}, \tag{2.14}$$

which shows that

$$\nu_{TC} = U\delta. \tag{2.15}$$

Suppressing the star notation for convenience,

$$E\left(x, \frac{\partial \overline{w}}{\partial y}, \frac{\partial^2 \overline{w}}{\partial y^2}\right) = \frac{\nu}{\nu + \nu_{TC}} + \frac{\nu_{TC}}{\nu + \nu_{TC}} l_1^2(x) \left[\left(\frac{\partial \overline{w}}{\partial y}\right)^2 + l_2^2(x) \left(\frac{\partial^2 \overline{w}}{\partial y^2}\right)^2 \right]^{1/2}. \tag{2.16}$$

Substituting (2.16) into (2.7) gives

$$\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left[\frac{\nu}{\nu + \nu_{TC}} + \frac{\nu_{TC}}{\nu + \nu_{TC}} l_1^2(x) \left[\left(\frac{\partial \overline{w}}{\partial y}\right)^2 + l_2^2(x) \left(\frac{\partial^2 \overline{w}}{\partial y^2}\right)^2 \right]^{1/2} \frac{\partial \overline{w}}{\partial y} \right]. \tag{2.17}$$

As an initial investigation, the kinematic viscosity is neglected. The first term on the right hand side of Equation (2.17) can be neglected, and $\nu + \nu_{TC} \approx \nu_{TC}$. Neglecting the kinematic viscosity leads to

$$\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left[l_1^2(x) \left[\left(\frac{\partial \overline{w}}{\partial y}\right)^2 + l_2^2(x) \left(\frac{\partial^2 \overline{w}}{\partial y^2}\right)^2 \right]^{1/2} \frac{\partial \overline{w}}{\partial y} \right]. \tag{2.18}$$

3 Similarity solutions

In this section, similarity solutions admitted by (2.18) are considered. The partial differential equation is then reduced to an ordinary differential equation. Expressions for l_1 and l_2 cannot be obtained when the kinematic viscosity is neglected. Instead, Prandtl's hypothesis that these mixing lengths are proportional to the width of the wake, which behaves as $\sqrt{2x}$, is imposed. This gives

$$l_1(x) = l_{01}\sqrt{2x}, \tag{3.1}$$

$$l_2(x) = l_{02}\sqrt{2x}, \tag{3.2}$$

where l_{01} and l_{02} are constants that can be obtained either numerically or from experimental results. Because there is no extrinsic length scale for this problem, it is reasonable to seek for similarity solutions. The width of the wake behaves like $\sqrt{2x}$, so the similarity variable

$$\xi(x,y) = \frac{y}{\sqrt{2x}}, \tag{3.3}$$

is defined. Let

$$\overline{w}(x,y) = \frac{F(\xi)}{\sqrt{2x}}, \tag{3.4}$$

where F is a function to be determined. Substituting (3.3) and (3.4) into (2.18) results in the ordinary differential equation

$$\frac{d}{d\xi}\left[l_{01}^2\left[\left(F'\right)^2 + l_{02}^2\left(F''\right)^2\right]^{1/2} F'\right] + \frac{d}{d\xi}\left[\xi F\right] = 0. \tag{3.5}$$

In terms of the similarity variables, the conserved quantity, (2.10), becomes

$$\int_0^{y_b(x)/\sqrt{2x}} F\,d\xi = \frac{D}{2}, \tag{3.6}$$

and because this is independent of x,

$$y_b(x) = \xi_b\sqrt{2x}, \tag{3.7}$$

where ξ_b is a constant that remains to be determined. The conserved quantity becomes

$$\int_0^{\xi_b} F\,d\xi = \frac{D}{2}. \tag{3.8}$$

The boundary conditions from Equations (2.8) and (2.9) are, in terms of F

$$F(\pm\xi_b) = 0, \quad F'(\pm\xi_b) = 0, \tag{3.9}$$

$$F'(0) = 0. \tag{3.10}$$

Equation (3.5) can be integrated once. Applying the boundary conditions, (3.9), results in a zero constant of integration. Thus,

$$l_{01}^2\left[\left(F'\right)^2 + l_{02}^2\left(F''\right)^2\right]^{1/2} F' + \xi F = 0. \tag{3.11}$$

4 Numerical results

In this section the numerical method used to solve the ordinary differential equation, (3.11), subject to the boundary conditions and the conserved quantity is presented. Because the wake is symmetric about the x–axis, it is convenient to consider only the upper half of the wake. First let

$$\overline{\xi} = \frac{\xi}{\xi_b}. \tag{4.1}$$

Substituting into (3.8)–(3.11) and omitting the bars for convenience gives

$$\frac{l_{01}^2}{\xi_b^3}\left[\left(F'\right)^2+\frac{l_{02}^2}{\xi_b^2}\left(F''\right)^2\right]^{1/2}F'+\xi F=0, \tag{4.2}$$

$$F\left(1\right)=0, \quad F'\left(1\right)=0, \tag{4.3}$$

$$F'\left(0\right)=0, \tag{4.4}$$

$$\int_0^1 F\left(\xi\right)d\xi=\frac{D}{2\xi_b}. \tag{4.5}$$

Now, in the upper half of the wake, $F'\leq 0$ and so (4.2) can be written in the form

$$l_{01}^2\left[\left(F'\right)^2+\frac{l_{02}^2}{\xi_b^2}\left(F''\right)^2\right]^{1/2}\left|F'\right|=\xi_b^3\xi F. \tag{4.6}$$

Squaring both sides and solving for F'' leads to

$$F''=\pm\frac{\xi_b^4}{l_{02}l_{01}^2}\left[\frac{\xi^2F^2}{\left(F'\right)^2}-\frac{l_{01}^4}{\xi_b^6}\left(F'\right)^2\right]^{1/2}. \tag{4.7}$$

Let

$$G=F'. \tag{4.8}$$

Then Equation (4.7) can be written as two first order differential equations:

$$F'=G, \tag{4.9}$$

$$G'=\pm\frac{\xi_b^4}{l_{02}l_{01}^2}\left[\frac{\xi^2F^2}{G^2}-\frac{l_{01}^4}{\xi_b^6}G^2\right]^{1/2}. \tag{4.10}$$

In terms of F and G, the boundary conditions (4.3) and (4.4) become

$$F(1)=0, \quad G(1)=0, \tag{4.11}$$

$$G(0)=0. \tag{4.12}$$

Although both boundary conditions on G are not required since the differential equation for G is of first order, using the condition $G(0)=0$ is convenient as solving for G results in solving an initial value problem. However, the only condition on F is at a boundary, and so a shooting method is required to solve for F.

In the upper half of the wake, $F'=G\leq 0$. From $G(0)=0$ and the fact that $G\leq 0$, it is seen that the negative root in Equation (4.10) must be taken. Using a forward difference scheme,

$$F_{n+1}=\Delta\xi G_n+F_n, \tag{4.13}$$

$$G_{n+1} = -\Delta\xi \frac{\xi_b^4}{l_{02} l_{01}^2} \left[\frac{\xi_n^2 F_n^2}{G_n^2} - \frac{l_{01}^4}{\xi_b^6} G_n^2 \right]^{1/2} + G_n, \tag{4.14}$$

where $\Delta\xi$ is the chosen step-size. The initial value for G is $G_1 = 0$ and at the end boundary, $F_N = 0$. As mentioned previously, a shooting method is required to solve for F. An initial guess for F_1 is obtained from the conserved quantity and the most basic approximation to it:

$$\int_0^1 F(\xi) d\xi \approx \frac{1}{2} (F_1 + F_N) = \frac{1}{2} F_1 = \frac{D}{2\xi_b}. \tag{4.15}$$

So,

$$F_1 = \frac{D}{\xi_b}. \tag{4.16}$$

The value of ξ_b is also not known and must be determined from the conserved quantity. Thus a double shooting method is required to determine the boundary value problem for F, and the value of ξ_b.

The process is as follows: An initial value for ξ_b is chosen. Initially, choose $\xi_b = 1$. The value for F_1 is obtained from (4.16). For the chosen ξ_b value, the boundary value problem for F is solved. Once the solutions for F and G are obtained, the conserved quantity is evaluated and the value of ξ_b is updated. This process is continued until convergence is achieved.

For illustration purposes, let

$$\frac{\xi_b^4}{l_{02} l_{01}^2} = 1, \quad \frac{l_{01}^4}{\xi_b^6} = 0.05, \quad D = 0.1. \tag{4.17}$$

A step size value of $\Delta\xi = 0.001$ is used. A plot of the similarity profile is shown in Figure 2.

5 Further work and conclusions

In this work, the extended Prandtl closure model was applied to the two-dimensional turbulent classical far wake. A similarity solution that left both the governing equation for the stream-wise mean velocity deficit in the x–direction and the conserved quantity invariant, was obtained. The governing partial differential equation was reduced to a second order ordinary differential equation. This second order differential equation was then expressed as two first order ordinary differential equations. Numerical methods were required to solve these two equations. The numerical method of choice involved using a double shooting method to solve a boundary value problem and the unknown value of the boundary which was determined from the conserved quantity. A plot of the similarity velocity profile was provided for illustrative purposes.

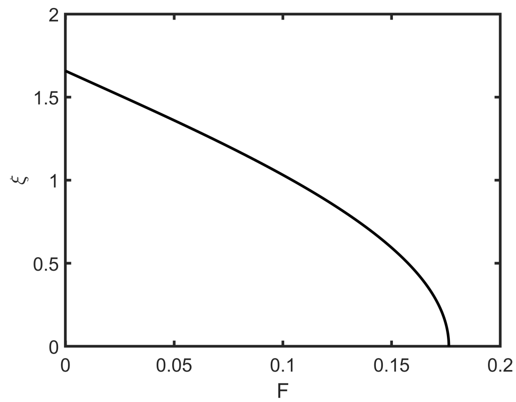

Fig. 2 Similarity profile of the mean velocity deficit.

The numerical scheme presented in Section 4 takes time to converge and is very sensitive to the initial choices for ξ_b and F_1. It also appears that convergence is only achieved for a limited range of values of l_{01} and l_{02}. A much more in-depth analysis of this numerical scheme is required. Alternative schemes need to be developed that allow for faster convergence without compromising accuracy.

The values for l_{01} and l_{02} were chosen arbitrarily. In order to obtain the correct order of magnitude for these values, the numerical result must be compared to experimental data. To date, the skills required for fitting parameters in a model to data are being investigated.

Once an improved numerical scheme is developed, and parameter fitting methods are well-understood, the aim is then to compare the different closure models.

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