Controlling stability of longwave oscillatory Marangoni patterns

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Abstract We apply nonlinear feedback control to govern the stability of long-wave oscillatory Marangoni patterns. We focus on the patterns caused by instability in thin liquid film heated from below with a deformable free surface. This instability emerges in the case of substrate of low thermal conductivity, when two monotonic long-wave instabilities, Pearson's and deformational, are coupled. We provide weakly nonlinear analysis within the amplitude equations, which govern the evolution of the layer thickness and the temperature deviation. The action of the nonlinear feedback control on the nonlinear interaction of two standing waves is investigated. It is shown that quadratic feedback control can produce additional stable structures (standing rolls and standing squares), which are subject to instability leading to traveling wave in the uncontrolled case.

1 Introduction

The onset and development of oscillatory Marangoni convection in a thin film heated from below without control was recently investigated by Shklyaev *et al.* [5]. Among all the variety of possible patterns, only a few were stable: one-dimensional traveling waves, traveling rectangles and alternating rolls. In this paper we aim at revealing more complex and exotic stable patterns, such as alternating rolls and standing squares.

We have recently considered the influence of the feedback control on the oscillatory Marangoni instability in a thin film heated from below. We have shown that

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a linear control gain can delay the onset of instability [3] and a quadratic control gain can eliminate the subcritical excitation of instability [4]. The analysis of pattern formation was done for an infinite region, nonlinear interaction of the traveling waves was considered. In the case of traveling waves we showed that quadratic feedback control can produce additional stable structures, besides conventional traveling rolls. However, in a realistic system the reflection of waves on the lateral boundaries results in emergence of standing waves, which can interact to each other. Extending our previous investigation, we examine here the effect of nonlinear feedback control on development of Marangoni instability in a system of standing waves propagating with a definite angle between the wave vectors.

The paper is organized as follows. We start with the mathematical formulation of the long-wave Marangoni convection problem in Sec. 2. There we present a set of coupled amplitude equations which governs the evolution of the layer thickness and the temperature deviation under nonlinear feedback control [4]. In Sec. 3 we perform the weakly nonlinear stability analysis of wave patterns within these amplitude equations. Nonlinear interaction of standing waves is investigated by means of the analysis of a system of four complex Landau equations. The paper concludes with summary in Sec. 4.

2 Amplitude Equations

We consider a horizontal liquid layer confined between a deformable free upper surface and a solid bottom wall. The layer is heated from below; the thermal conductivity of the liquid λ is assumed to be large in comparison with that of the substrate, so that the vertical component of the heat flux λA is fixed. The unperturbed layer thickness H is assumed sufficiently small, so that the influence of buoyancy is negligible and the free surface deformation is important. The surface tension decreases linearly with the temperature: $\sigma = \sigma_0 - \sigma_T T$, where T is the deviation of the temperature from a reference one, which is the temperature of the gas above the liquid layer. The heat flux from the free surface is governed by Newton's law of cooling, which describes the rate of heat transfer from the liquid to the ambient gas phase with the heat transfer coefficient q. The Cartesian reference frame is chosen in such a way that the x- and y-axes are in the substrate plane and the z-axis is normal to the substrate.

The problem of convective instability in the given system is characterized by the following dimensionless parameters,

$$Ca = \frac{\sigma_0 H}{\rho \nu \chi},\ Bi = \frac{qH}{\lambda},\ Ga = \frac{gH^3}{\nu \chi},\ Ma = \frac{\sigma_T A H^2}{\rho \nu \chi},$$

which are the capillary, Biot, Galileo and Marangoni numbers, respectively. Here g is the gravitational acceleration, χ is the thermal diffusivity, ρ is the density, and ν is the kinematic viscosity.

In the uncontrolled case, the oscillatory long-wave Marangoni instability was revealed in [5]. To govern this instability we apply the feedback control based on the measurement of the temperature deviation on the free surface from its value in the conductive state. This feedback control strategy was recently demonstrated as the most effective one to delay the onset of instability under consideration [3].The heat flux applied on the solid substrate is changed as

$$\left.\frac{\partial T}{\partial z}\right|_{z=0} = -1 - K(f)f,\; f = T|_{z=h} - \left.T^{(0)}\right|_{z=1}, \tag{1}$$

where $T^{(0)}$ is the temperature of no-motion state, h is the local layer thickness, K is the non-dimensional scalar control gain.

Within the lubrication approximation we employ a standard long-wave scaling

$$x = \varepsilon^{-1}X,\; y = \varepsilon^{-1}Y,\; t = \varepsilon^{-2}\tau \tag{2}$$

and restrict ourselves to following assumptions

$$Ca = \varepsilon^{-2}C,\; Bi = \varepsilon^{2}\beta,\; K = \varepsilon^{2}\varkappa, \tag{3}$$

where $\varepsilon \ll 1$ can be thought of as the ratio of H to a typical horizontal lengthscale.

The long-wave Marangoni convection in this layer is governed by the following system of dimensionless amplitude equations [4]

$$\frac{\partial h}{\partial \tau} = \nabla\cdot\left(\frac{h^3}{3}\nabla P + Ma\frac{h^2}{2}\nabla f\right) \equiv \nabla\cdot\vec{j}, \tag{4}$$

$$h\frac{\partial \Theta}{\partial \tau} = \nabla\cdot(h\nabla\Theta) - \frac{1}{2}(\nabla h)^2 - (\beta - \varkappa(f))f + \vec{j}\cdot\nabla f + \nabla\cdot\left(\frac{h^4}{8}\nabla P + \frac{h^3}{6}Ma\nabla f\right), \tag{5}$$

where $\Theta(X,Y,\tau)$ is the temperature deviation from its conductive value

$$T = -z + \frac{1}{Bi} + \Theta. \tag{6}$$

Here $P = Gah - C\nabla^2 h,\; f = \Theta - h$ has a meaning of perturbation of the free surface temperature; $\nabla = (\partial/\partial X, \partial/\partial Y, 0)$. The vector $-\vec{j}$ has a meaning of the longitudinal flux of a liquid integrated across the layer.

Hereinafter we assume that the term corresponding to the feedback control in (5) is a quadratic polynomial of the free surface temperature perturbation:

$$\varkappa(f)f = \varkappa_l f + \varkappa_q f^2, \tag{7}$$

where $\varkappa_l$ and $\varkappa_q$ are constant.

The influence of the linear part of control gain $\varkappa_l$ can be expressed as replacement $\beta \to \beta - \varkappa_l$ in formulas describing the instability threshold [3]. The quadratic part of control gain $\varkappa_q$ affects the nonlinear development of instability. In the following sections we investigate the influence of a nonlinear feedback control on the pattern formation (the linear part $\varkappa_l$ will be omitted). Specifically, we are interested in the elimination of subcritical instability.

3 Weakly Nonlinear Analysis

Below we study the nonlinear dynamics of small perturbations close to the threshold of the oscillatory instability Ma_0

$$Ma - Ma_0 = \delta^2 Ma_2, \delta \ll 1 \, , \tag{8}$$

where $Ma_0 = 3 + Ga + Ck^2 + 3\beta/k^2$ is obtained from the linear analysis [3].

3.1 Basic Expansions

We present h, Θ, Ma and the time derivative as a series in power of the small parameter δ:

$$h = 1 + \delta\xi_1 + \delta^2\xi_2 + \dots \, , \Theta = 1 + \delta\theta_1 + \delta^2\theta_2 + \dots \, , \frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_0} + \delta^2\frac{\partial}{\partial\tau_2} + \dots , \tag{9}$$

where two time variables, τ_0 and τ_2, are introduced according to the multiscale approach [2] as the dynamics of wave patterns is characterized by two different time scales. The frequency of oscillations is of order of 1, while the growth rate of disturbances is of the order of $Ma - Ma_0$, i.e. $O(\delta^2)$

Substituting the ansatz (9) into equations (4)-(5), and collecting the terms of equal powers in δ, we obtain at the first order the linear stability problem. Its solution can be presented as

$$\xi_1 = \sum_{j=1}^{n} A_j(\tau_2)\, exp\left(\mathrm{i}\vec{k}_j \cdot \vec{r} - \mathrm{i}\omega\tau_0\right) + c.c. \, , \tag{10}$$

$$\theta_1 = (\alpha + 1) \sum_{j=1}^{n} A_j(\tau_2)\, exp\left(\mathrm{i}\vec{k}_j \cdot \vec{r} - \mathrm{i}\omega\tau_0\right) + c.c. \, , \tag{11}$$

where c.c. denotes complex conjugate terms, $|\vec{k}_j| = k$ is the wavenumber, $\alpha = -2\left(Ga + Ck^2\right)/3Ma_0 + 2\mathrm{i}\omega/Ma_0 k^2$. Frequency of neutral perturbations is determined by formula

$$\omega = \frac{k^2}{12}\sqrt{(72+Ga+Ck^2)\,(Ma_{mon}-Ma_0)},$$

where

$$Ma_{mon} = \frac{48\left(\beta+k^2\right)\left(Ga+Ck^2\right)}{k^2\,(72+Ga+Ck^2)}$$

is the threshold of a monotonic instability [3].

The analysis can be done for any k, but the case of the critical wavenumber k_c, corresponding to the minimum of the neutral curve, is especially important, because one can expect that patterns with the wavenumber k_c will appear in the natural way by the growth of Ma. Below we consider the nonlinear interaction of disturbances and the wave patterns supported by that interaction. The computations will be done for $k = k_c$, $k_c^2 = \sqrt{3\beta}$.

3.2 Interaction of Waves

In order to investigate the nonlinear interaction of waves, consider the class of solutions corresponding to two pairs of waves with the wave vectors $\pm\vec{k_1}$, $\pm\vec{k_2}$, where $\vec{k_1} = (k,0)$ and $\vec{k_2} = (k\cos\phi, k\sin\phi)$, that propagate with a phase velocity ω/k and complex amplitudes $A_{1,2}$ and $B_{1,2}$

$$\xi_1 = \left[A_1(\tau_2)\mathrm{e}^{\mathrm{i}kX} + A_2(\tau_2)\mathrm{e}^{-\mathrm{i}kX} + B_1(\tau_2)\mathrm{e}^{\mathrm{i}\vec{k_2}\cdot\vec{r}} + B_2(\tau_2)\mathrm{e}^{-\mathrm{i}\vec{k_2}\cdot\vec{r}}\right]\mathrm{e}^{\mathrm{i}\omega\tau_0} + c.c. \quad (12)$$

Here ϕ is an arbitrary angle different from 0 and π. That class of solutions includes travelling and standing waves as particular cases.

At the second order we obtain

$$\frac{\partial\xi_2}{\partial\tau_0} - \Delta\left(\frac{1}{3}P_2 + \frac{Ma_0}{2}f_2\right) = \nabla\cdot\left(\xi_1\nabla P_1 + Ma_0\xi_1\nabla f_1\right)\,, \quad (13)$$

$$\frac{\partial\theta_2}{\partial\tau_0} - \Delta\left(\theta_2 + \frac{1}{8}P_2 + \frac{Ma_0}{6}f_2\right) + \beta f_2 = -\xi_1\frac{\partial\theta_1}{\partial\tau_0} + \nabla\cdot(\xi_1\nabla\theta_1) - \frac{1}{2}(\nabla\xi_1)^2 + \varkappa_q f_1^2 + \left(\frac{1}{3}P_1 + \frac{Ma_0}{2}f_1\right)\cdot\nabla f_1 + \nabla\cdot\left(\frac{\xi_1}{2}\nabla P_1 + \frac{Ma_0}{2}\xi_1\nabla f_1\right), \quad (14)$$

where $P_{1,2} = Ga\xi_{1,2} - C\Delta\xi_{1,2}$, $f_{1,2} = \theta_{1,2} - \xi_{1,2}$. The solution can be chosen in the form

$$\xi_2 = a_{10}\left(A_1B_2^* + A_2B_1^*\right)\mathrm{e}^{\mathrm{i}\psi_+} + a_{1-0}\left(A_1B_1^* + A_2B_1^*\right)\mathrm{e}^{\mathrm{i}\psi_-}$$
$$+\Big[a_{11}\left(A_1B_1\mathrm{e}^{\mathrm{i}\psi_+} + A_2B_2\mathrm{e}^{-\mathrm{i}\psi_+}\right) + a_{1-1}\left(A_1B_2\mathrm{e}^{\mathrm{i}\psi_-} + A_2B_1\mathrm{e}^{-\mathrm{i}\psi_-}\right)$$
$$+ a_{22}\left(A_1^2\mathrm{e}^{2\mathrm{i}kX} + A_2^2\mathrm{e}^{-2\mathrm{i}kX} + B_1^2\mathrm{e}^{2\mathrm{i}\vec{k}_2\cdot\vec{r}} + B_2^2\mathrm{e}^{-2\mathrm{i}\vec{k}_2\cdot\vec{r}}\right)\Big]\mathrm{e}^{2\mathrm{i}\omega\tau_0}$$
$$+a_{20}\left(A_1A_2^*\mathrm{e}^{2\mathrm{i}kX} + B_1B_2^*\mathrm{e}^{2\mathrm{i}\vec{k}_2\cdot\vec{r}}\right) + c.c. \qquad (15)$$

$$\theta_2 = b_{20}\left(A_1A_2^*\mathrm{e}^{2\mathrm{i}kX} + B_1B_2^*\mathrm{e}^{2\mathrm{i}\vec{k}_2\cdot\vec{r}}\right) + b_{02}\left(A_1A_2 + B_1B_2\right)\mathrm{e}^{2\mathrm{i}\omega\tau_0}$$
$$+b_{10}\left(A_1B_2^* + A_2B_1^*\right)\mathrm{e}^{\mathrm{i}\psi_+} + B_{1-0}\left(A_1B_1^* + A_2B_1^*\right)\mathrm{e}^{\mathrm{i}\psi_-}$$
$$+\Big[b_{11}\left(A_1B_1\mathrm{e}^{\mathrm{i}\psi_+} + A_2B_2\mathrm{e}^{-\mathrm{i}\psi_+}\right) + B_{1-1}\left(A_1B_2\mathrm{e}^{\mathrm{i}\psi_-} + A_2B_1\mathrm{e}^{-\mathrm{i}\psi_-}\right)$$
$$+ b_{22}\left(A_1^2\mathrm{e}^{2\mathrm{i}kX} + A_2^2\mathrm{e}^{-2\mathrm{i}kX} + B_1^2\mathrm{e}^{2\mathrm{i}\vec{k}_2\cdot\vec{r}} + B_2^2\mathrm{e}^{-2\mathrm{i}\vec{k}_2\cdot\vec{r}}\right)\Big]\mathrm{e}^{2\mathrm{i}\omega\tau_0}$$
$$+b_{00}\left(|A_1|^2 + |A_2|^2 + |B_1|^2 + |B_2|^2\right) + c.c., \qquad (16)$$

where $\psi_+ = kX + \vec{k}_2\cdot\vec{r}$, $\psi_- = kX - \vec{k}_2\cdot\vec{r}$. Hereafter the asterisk denotes the complex-conjugate term; b_{00}, b_{02}, a_{10}, $b_{10}, \ldots$, b_{1-1} are constants, which are very cumbersome and therefore they are not given here.

At the third order in δ, we obtain

$$\frac{\partial \xi_3}{\partial \tau_0} - \Delta\left(\frac{1}{3}P_3 + \frac{Ma_0}{2}f_3\right) = F^{(1)}\,, \qquad (17)$$

$$\frac{\partial \theta_3}{\partial \tau_0} - \Delta\left(\theta_3 + \frac{1}{8}P_3 + \frac{Ma_0}{6}f_3\right) + \beta f_3 = F^{(2)}\,, \qquad (18)$$

where $P_3 = Ga\xi_3 - C\Delta\xi_3$, $f_3 = \theta_3 - \xi_3$; inhomogeneities $F^{(1,2)}$ are defined as

$$F^{(1)} = -\frac{\partial \xi_1}{\partial \tau_2} + \frac{1}{2}Ma_2\Delta f_1 + \nabla\cdot\left(Ma_0\xi_1\nabla f_2 + \xi_1\nabla P_2\right)$$
$$+\nabla\cdot\left[\xi_1^2\left(\nabla P_1 + \frac{Ma_0}{2}\nabla f_1\right) + \xi_2\left(\nabla P_2 + Ma_0\nabla f_1\right)\right], \qquad (19)$$

$$F^{(2)} = -\frac{\partial \theta_1}{\partial \tau_2} - \xi_2\frac{\partial \theta_1}{\partial \tau_0} - \xi_1\frac{\partial \theta_2}{\partial \tau_0} + 2\varkappa_q f_1 f_2 + \frac{1}{6}Ma_2\Delta f_1$$
$$-\nabla\xi_1\cdot\nabla\xi_2 + \nabla\cdot\left(\xi_1\nabla\theta_2 + \xi_2\nabla\theta_1\right) + \frac{1}{3}\nabla P_2\cdot\nabla f_1$$
$$+\nabla P_1\cdot\left(\xi_1\nabla f_1 + \frac{1}{3}\nabla f_2\right) + \frac{3}{4}\nabla\cdot\left(\xi_1^2\nabla P_1\right) + \frac{1}{2}\nabla\cdot\left(\xi_1\nabla P_2 + \xi_2\nabla P_1\right)$$
$$+Ma_0\left[\xi_1{\nabla f_1}^2 + \nabla f_1\cdot\nabla f_2 + \frac{1}{2}\nabla\cdot\left[\left(\xi_1^2 + \xi_2\right)\nabla f_1\right] + \frac{1}{2}\nabla\cdot\left(\xi_1\nabla f_2\right)\right]\,. \qquad (20)$$

The solvability condition at the third order can be formulated as

$$\left(\mathrm{i}\omega+\frac{Ma_0k^2}{6}+k^2+\beta\right)F_{sec}^{(1)}=\frac{Ma_0k^2}{2}F_{sec}^{(2)}\,, \tag{21}$$

where $F_{sec}^{(1,2)}$ are secular parts of inhomogeneities. It yields a set of four complex differential equations that govern the evolution of wave amplitudes $A_{1,2}$ and $B_{1,2}$

$$\begin{aligned}
\frac{dA_1}{d\tau_2}&=\left(\gamma-K_0|A_1|^2-K_1|A_2|^2-K_2(\phi)|B_1|^2-K_2(\pi-\phi)|B_2|^2\right)A_1-K_3(\phi)A_2{}^*B_1B_2\\
\frac{dA_2}{d\tau_2}&=\left(\gamma-K_0|A_2|^2-K_1|A_1|^2-K_2(\phi)|B_2|^2-K_2(\pi-\phi)|B_1|^2\right)A_2-K_3(\phi)A_1{}^*B_1B_2\\
\frac{dB_1}{d\tau_2}&=\left(\gamma-K_0|B_1|^2-K_1|B_2|^2-K_2(\phi)|A_1|^2-K_2(\pi-\phi)|A_2|^2\right)B_1-K_3(\phi)B_2{}^*A_1A_2\\
\frac{dB_2}{d\tau_2}&=\left(\gamma-K_0|B_2|^2-K_1|B_1|^2-K_2(\phi)|A_2|^2-K_2(\pi-\phi)|A_1|^2\right)B_2-K_3(\phi)B_1{}^*A_1A_2
\end{aligned} \tag{22}$$

Here

$$\gamma=\frac{k^2Ma_2}{2}\left(1-\mathrm{i}\frac{3k^2\left(Ga+Ck^2+72\right)}{2\omega_0}\right),$$

expressions for Landau coefficients K_0, K_1, $K_2(\phi)$ and $K_3(\phi)$ are very cumbersome and therefore they are not given here.

Equations (22) were studied in detail by [6] in the case of square symmetry, i.e. for $\phi=\pi/2$. They found six types of solutions.

(i)	Traveling rolls (TR)	$\lvert A_1\rvert^2=\gamma_r/K_{0r}$, $A_2=B_1=B_2=0$.
(ii)	Standing rolls (SR)	$A_1=A_2,\ \lvert A_1\rvert^2=\gamma_r/\left(K_{0r}+K_{1r}\right),\ B_1=B_2=0$.
(iii)	Traveling squares (TS)	$A_1=B_1,\ \lvert A_1\rvert^2=\gamma_r/\left(K_{0r}+K_{2r}\right),\ A_2=B_2=0$.
(iv)	Standing squares (SSq)	$A_1=A_2=B_1=B_2$, $\lvert A_1\rvert^2=\gamma_r/\left(K_{0r}+K_{1r}+2K_{2r}+K_{3r}\right)$.
(v)	Alternating rolls (AR)	$A_1=A_2=\mathrm{i}B_1=\mathrm{i}B_2$, $\lvert A_1\rvert^2=\gamma_r/\left(K_{0r}+K_{1r}+2K_{2r}-K_{3r}\right)$.
(vi)	Standing cross-rolls (SCR)	$A_1=A_2,\ B_1=B_2,\ \lvert A_1\rvert\neq\lvert B_1\rvert$.

For any parameters, we use notation $K_r=\mathrm{Re}K$, $K_i=\mathrm{Im}K$.

A stability analysis for the patterns on the square lattice shows that they are selected if they emerge through the direct Hopf bifurcation ($\gamma_r>0$). The remaining stability conditions also obtained by [6], are as follows

(TR): $K_{0r} < K_{1r},\ K_{0r} < K_{2r}$

(SR): $K_{0r} > K_{1r},\ K_{0r} + K_{1r} - 2K_{2r} < 0,\ |K_0 + K_1 - 2K_2|^2 > |K_3|^2$.

(TS): $K_{0r} > K_{2r},\ K_{0r} - K_{1r} - K_{3r} < 0,\ K_{0r} - K_{1r} + K_{3r} < 0$.

(SSq): $K_{0r} + K_{1r} - 2K_{2r} - 3K_{3r} > 0,\ K_{0r} - K_{1r} - K_{3r} > 0,$
$[K_3^* (K_0 + K_1 - 2K_2)]_r < |K_3|^2$.

(AR): $K_{0r} + K_{1r} - 2K_{2r} + 3K_{3r} > 0,\ K_{0r} - K_{1r} + K_{3r} > 0,$
$-[K_3^* (K_0 + K_1 - 2K_2)]_r < |K_3|^2$.

(SCR) is always unstable.

These conditions provide boundaries of selection between two stable patterns. Obviously, equation $K_0 = K_{2r}$ defines boundary between stable TR and TS; equation $K_{0r} = K_{1r}$ – between stable TR and SR. Equations $K_{0r} - K_{1r} = K_{3r}$ and $K_{0r} - K_{1r} = -K_{3r}$ define selection between stable TS and SSq or AR, respectively.

Below we apply the general results described above to the particular problem, which is the subject of the present paper. Our goal is the computation of coefficients K_{0r}, K_{1r}, $K_{2r}(\pi/2)$ and $K_{3r}(\pi/2)$ as functions of the problem parameters, which are β, Ga and $\varkappa_q$.

For uncontrolled convection, pattern selection was investigated previously in the case $\phi = \pi/2$ [5]. It was shown that a small area of stable alternating rolls was discovered (see Fig.1 (a)). However, this area intersects with the domain of subcritical traveling rolls, so here depending on the initial condition the system either approaches AR or demonstrates the infinite growth of one of the amplitudes. Note, that the boundary of stability for alternating rolls here is defined by condition $K_{0r} - K_{1r} + K_{3r} < 0$ corresponding to the boundary between AR and TS. Thus, alternating rolls first become unstable against traveling squares, that in turn become unstable against traveling rolls.

3.3 Nonlinear Feedback Control

Quadratic control gain varies Landau coefficients, resulting in a change of stability boundaries for the patterns.

Influence of the quadratic feedback control on pattern selection for $\phi = \pi/2$ is presented in Fig.1 (b). Recall that the oscillatory instability is critical only inside the domain bounded by the dashed line in Fig.1.

Positive control gain reduces the domain of stability for traveling wave, whereas the domain of subcriticality for traveling wave is enlarged. Additional domain of subcriticality arises due to the standing squares. Stable standing squares emerge for $\varkappa_q = 0.1$ instead of alternating rolls in the uncontrolled case. However, the domain of stable standing squares intersects with the domain of subcritical traveling rolls, so here depending on the initial condition the system either approaches SSq or

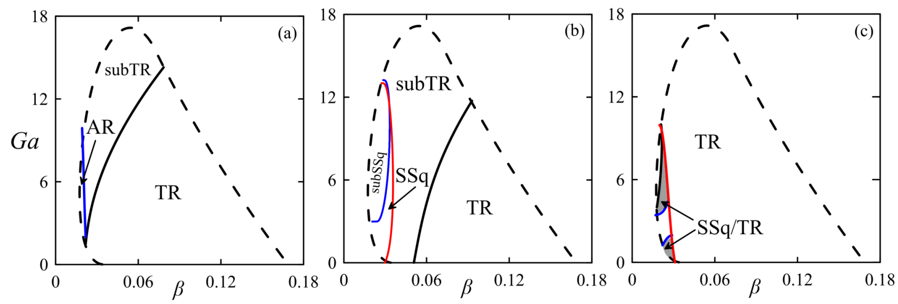

Fig. 1 Pattern selection for $\phi = \pi/2$ in the case of uncontrolled convection **(a)**,for $\varkappa_q = 0.1$ **(b)** and for $\varkappa_q = -0.1$. Domains of stability for traveling rolls, standing squares and alternating rolls are marked by "TR", "SSq" and "AR", respectively. The domains of subcriticality for traveling rolls and standing squares are marked by "subTR" and "subSSq", respectively. Domains of bistability of traveling rolls and standing squares is shaded and marked by "SSq/TR".

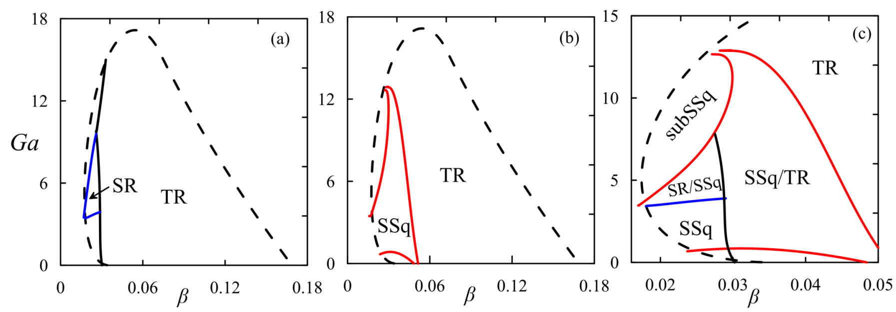

Fig. 2 Pattern selection for $\phi = \pi/2$, $\varkappa_q = -0.2$. Domains of stability for traveling rolls, standing rolls and standing squares are marked "TR", "SR" and "SSq", respectively. Panel **(c)** shows zoomed-in domains of bistability (marked "SR/SSq" and SSq/TR). "subSSq" marks domain of subcriticality for standing squares.

demonstrates the infinite growth of one of the amplitudes. Note, that the boundary of stability for standing squares here is defined by condition $K_{0r} - K_{1r} - K_{3r} < 0$ corresponding to the boundary between SSq and TS. Thus, standing squares first become unstable against traveling squares, that in turn become unstable against traveling rolls.

For negative control gain traveling rolls are stable within the whole domain, where the oscillatory mode is critical, see Fig.1 (c). However, there is a domain of subcriticality for standing squares. Moreover, there are two small areas of stable standing squares, which intersect the domain of stable traveling rolls, resulting in the bistability.

Pattern selection for $\phi = \pi/2$ under the control gain $\varkappa_q = -0.2$ is presented in Fig. 2. Small areas of subcritical traveling rolls, standing rolls and standing squares exist for a small values of β.

Traveling rolls are stable in most of the domain, where the oscillatory mode is critical. But there are also domains of stability for standing rolls and standing squares, see Figs.2(a) and (b), respectively. Note that domains of stability for TR and SSq, SR and SSq intersect partially, resulting in bistability.

4 Conclusions and Discussion

We have studied pattern formation of oscillatory Marangoni instability in a thin film under nonlinear feedback control.

We have performed a weakly nonlinear analysis within the amplitude equations, which describe coupled evolution of the thickness and temperature of thin film in the presence of the nonlinear control. Our analysis is based on the consideration of the nonlinear interaction of a pair of standing waves propagating at the angle ϕ between the wave vectors. That consideration leads to a set of four complex Landau equations that govern the evolution of wave amplitudes. The coefficients of Landau equations, which define pattern formation, have been calculated in the case $\phi = \pi/2$ for different values of the control gain, Galileo and Biot numbers. We have demonstrated, that besides conventional traveling rolls an additional stable patterns (such as standing rolls and standing squares) emerges under nonlinear feedback control. In the case of negative control gain, we have shown that a quadratic control can eliminate the subcritical excitation of instability within entire domain, where oscillatory mode is critical.

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