Compressibility Effect on Markstein Number for a Flame Front in Long-Wavelength Approximation

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Abstract The effect of compressibility on the Markstein number for a planar front of a premixed flame is examined, at small Mach numbers, in the form of M^2-expansions. The method of matched asymptotic expansions is used to analyze the solution in the preheat zone in a power series in two small parameters, the relative thickness of the preheat zone and the Mach number. We employ a specific form of perturbations, valid at long wavelengths, for the thermodynamic variables, which produces the correction term, to the Markstein number, of second order in the Mach number in drastically simple form. Our analysis accounts for the pressure variation as a source term in the heat-conduction equation and calls for the Navier-Stokes equation. The suppression effect of the front curvature on the Darrieus-Landau instability is enhanced by the viscous effect if $Pr > 4/3$, but is weakened if otherwise.

1 Introduction

The pioneering work of the linear stability analysis of a planar front of a premixed flame was made in the low-Mach-number limit by Darrieus [6] and Landau [13, 14] independently. They treated a flame front as a density discontinuity interface accompanied by an essential parameter of the thermal expansion, or the heat release. Their conclusion is that a planar flame front is unstable for small perturbations of any wavelength. This result is now called the Darrieus-Landau instability (DLI),

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or the hydrodynamic instability. Although the DLI effectively explains the intrinsic instability of a planar flame front, stable flame fronts are observed in the laboratory, some contradiction between the DLI and the observation. The DLI imposes a major assumption on the boundary condition that the flame speed S_f is constant, which is expressed in non-dimensional form as

$$S_f = 1. \tag{1}$$

The flame speed S_f is defined as the incoming normal velocity of a gas relative to a flame front and evaluated at the edge of a flame front on the unburned side. The constancy of S_f was modified phenomenologically by Markstein [16]. It is regarded as the curvature effect [5].

$$S_f = 1 - Mr\Delta F, \tag{2}$$

where Mr is the Markstein number and F is the infinitesimal displacement of a planar flame front. Based on (2), not (1), Markstein showed that the DLI is stabilized at sufficiently large values of wavenumber, or small wavelengths, of perturbations. In order to find the expression of the Markstein number Mr in (2), many subsequent works have investigated the transport process inside a flame front in detail. For example, the effect of diffusion properties of the mixture on the flame speed was clarified by Eckhaus [7, 8]. These results were generalized to a more comprehensive concept of flame stretch [2, 17, 18, 22]. However, most of the previous research stayed at the low-Mach-number limit. In such a treatment, the isobaric condition prevails, without having to consider pressure variation.

In this paper, we highlight the compressibility effect and shall derive S_f corrected by the compressibility effect in a tidy form as

$$S_f = 1 - Mr_M\Delta F, \tag{3}$$

with

$$Mr_M = \delta\left\{1 + Ma^2(\gamma - 1)\left(\frac{4}{3}Pr - 1\right)\right\}, \tag{4}$$

where, as will be defined in (11), δ, Ma, γ and Pr are respectively the scale factor of a preheat zone which is an inner structure of a flame front, the Mach number, the specific heats ratio and the Prandtl number (= kinematic viscosity/thermal diffusivity). We reveal that the compressibility effect is accompanied with the viscous effect as evidenced by Pr in (4) for Mr_M which is the extension of the Markstein number Mr to the compressible case. If the value of Pr is larger than $3/4$, the compressibility acts to weaken the DLI for any values of the Mach number. On the other hand, if the value of Pr lies in the range from 0 to $3/4$, then the compressibility reduces the curvature effect.

The flame speed condition (3) is derived from the study of the preheat zone which is the inner structure of a flame front. In the preheat zone, the transport process of the heat and the mass is dominant rather than the exothermic chemical reaction. The effect of the chemical kinetics is confined in the reaction zone, which is the innermost structure of a flame front and is sandwiched by the preheat and the burned zones. Although the investigation of the preheat zone requests the jump (boundary) conditions across the reaction zone contained in it [20, 21], we may dispense with such conditions by postulating the long-wavelength approximation [3], or the translational symmetry. Thanks to the collaboration of the matched asymptotic expansions with respect to δ and the long-wavelength approximation, we reach the substantially compact representation of the Markstein number (4) affected by the compressibility effect.

There are several works on the DLI with the compressibility effect incorporated. Some employed the same assumption as (1) with the density perturbation omitted and concluded that the DLI is enhanced at small values of the Mach number [9, 11, 12]. In contrast to them, works based on the long-wavelength approximation [3, 15] successfully included the density perturbation in the flame speed condition, although the viscous effect is ignored. In [15], the second-order effect of the wavenumber is analysed and the suppression of the DLI by the increasing wavenumber is shown numerically. In this paper, we reveal that the effect of viscosity comes into play for the DLI, with the Navier-Stokes equations coupled to the heat-conduction equations via pressure variation.

We explore the compressibility effect in the form of the M^2 expansions for small Mach numbers Ma ($Ma^2 \ll 1$), under the long-wavelength approximation, as will be exposed in Sect. 2. The scheme of the matched asymptotic expansions with respect to δ ($\ll 1$) and Ma, for deriving the condition of the flame speed based on the first principle, is sketched in Sect. 3. This paper sidesteps handling the reaction term in the heat-conduction equation, but instead, resort to the large activation energy asymptotics. The detailed analysis of the translational symmetry is performed to gain (3), the flame speed with a correction from the weak compressibility effect, in Sect. 4. In Sect. 5, the dispersion relation of a planar flame front is calculated, showing that the DLI can be suppressed depending on the Prandtl number Pr and the Mach number Ma. This paper is closed with a summary in Sect. 6.

2 Non-Dimensional Governing Equations

The Cartesian coordinate system (x, y, z) and the velocity field are made dimensionless by use of the hydrodynamic length scale $\tilde{L}$ and the laminar flame speed $\tilde{S}_L$, the speed of a flat flame. We consider the situation where a planar flame front propagates in the negative z-direction. The velocity field is partitioned into the tangential and normal components as $\vec{V} + W\vec{e}_z$. Besides, the differential operator is defined for the x-y plane as $\nabla = (\partial/\partial x, \partial/\partial y)$ and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Then we deal with the following equations governing dimensionless hydrodynamic variables [18, 24]:

$$\frac{\partial R}{\partial t}+\nabla\cdot(R\mathbf{V})+\frac{\partial}{\partial z}(RW)=0, \tag{5}$$

$$R\left(\frac{\partial \mathbf{V}}{\partial t}+\left(\mathbf{V}\cdot\nabla+W\frac{\partial}{\partial z}\right)\mathbf{V}\right)$$
$$=-\frac{1}{\gamma Ma^2}\nabla P+\delta Pr\left(\left(\Delta+\frac{\partial^2}{\partial z^2}\right)\mathbf{V}+\frac{1}{3}\nabla\left(\nabla\cdot\mathbf{V}+\frac{\partial W}{\partial z}\right)\right), \tag{6}$$

$$R\left(\frac{\partial W}{\partial t}+\left(\mathbf{V}\cdot\nabla+W\frac{\partial}{\partial z}\right)W\right)$$
$$=-\frac{1}{\gamma Ma^2}\frac{\partial P}{\partial z}+\delta Pr\left(\left(\Delta+\frac{\partial^2}{\partial z^2}\right)W+\frac{1}{3}\frac{\partial}{\partial z}\left(\nabla\cdot\mathbf{V}+\frac{\partial W}{\partial z}\right)\right), \tag{7}$$

$$R\left(\frac{\partial T}{\partial t}+\left(\mathbf{V}\cdot\nabla+W\frac{\partial}{\partial z}\right)T\right)$$
$$=\delta\left(\Delta+\frac{\partial^2}{\partial z^2}\right)T+\frac{\gamma-1}{\gamma}\left(\frac{\partial P}{\partial t}+\left(\mathbf{V}\cdot\nabla+W\frac{\partial}{\partial z}\right)P\right)+qQ, \tag{8}$$

$$P=RT, \tag{9}$$

where the non-dimensional variables, R, T and P are the density of the mixture, the temperature and the pressure, respectively. All of the variables are made dimensionless with use of those of the fresh mixture at a position far from a flame front, where the flow field is assumed to be uniform with its velocity identified with $\tilde{S}_L$ in the coordinate frame relative to the flame front. Therefore, each quantity is subject to the following boundary condition in the far field on the unburned side ($z<0$).

$$R=W=T=P=1, \quad \vec{V}=\vec{0} \quad \text{as} \quad z\to-\infty. \tag{10}$$

Our aim is to derive the condition of the flame speed (3). For this, we need to investigate the preheat zone whose length scale is represented by $\tilde{l}_d=\tilde{D}_{th}/\tilde{S}_L$, with $\tilde{D}_{th}$ being the thermal diffusivity. We find that (3) is highly influenced by the scale factor of the preheat zone δ, the Mach number Ma, the Prandtl number Pr and the specific heats ratio γ. These non-dimensional parameters are defined as

$$\delta=\frac{\tilde{l}_d}{\tilde{L}}, \quad Ma=\frac{\tilde{S}_L}{\tilde{c}_s}, \quad Pr=\frac{\tilde{\nu}}{\tilde{D}_{th}} \quad \gamma=\frac{\tilde{c}_p}{\tilde{c}_v}, \tag{11}$$

where $\tilde{c}_s$, $\tilde{\nu}$, $\tilde{c}_p$ and $\tilde{c}_v$ are the adiabatic sound speed defined in the fresh mixture, the kinematic viscosity, the specific heats at constant pressure and volume, respectively.

The reaction term Q in (8) is not dealt with in this paper by resorting to the large-activation-energy asymptotics. The coefficient q represents the non-dimensional heat release whose value is positive for an exothermic chemical reaction. A detailed analysis, with the compressibility effect taken into consideration, is relegated to an independent investigation [23].

It is noteworthy that Bychkov *et al.* [3] ignored the viscous terms in (7), though it naturally enters through the coupling of the heat-conduction equations with the Navier-Stokes equation, via pressure variation. However, as seen from (4), the viscous effect is indispensable for the compressible correction to the Markstein number and thence to the DLI. The compressible correction is sensitive to the value of Pr.

2.1 Perturbations in Hydrodynamic Zone

We superimpose an infinitesimal perturbation to a plane flame front, which coincides with the x-y plane parametrized by $\vec{x} = (x, y)$.

$$F(x, y, t) = f \exp(\mathrm{i}\vec{x} \cdot \vec{k} + \Omega t),$$

where $\vec{k} = (k_x, k_y)$ and Ω are the wavenumber, with $k = (k_x^2 + k_y^2)^{1/2}$, and the growth rate of the perturbation, which are made dimensionless as follows.

$$\vec{k} = \tilde{\vec{k}}\tilde{L}, \quad \Omega = \tilde{\Omega}\tilde{L}/\tilde{S_L}. \tag{12}$$

Any hydrodynamic variable Φ is partitioned into a steady planar solution $\bar{\Phi}(z)$ and a small perturbation $\Phi'(x, y, z, t)$ to it as

$$\Phi = \bar{\Phi}(z) + \Phi'(x, y, z, t), \tag{13}$$

with

$$\Phi' = \tilde{\Phi}(z) \exp(\mathrm{i}\vec{x} \cdot \vec{k} + \Omega t) \ . \tag{14}$$

For our purpose of taking compressibility into account, we expand all the functions in powers of a small parameter $Ma^2 (\ll 1)$, up to $O(Ma^2)$. For the basic flow, M^2 expansions tale the form as

$$\begin{aligned} &\bar{R} = \bar{R}_{0M} + Ma^2 \bar{R}_{2M}, \quad \bar{W} = \bar{W}_{0M} + Ma^2 \bar{W}_{2M}, \quad \bar{M} = \bar{M}_{0M} + Ma^2 \bar{M}_{2M}, \\ &\bar{T} = \bar{T}_{0M} + Ma^2 \bar{T}_{2M}, \qquad \bar{P} = 1 + \gamma Ma^2 \bar{P}_{2M}, \end{aligned} \tag{15}$$

and, for the perturbations, as

$$\begin{aligned} &R' = R'_{0M} + Ma^2 R'_{2M}, \quad W' = W'_{0M} + Ma^2 W'_{2M}, \quad M' = M'_{0M} + Ma^2 M'_{2M}, \\ &T' = T'_{0M} + Ma^2 T'_{2M}, \qquad P' = \gamma Ma^2 P'_{2M} + \dots, \qquad \vec{V}' = \vec{V}'_{0M} + \dots, \\ &F = F_{0M} + Ma^2 F_{2M}, \end{aligned} \tag{16}$$

where the mass flux perpendicular to a plane flame front is defined by

$$M = RW. \tag{17}$$

We note that the leading term of the pressure is constant under $Ma^2 \ll 1$ because of the first term on the right hand side of (6) and (7) with the boundary condition (10). Furthermore, each quantity is expanded with respect to δ as, for the basic flow,

$$\bar{R}_{0M} = \bar{R}_0 + \delta \bar{R}_1, \qquad \bar{W}_{0M} = \bar{W}_0 + \delta \bar{W}_1,$$
$$\bar{R}_{2M} = \bar{R}_{2M,0} + \delta \bar{R}_{2M,1}, \quad \bar{W}_{2M} = \bar{W}_{2M,0} + \delta \bar{W}_{2M,1}, \tag{18}$$

and, for the perturbations,

$$R'_{0M} = R'_0 + \delta R'_1, \qquad W'_{0M} = W'_0 + \delta W'_1, \qquad F_{0M} = F_0 + \delta F_1,$$
$$R'_{2M} = R'_{2M,0} + \delta R'_{2M,1}, \quad W'_{2M} = W'_{2M,0} + \delta W'_{2M,1}, \quad F_{2M} = F_{2M,0} + \delta F_{2M,1}. \tag{19}$$

The steady planar state of (5), (7) and (8) should satisfy, in the language of the notation (13),

$$\frac{d\bar{M}}{dz} = 0, \tag{20}$$
$$\bar{M}\frac{d\bar{W}}{dz} = -\frac{1}{\gamma Ma^2}\frac{d\bar{P}}{dz} + \frac{4}{3}\delta Pr \frac{d^2\bar{W}}{dz^2}, \tag{21}$$
$$\bar{M}\frac{d\bar{T}}{dz} = \delta \frac{d^2\bar{T}}{dz^2} + \frac{\gamma - 1}{\gamma}\bar{W}\frac{d\bar{P}}{dz}, \tag{22}$$

where the reaction term Q is omitted from (23) by use of the assumption of the large-activation-energy asymptotics. We easily find from (20) that the steady planar mass flux is constant. Especially, $\bar{M} = 1$ from the boundary condition (10) on the unburned side of a flame front.

The perturbed heat-conduction equation is deduced, from (8) with substitution from (13), as

$$\bar{R}\frac{\partial T'}{\partial t} + \frac{\partial T'}{\partial z} + M'\frac{d\bar{T}}{dz}$$
$$= \delta\left(\frac{\partial^2}{\partial z^2} + \Delta\right)T' + \frac{\gamma - 1}{\gamma}\left(\frac{\partial P'}{\partial t} + \bar{W}\frac{\partial P'}{\partial z} + W'\frac{d\bar{P}}{dz}\right). \tag{23}$$

In deriving the condition for a flame speed, we need to explore (23) in the preheat zone scale. We omit the detailed derivation of the jump conditions for the hydrodynamic variables, gained by taking the outer limit of the solution of (23), which is treated in [15] (See also refs [1, 4, 10, 19]).

2.2 Long-Wavelength Approximation

We focus on the specific form of perturbations of the temperature and the density posed by Bychkov *et al.* [3].

$$T' = -F\frac{d\bar{T}}{dz}, \quad R' = -F\frac{d\bar{R}}{dz}, \tag{24}$$

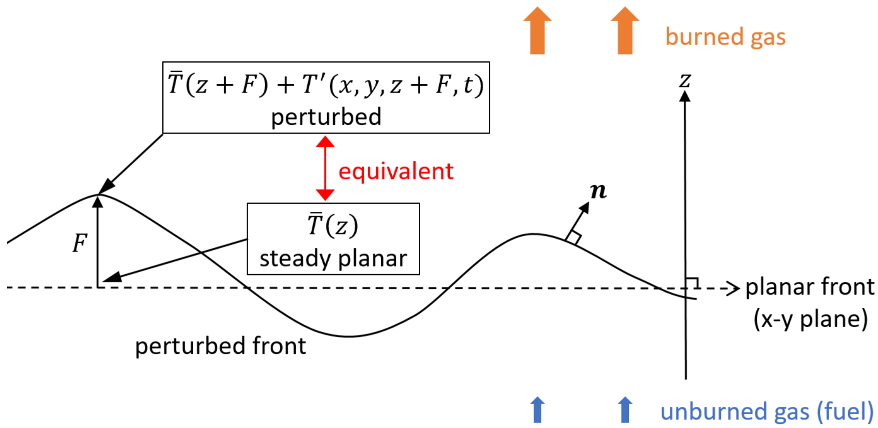

Fig. 1 Schematic illustration of translational symmetry for small perturbations ($F \ll 1$).

where $F(x,y,t)$ is a small-amplitude displacement of the perturbed flame front as shown in Fig. 1. This form is interpreted to come from the translational symmetry possessed by the temperature and the density as illustrated by Fig. 1. It follows from (9), the equation of state, that the pressure inherits the same symmetry.

$$P' = R'\bar{T} + \bar{R}T' = -F\frac{d\bar{R}}{dz}\bar{T} - \bar{R}F\frac{d\bar{T}}{dz} = -F\frac{d\bar{P}}{dz}. \tag{25}$$

Bychkov *et al.* [3] proved that these forms of the perturbations lead to the certain relation for the mass flux valid in the compressible case. This provides the missing boundary condition for the DLI.

By substitution from the asymptotic expansions (15), (16), (18) and (19), we rewrite the perturbation of density in (24) as

$$\begin{aligned}
R'_0 &= -F_0\frac{d\bar{R}_0}{dz}, \quad R'_1 = -\left(F_1\frac{d\bar{R}_0}{dz} + F_0\frac{d\bar{R}_1}{dz}\right), \\
R'_{2M,0} &= -\left(F_{2M,0}\frac{d\bar{R}_0}{dz} + F_0\frac{d\bar{R}_{2M,0}}{dz}\right), \\
R'_{2M,1} &= -\left(F_{2M,1}\frac{d\bar{R}_0}{dz} + F_1\frac{d\bar{R}_{2M,0}}{dz} + F_{2M,0}\frac{d\bar{R}_1}{dz} + F_0\frac{d\bar{R}_{2M,1}}{dz}\right).
\end{aligned} \tag{26}$$

3 Equations in Preheat Zone

In this section, we write down the equations for a steady planar flow and perturbations to it in the preheat zone which are used to derive the boundary condition of the mass flux in Sect. 4. Our approach successfully links the heat-conduction equation to the Navier-Stokes equation, via pressure variation, which is absent in the low-Mach-number limit. In accordance, the viscous effect was neglected in the previous work [3]. Under the long-wavelength approximation, the stretching transformation $z - F = \delta\zeta$ is employed to analyse the preheat zone of a planar flame front by the inner variable ζ with the assumption $\delta \ll 1$.

3.1 Steady Planar Flow

From (9) and (20)-(22), the governing equations for a steady planar flow are

$$\bar{M} = 1, \tag{27}$$

$$\frac{d\bar{W}}{d\zeta} = -\frac{1}{\gamma Ma^2}\frac{d\bar{P}}{d\zeta} + \frac{4}{3}Pr\frac{d^2\bar{W}}{d\zeta^2}, \tag{28}$$

$$\frac{d\bar{T}}{d\zeta} = \frac{d^2\bar{T}}{d\zeta^2} + \frac{\gamma-1}{\gamma}\bar{W}\frac{d\bar{P}}{d\zeta}, \tag{29}$$

$$\bar{P} = \bar{R}\bar{T}. \tag{30}$$

It follows from (15), (17), (27) and (30) that

$$\begin{gathered}\bar{R}_{0M}\bar{W}_{0M} = 1, \quad 1 = \bar{R}_{0M}\bar{T}_{0M}, \\ \bar{R}_{2M}\bar{W}_{0M} + \bar{R}_{0M}\bar{W}_{2M} = 0, \quad 0 = \bar{R}_{2M}\bar{T}_{0M} + \bar{R}_{0M}\bar{T}_{2M}.\end{gathered} \tag{31}$$

Remembering that $\bar{P}_{0M} = 1$, (29) is rewritten as

$$\frac{d\bar{T}_{0M}}{d\zeta} = \frac{d^2\bar{T}_{0M}}{d\zeta^2}. \tag{32}$$

Because of $\bar{W}_{0M} = \bar{T}_{0M}$ from (31), (28) yields

$$\frac{d\bar{P}_{2M}}{d\zeta} = \left(\frac{4}{3}Pr - 1\right)\frac{d\bar{T}_{0M}}{d\zeta}. \tag{33}$$

The asymptotic expansions of the steady planar solutions and small amplitude of perturbation with respect to δ in the preheat zone, by use of (15) and (16), are

$$\begin{aligned}
&\bar{T}_{0M} = \bar{\theta}_0 + \delta\bar{\theta}_1 + \delta^2\bar{\theta}_2 + \delta^3\bar{\theta}_3,\ \bar{R}_{0M} = \bar{\rho}_0 + \delta\bar{\rho}_1 + \delta^2\bar{\rho}_2 + \delta^3\bar{\rho}_3,\\
&F_{0M} = F_0 + \delta F_1 + \delta^2 F_2 + \delta^3 F_3,\ \bar{W}_{0M} = \bar{w}_0 + \delta\bar{w}_1 + \delta^2\bar{w}_2 + \delta^3\bar{w}_3,\\
&\bar{T}_{2M} = \bar{\theta}_{2M,0} + \delta\bar{\theta}_{2M,1} + \delta^2\bar{\theta}_{2M,2} + \delta^3\bar{\theta}_{2M,3},\\
&\bar{R}_{2M} = \bar{\rho}_{2M,0} + \delta\bar{\rho}_{2M,1} + \delta^2\bar{\rho}_{2M,2} + \delta^3\bar{\rho}_{2M,3},\\
&\bar{W}_{2M} = \bar{w}_{2M,0} + \delta\bar{w}_{2M,1} + \delta^2\bar{w}_{2M,2} + \delta^3\bar{w}_{2M,3},\\
&\bar{P}_{2M} = \bar{p}_{2M,0} + \delta\bar{p}_{2M,1} + \delta^2\bar{p}_{2M,2} + \delta^3\bar{p}_{2M,3},\\
&F_{2M} = F_{2M,0} + \delta F_{2M,1} + \delta^2 F_{2M,2} + \delta^3 F_{2M,3}.
\end{aligned} \tag{34}$$

By introducing (34), (31) and (33) reduce to

$$\bar{\rho}_0\bar{w}_0 = 1, \quad 1 = \bar{p}_0 = \bar{\rho}_0\bar{\theta}_0, \tag{35}$$

$$\bar{\rho}_{2M,0}\bar{w}_0 + \bar{\rho}_0\bar{w}_{2M,0} = 0, \quad 0 = \bar{\rho}_{2M,0}\bar{\theta}_0 + \bar{\rho}_0\bar{\theta}_{2M,0}, \tag{36}$$

$$\frac{d\bar{p}_{2M,0}}{d\zeta} = \left(\frac{4}{3}Pr - 1\right)\frac{d\bar{\theta}_0}{d\zeta}. \tag{37}$$

3.2 Perturbations with Translational Symmetry

The linearised heat-conduction equation (23), valid in the preheat zone, is

$$\begin{aligned}
&\bar{R}\frac{\partial T'}{\partial t} + \frac{\bar{M}}{\delta}\frac{\partial T'}{\partial \zeta} + \frac{M'}{\delta}\frac{d\bar{T}}{d\zeta}\\
&= \delta\left(\frac{1}{\delta^2}\frac{\partial^2}{\partial \zeta^2} + \Delta\right)T' + \frac{\gamma-1}{\gamma}\left(\frac{\partial P'}{\partial t} + \frac{\bar{W}}{\delta}\frac{\partial P'}{\partial \zeta} + \frac{W'}{\delta}\frac{d\bar{P}}{d\zeta}\right).
\end{aligned} \tag{38}$$

The perturbations (24) and (25) are written in terms of the inner variable as

$$T' = -\frac{F}{\delta}\frac{d\bar{T}}{d\zeta}, \quad R' = -\frac{F}{\delta}\frac{d\bar{R}}{d\zeta}, \quad P' = -\frac{F}{\delta}\frac{d\bar{P}}{d\zeta}. \tag{39}$$

Furthermore, (39) is expanded in powers of Ma^2, with the help of (15) and (16), as

$$\begin{aligned}
&T'_{0M} = -\frac{F_{0M}}{\delta}\frac{d\bar{T}_{0M}}{d\zeta}\ ,\ T'_{2M} = -\frac{F_{2M}}{\delta}\frac{d\bar{T}_{0M}}{d\zeta} - \frac{F_{0M}}{\delta}\frac{d\bar{T}_{2M}}{d\zeta},\\
&R'_{0M} = -\frac{F_{0M}}{\delta}\frac{d\bar{R}_{0M}}{d\zeta}\ ,\ R'_{2M} = -\frac{F_{2M}}{\delta}\frac{d\bar{R}_{0M}}{d\zeta} - \frac{F_{0M}}{\delta}\frac{d\bar{R}_{2M}}{d\zeta},\\
&P'_{2M} = -\frac{F_{0M}}{\delta}\frac{d\bar{P}_{2M}}{d\zeta}.
\end{aligned} \tag{40}$$

By substitution from the asymptotic expansions (34), we rewrite the perturbations (40) as

$$
\begin{aligned}
T'_{0M} &= \frac{\theta_{-1}}{\delta} + \theta_0 + \delta\theta_1 + \delta^2\theta_2,\\
R'_{0M} &= \frac{\rho_{-1}}{\delta} + \rho_0 + \delta\rho_1 + \delta^2\rho_2,\\
T'_{2M} &= \frac{\theta_{2M,-1}}{\delta} + \theta_{2M,0} + \delta\theta_{2M,1} + \delta^2\theta_{2M,2},\\
R'_{2M} &= \frac{\rho_{2M,-1}}{\delta} + \rho_{2M,0} + \delta\rho_{2M,1} + \delta^2\rho_{2M,2},\\
P'_{2M} &= \frac{p_{2M,-1}}{\delta} + p_{2M,0} + \delta p_{2M,1} + \delta^2 p_{2M,2},
\end{aligned}
\tag{41}
$$

where each term is written, for instance, as

$$
\begin{aligned}
&\theta_{-1} = -F_0\frac{d\bar\theta_0}{d\zeta}, \quad \theta_0 = -\left(F_1\frac{d\bar\theta_0}{d\zeta} + F_0\frac{d\bar\theta_1}{d\zeta}\right),\\
&\theta_1 = -\left(F_2\frac{d\bar\theta_0}{d\zeta} + F_1\frac{d\bar\theta_1}{d\zeta} + F_0\frac{d\bar\theta_2}{d\zeta}\right),\\
&\theta_2 = -\left(F_3\frac{d\bar\theta_0}{d\zeta} + F_2\frac{d\bar\theta_1}{d\zeta} + F_1\frac{d\bar\theta_2}{d\zeta} + F_0\frac{d\bar\theta_3}{d\zeta}\right),\\
&\theta_{2M,-1} = -\left(F_{2M,0}\frac{d\bar\theta_0}{d\zeta} + F_0\frac{d\bar\theta_{2M,0}}{d\zeta}\right),\\
&\theta_{2M,0} = -\left(F_{2M,1}\frac{d\bar\theta_0}{d\zeta} + F_1\frac{d\bar\theta_{2M,0}}{d\zeta} + F_{2M,0}\frac{d\bar\theta_1}{d\zeta} + F_0\frac{d\bar\theta_{2M,1}}{d\zeta}\right),\\
&\theta_{2M,1} = -\left(F_{2M,2}\frac{d\bar\theta_0}{d\zeta} + F_2\frac{d\bar\theta_{2M,0}}{d\zeta} + F_{2M,1}\frac{d\bar\theta_1}{d\zeta}\right.\\
&\qquad\qquad \left. + F_1\frac{d\bar\theta_{2M,1}}{d\zeta} + F_{2M,0}\frac{d\bar\theta_2}{d\zeta} + F_0\frac{d\bar\theta_{2M,2}}{d\zeta}\right).
\end{aligned}
\tag{42}
$$

The similar is true for other quantities. Following ref [3], the form associated with he translational symmetry, like (24), is not postulated for the disturbance of the normal component of the velocity field. Formally we pose the following expansions.

$$
\begin{aligned}
W'_{0M} &= w_0 + \delta w_1 + \delta^2 w_2,\\
W'_{2M} &= w_{2M,0} + \delta w_{2M,1} + \delta^2 w_{2M,2}.
\end{aligned}
\tag{43}
$$

For the perturbation of the mass flux which is introduced in (16), we have, upon substitution from (15), (41) and (43),

$$
\begin{aligned}
M'_{0M} &= \frac{m_{-1}}{\delta} + m_0 + \delta m_1 + \delta^2 m_2,\\
M'_{2M} &= \frac{m_{2M,-1}}{\delta} + m_{2M,0} + \delta m_{2M,1} + \delta^2 m_{2M,2},
\end{aligned}
\tag{44}
$$

where each term is expressed as

$$
\begin{aligned}
m_{-1} &= \rho_{-1}\bar{w}_0 = -F_0\frac{d\bar{\rho}_0}{d\zeta}\bar{w}_0,\\
m_0 &= \rho_0\bar{w}_0 + \rho_{-1}\bar{w}_1 + \bar{\rho}_0 w_0\\
&= -\left(F_1\frac{d\bar{\rho}_0}{d\zeta} + F_0\frac{d\bar{\rho}_1}{d\zeta}\right)\bar{w}_0 - F_0\frac{d\bar{\rho}_0}{d\zeta}\bar{w}_1 + \bar{\rho}_0 w_0,
\end{aligned} \tag{45}
$$

and the same procedure is repeated for the higher-order terms.

4 Boundary Condition of Mass Flux

We seek the compressibility effect on the flame speed condition, whose original form of the DLI is given by (1). For this purpose, by leaving the detailed analysis of the preheat zone to our next paper, we treat, in this investigation, restricted solutions valid for long wavelengths brought by the constraint of translational symmetry dictated by the previous section. We extend the mass-flux condition by [3] to $O(\delta Ma^2)$. At this order, the viscous effect switches on by coupling the Navier-Stokes equation with the heat-conduction equation. The end product, equation (74), reflects, remarkably in a tidy form, the compressible effect on the Markstein number.

4.1 Matching Condition

In order to calculate the mass-flux condition, we use the matching conditions in the overlapping region between the preheat and the hydrodynamic zones. For the density, these conditions read, on the unburned side,

$$
\bar{\rho}_0|_{\zeta\to-\infty} = \bar{R}_0|_{z\to F_-}, \qquad \left.\frac{d\bar{\rho}_1}{d\zeta}\right|_{-\infty} = \left.\frac{d\bar{R}_0}{dz}\right|_{-}, \tag{46}
$$

$$
\bar{\rho}_{1-\infty} = \bar{R}_{1-} + \zeta\left.\frac{d\bar{R}_0}{dz}\right|_{-}, \quad \left.\frac{d\bar{\rho}_2}{d\zeta}\right|_{-\infty} = \left.\frac{d\bar{R}_1}{dz}\right|_{-} + \zeta\left.\frac{d^2\bar{R}_0}{dz^2}\right|_{-}, \tag{47}
$$

$$
\left.\frac{d^2\bar{\rho}_1}{d\zeta^2}\right|_{-\infty} = 0, \qquad \left.\frac{d^2\bar{\rho}_2}{d\zeta^2}\right|_{-\infty} = \left.\frac{d^2\bar{R}_0}{dz^2}\right|_{-}. \tag{48}
$$

The similar conditions apply to the other quantities. Matching conditions (46)-(48) also hold at $O(Ma^2)$, for instance, $\bar{\rho}_{2M,0-\infty} = \bar{R}_{2M,0-}$, $d\bar{\rho}_{2M,1}/d\zeta|_{-\infty} = d\bar{R}_{2M,0}/dz|_-$ and so on.

4.2 Low-Mach-Number Limit

We begin with the derivation of the mass-flux condition in the incompressible limit. Collecting the terms of $O(\delta^{-2}Ma^0)$ in (38), we get

$$\frac{\partial\theta_{-1}}{\partial\zeta}+m_{-1}\frac{d\bar{\theta}_0}{d\zeta}=\frac{\partial^2\theta_{-1}}{\partial\zeta^2}. \tag{49}$$

Substitution from (42) and (45), (49) becomes

$$F_0\bar{w}_0\left(\frac{d\bar{\theta}_0}{d\zeta}\right)^2=-F_0\frac{d}{d\zeta}\left(\frac{d^2\bar{\theta}_0}{d\zeta^2}-\frac{d\bar{\theta}_0}{d\zeta}\right), \tag{50}$$

by virtue of (35). The right-hand side of (50) is zero, because the last term of (29) vanishes by $\bar{p}_0=1$, the second of (35). Requirement of $\bar{w}_0\neq 0$ enforces

$$\frac{d\bar{\theta}_0}{d\zeta}=0. \tag{51}$$

In view of (35) and (37), (51) leads to

$$\frac{d\bar{w}_0}{d\zeta}=\frac{d\bar{\rho}_0}{d\zeta}=\frac{d\bar{p}_{2M,0}}{d\zeta}=0. \tag{52}$$

The terms of $O(\delta^{-1}Ma^0)$ do not bring any new information. We proceed to the next order. Collecting the terms of $O(\delta^0Ma^0)$ in (38), we have

$$\bar{\rho}_1\frac{\partial\theta_{-1}}{\partial t}+\bar{\rho}_0\frac{\partial\theta_0}{\partial t}+\frac{\partial\theta_1}{\partial\zeta}+m_1\frac{d\bar{\theta}_0}{d\zeta}+m_0\frac{d\bar{\theta}_1}{d\zeta}+m_{-1}\frac{d\bar{\theta}_2}{d\zeta}=\frac{\partial^2\theta_1}{\partial\zeta^2}+\Delta\theta_{-1}. \tag{53}$$

By substitution from (32), (42), (45), (51) and (52), we are left with

$$-F_0\frac{d\bar{\rho}_1}{d\zeta}\bar{w}_0+\bar{\rho}_0\left(w_0-\frac{\partial F_0}{\partial t}\right)=0. \tag{54}$$

Taking the outer limit $\zeta\to-\infty$, (46) gives rise to the matching condition for the hydrodynamic zone on the unburned side, resulting in

$$R'_{0-}\bar{W}_{0-}+\bar{R}_{0-}\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)=0, \tag{55}$$

where use has been made of (26) for the density perturbation. This is the desired mass-flux condition on the unburned side of a flame front in the hydrodynamic zone, correcting Landau's assumption [13, 14] with the first term incorporating the compressibility effect. This condition coincides with that of Bychkov *et al*. [3].

We are ready to go on to the first-order solution in δ, in the preheat zone, to deal with the curvature effect, embodying the Markstein effect [16]. Collecting the terms of $O(\delta Ma^0)$ in (38), we have, using (29), (42), (45), (51), (52) and (54),

$$\begin{aligned}
&-\left(F_1\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_2}{d\zeta}\right)\bar{w}_0-F_0\frac{d\bar{\rho}_1}{d\zeta}\bar{w}_1\\
&+\bar{\rho}_1\left(w_0-\frac{\partial F_0}{\partial t}\right)+\bar{\rho}_0\left(w_1-\frac{\partial F_1}{\partial t}\right)+\Delta F_0=0.
\end{aligned} \tag{56}$$

Matching with the hydrodynamic zone, by taking the limit $\zeta\to-\infty$ of (56), with use of (46) and (47), leads to

$$\begin{aligned}
&-\left(F_1\frac{d\bar{R}_0}{dz}\bigg|_-+F_0\frac{d\bar{R}_1}{dz}\bigg|_-\right)\bar{W}_{0-}-F_0\frac{d\bar{R}_0}{dz}\bigg|_-\bar{W}_{1-}\\
&+\bar{R}_{1-}\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)+\bar{R}_{0-}\left(W'_{1-}-\frac{\partial F_1}{\partial t}\right)+\Delta F_0\\
&=\zeta\left\{F_0\frac{d^2\bar{R}_0}{dz^2}\bigg|_-\bar{W}_{0-}+F_0\frac{d\bar{R}_0}{dz}\bigg|_-\frac{d\bar{W}_0}{dz}\bigg|_-\right.\\
&\qquad\left.-\frac{d\bar{R}_0}{dz}\bigg|_-\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)-\bar{R}_{0-}\frac{dW'_0}{dz}\bigg|_-\right\}.
\end{aligned} \tag{57}$$

The right-hand side of (57) diverges in the limit of $\zeta\to-\infty$. But this difficulty is rescued by the equation obtained from the derivative of (56) with respect to ζ,

$$\begin{aligned}
&-\left(F_1\frac{d^2\bar{\rho}_1}{d\zeta^2}+F_0\frac{d^2\bar{\rho}_2}{d\zeta^2}\right)\bar{w}_0-\left(F_1\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_2}{d\zeta}\right)\frac{d\bar{w}_0}{d\zeta}-F_0\frac{d^2\bar{\rho}_1}{d\zeta^2}\bar{w}_1-F_0\frac{d\bar{\rho}_1}{d\zeta}\frac{d\bar{w}_1}{d\zeta}\\
&+\frac{d\bar{\rho}_1}{d\zeta}\left(w_0-\frac{\partial F_0}{\partial t}\right)+\bar{\rho}_1\frac{d\bar{w}_0}{d\zeta}+\frac{d\bar{\rho}_0}{d\zeta}\left(w_1-\frac{\partial F_1}{\partial t}\right)+\bar{\rho}_0\frac{d\bar{w}_1}{d\zeta}=0.
\end{aligned} \tag{58}$$

The outer limit ($\zeta\to-\infty$) of (58), with application of (46), (47) and (48), results in

$$-F_0\frac{d^2\bar{R}_0}{dz^2}\bigg|_-\bar{W}_{0-}-F_0\frac{d\bar{R}_0}{dz}\bigg|_-\frac{d\bar{W}_0}{dz}\bigg|_-+\frac{d\bar{R}_0}{dz}\bigg|_-\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)+\bar{R}_{0-}\frac{dW'_0}{dz}\bigg|_-=0.$$

Thus, the diverging terms in (57) cancel each other and we eventually reach the mass flux of $O(\delta)$ by using (26) for the density perturbation.

$$R'_{1-}\bar{W}_{0-}+R'_{0-}\bar{W}_{1-}+\bar{R}_{1-}\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)+\bar{R}_{0-}\left(W'_{1-}-\frac{\partial F_1}{\partial t}\right)=-\Delta F_0. \tag{59}$$

This is the desired mass-flux condition reflecting the curvature of a flame front, a feature of the Markstein condition [16]. In the context of the long-wave approximation, the previous investigation [3] did not enter into $O(\delta)$, and the condition (59) is new.

4.3 Compressibility Effect

We make headway to deduce the mass flux on the unburned side of a flame front, with the compressibility effect taken into account, based only on the heat-conduction equation. Collecting the terms of $O(\delta^{-2}Ma^2)$ in (38), we have

$$\frac{\partial \theta_{2M,-1}}{\partial \zeta} + m_{2M,-1}\frac{d\bar{\theta}_0}{d\zeta} + m_{-1}\frac{d\bar{\theta}_{2M,0}}{d\zeta} = \frac{\partial^2 \theta_{2M,-1}}{\partial \zeta^2} + (\gamma-1)\bar{w}_0\frac{\partial p_{2M,-1}}{\partial \zeta}. \quad (60)$$

The last term vanishes because the analogue of the first of (42) reads $p_{2M,-1} = -F_0 d\bar{p}_{2M,0}/d\zeta = 0$, the latter equality coming from (52). By use of (45), (51), (52) and the variant of (42), (60) becomes

$$\frac{\partial}{\partial \zeta}\left(-F_0\frac{d\bar{\theta}_{2M,0}}{d\zeta}\right) = \frac{\partial^2}{\partial \zeta^2}\left(-F_0\frac{d\bar{\theta}_{2M,0}}{d\zeta}\right),$$

which is satisfied by (29) because of (52).

Next, collecting the terms of $O(\delta^{-1}Ma^2)$ in (38), we have

$$\begin{aligned}&\bar{\rho}_{2M,0}\frac{\partial \theta_{-1}}{\partial t} + \bar{\rho}_0\frac{\partial \theta_{2M,-1}}{\partial t} + \frac{\partial \theta_{2M,0}}{\partial \zeta} + m_{2M,0}\frac{d\bar{\theta}_{0M}}{d\zeta}\\ &+ m_{2M,-1}\frac{d\bar{\theta}_1}{d\zeta} + m_0\frac{d\bar{\theta}_{2M,0}}{d\zeta} + m_{-1}\frac{d\bar{\theta}_{2M,1}}{d\zeta}\\ &= \frac{\partial^2 \theta_{2M,0}}{\partial \zeta^2} + (\gamma-1)\left(\frac{\partial p_{2M,-1}}{\partial t} + \bar{w}_1\frac{\partial p_{2M,-1}}{\partial \zeta} + \bar{w}_0\frac{\partial p_{2M,0}}{\partial \zeta} + w_0\frac{d\bar{p}_{2M,0}}{d\zeta}\right).\end{aligned}$$

By use of (29), (42), (45), (51), (52) and (54), we are left only with

$$-F_0\frac{d\bar{\rho}_{2M,0}}{d\zeta}\bar{w}_0\frac{d\bar{\theta}_1}{d\zeta} = 0.$$

The temperature gradient $d\bar{\theta}_1/d\zeta$ should not be zero because the θ_0 term in (42) should not be zero due to the matching condition $\theta_0|_{\zeta\to-\infty} \to T_{0-} \neq 0$ and $d\bar{\theta}_0/d\zeta = 0$ by (51). Consequently, we have no choice but to put the density gradient zero.

$$\frac{d\bar{\rho}_{2M,0}}{d\zeta} = 0. \quad (61)$$

It follows from (36) that

$$\frac{d\bar{w}_{2M,0}}{d\zeta} = 0, \quad \frac{d\bar{\theta}_{2M,0}}{d\zeta} = 0, \quad (62)$$

with the help of (51) and (52).

Collecting the terms of $O(\delta^0 Ma^2)$ in (38), taking account of (29), (42), (45), (51), (52), (61) and (62), we have

$$\left\{-\bar{\rho}_{2M,0}\frac{\partial F_0}{\partial t}-\bar{\rho}_0\frac{\partial F_{2M,0}}{\partial t}-\left(F_{2M,0}\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_{2M,1}}{d\zeta}\right)\bar{w}_0\right. \tag{63}$$

$$\left.-F_0\frac{d\bar{\rho}_1}{d\zeta}\bar{w}_{2M,0}+\bar{\rho}_{2M,0}w_0+\bar{\rho}_0 w_{2M,0}\right\}\frac{d\bar{\theta}_1}{d\zeta} \tag{64}$$

$$=\frac{\gamma-1}{\bar{\rho}_0}\left(\bar{\rho}_0 w_0-\bar{\rho}_0\frac{\partial F_0}{\partial t}-\bar{w}_0 F_0\frac{d\bar{\rho}_1}{d\zeta}\right)\frac{d\bar{p}_{2M,1}}{d\zeta}, \tag{65}$$

where, in the same manner as $O(\delta^0 Ma^0)$, the first of (42), valid in the long-wavelength approximation, has dictated vanishing of $\theta_{2M,-1}$ and therefore of $\Delta\theta_{2M,-1}$ because of (62). The right-hand side is eliminated owing to (54), and (65) further simplifies to

$$-\left(F_{2M,0}\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_{2M,1}}{d\zeta}\right)\bar{w}_0-F_0\frac{d\bar{\rho}_1}{d\zeta}\bar{w}_{2M,0}$$
$$+\bar{\rho}_{2M,0}\left(w_0-\frac{\partial F_0}{\partial t}\right)+\bar{\rho}_0\left(w_{2M,0}-\frac{\partial F_{2M,0}}{\partial t}\right)=0. \tag{66}$$

Taking the outer limit $\zeta\to-\infty$, with use of (46), gives the boundary condition on the hydrodynamic zone.

$$R'_{2M,0-}\bar{W}_{0-}+R'_{0-}\bar{W}_{2M,0-}$$
$$+\bar{R}_{2M,0-}\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)+\bar{R}_{0-}\left(W'_{2M,0-}-\frac{\partial F_{2M,0}}{\partial t}\right)=0, \tag{67}$$

where we notice from the second of (26) and (46) that the first two terms of (66) become the first term of (67) as $\zeta\to-\infty$. This implies that the perturbed mass flux on the unburned side of the flame front is absent, an extension of Landau's assumption to the compressible case. The condition (67) coincides with that of Bychkov *et al.* [3].

In the long-wave approximation admitting the translational symmetry, to the leading order in δ, the perturbed mass flux is zero at the flame front even when the compressibility effect is included. The compressibility has a non-trivial influence on the mass flux at the next order in δ, at which the contributions from the curvature of the flame front and the viscosity play an vital role. Collecting the terms of $O(\delta Ma^2)$ in (38), using the conditions (29), (42), (45), (51), (52), (61) and (62), we have

$$
\begin{aligned}
&\frac{d\bar{\theta}_1}{d\zeta}\Bigg\{-\bar{\rho}_{2M,1}\frac{\partial F_0}{\partial t}-\bar{\rho}_{2M,0}\frac{\partial F_1}{\partial t}-\bar{\rho}_1\frac{\partial F_{2M,0}}{\partial t}-\bar{\rho}_0\frac{\partial F_{2M,1}}{\partial t}\\
&-\left(F_{2M,1}\frac{d\bar{\rho}_1}{d\zeta}+F_{2M,0}\frac{d\bar{\rho}_2}{d\zeta}+F_1\frac{d\bar{\rho}_{2M,1}}{d\zeta}+F_0\frac{d\bar{\rho}_{2M,2}}{d\zeta}\right)\bar{w}_0\\
&-\left(F_{2M,0}\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_{2M,1}}{d\zeta}\right)\bar{w}_1-\left(F_1\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_2}{d\zeta}\right)\bar{w}_{2M,0}\\
&-F_0\frac{d\bar{\rho}_1}{d\zeta}\bar{w}_{2M,1}+\bar{\rho}_{2M,1}w_0+\bar{\rho}_{2M,0}w_1+\bar{\rho}_1 w_{2M,0}+\bar{\rho}_0 w_{2M,1}\Bigg\}\\
&=-\Delta F_{2M,0}\frac{d\bar{\theta}_1}{d\zeta}+(\gamma-1)\Bigg\{\left(-\frac{\partial F_1}{\partial t}+F_0\frac{d\bar{w}_2}{d\zeta}+F_1\frac{d\bar{w}_1}{d\zeta}+w_1\right)\frac{d\bar{p}_{2M,1}}{d\zeta}\\
&\qquad\qquad+\left(-\frac{\partial F_0}{\partial t}+F_0\frac{d\bar{w}_1}{d\zeta}+w_0\right)\frac{d\bar{p}_{2M,2}}{d\zeta}\Bigg\}. \qquad (68)
\end{aligned}
$$

In order to reduce the right-hand side of (68), we invoke (31) and (33). We see from (31) that

$$
\begin{aligned}
\bar{\rho}_1\bar{w}_0+\bar{\rho}_0\bar{w}_1&=0,\\
\bar{\rho}_2\bar{w}_0+\bar{\rho}_1\bar{w}_1+\bar{\rho}_0\bar{w}_2&=0.
\end{aligned}
$$

Then, because of (52), $d\bar{w}_1/d\zeta$ and $d\bar{w}_2/d\zeta$ are rewritten as

$$
\begin{aligned}
\frac{d\bar{w}_1}{d\zeta}&=-\frac{\bar{w}_0}{\bar{\rho}_0}\frac{d\bar{\rho}_1}{d\zeta},\\
\frac{d\bar{w}_2}{d\zeta}&=-\frac{\bar{w}_0}{\bar{\rho}_0}\frac{d\bar{\rho}_2}{d\zeta}-\frac{d\bar{\rho}_1}{d\zeta}\frac{\bar{w}_1}{\bar{\rho}_0}-\frac{\bar{\rho}_1}{\bar{\rho}_0}\frac{d\bar{w}_1}{d\zeta}. \qquad (69)
\end{aligned}
$$

Upon substitution of (34) into (33), we get

$$
\frac{d\bar{p}_{2M,1}}{d\zeta}=\left(\frac{4}{3}Pr-1\right)\frac{d\bar{\theta}_1}{d\zeta}. \qquad (70)
$$

By taking advantage of (56), (69) and (70), we reduce (68) to

$$
\begin{aligned}
&-\bar{\rho}_{2M,1}\frac{\partial F_0}{\partial t}-\bar{\rho}_{2M,0}\frac{\partial F_1}{\partial t}-\bar{\rho}_1\frac{\partial F_{2M,0}}{\partial t}-\bar{\rho}_0\frac{\partial F_{2M,1}}{\partial t}\\
&-\left(F_{2M,1}\frac{d\bar{\rho}_1}{d\zeta}+F_{2M,0}\frac{d\bar{\rho}_2}{d\zeta}+F_1\frac{d\bar{\rho}_{2M,1}}{d\zeta}+F_0\frac{d\bar{\rho}_{2M,2}}{d\zeta}\right)\bar{w}_0\\
&-\left(F_{2M,0}\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_{2M,1}}{d\zeta}\right)\bar{w}_1-\left(F_1\frac{d\bar{\rho}_1}{d\zeta}+F_0\frac{d\bar{\rho}_2}{d\zeta}\right)\bar{w}_{2M,0}\\
&-F_0\frac{d\bar{\rho}_1}{d\zeta}\bar{w}_{2M,1}+\bar{\rho}_{2M,1}w_0+\bar{\rho}_{2M,0}w_1+\bar{\rho}_1 w_{2M,0}+\bar{\rho}_0 w_{2M,1}\\
&=-\Delta F_{2M,0}-\frac{\gamma-1}{\bar{\rho}_0}\left(\frac{4}{3}Pr-1\right)\Delta F_0. \qquad (71)
\end{aligned}
$$

The outer limit $\zeta \to -\infty$ of (71) produces terms proportional to ζ, and we are requested to show cancellation of these, otherwise diverging, terms. To confirm this, it suffices to take the derivative of (71) with respect to ζ, to take the outer limit, and to impose the matching conditions (46), (47) and (48). The resulting equation is

$$
\begin{aligned}
&-\left.\frac{d\bar{R}_{2M,0}}{dz}\right|_{-}\frac{\partial F_0}{\partial t}-\left.\frac{d\bar{R}_0}{dz}\right|_{-}\frac{\partial F_{2M,0}}{\partial t}-\left(F_{2M,0}\left.\frac{d^2\bar{R}_0}{dz^2}\right|_{-}+F_0\left.\frac{d^2\bar{R}_{2M,0}}{dz^2}\right|_{-}\right)\bar{W}_{0-}\\
&-\left(F_{2M,0}\left.\frac{d\bar{R}_0}{dz}\right|_{-}+F_0\left.\frac{d\bar{R}_{2M,0}}{dz}\right|_{-}\right)\left.\frac{d\bar{W}_0}{dz}\right|_{-}-F_0\left.\frac{d^2\bar{R}_0}{dz^2}\right|_{-}\bar{W}_{2M,0-}-F_0\left.\frac{d\bar{R}_0}{dz}\right|_{-}\left.\frac{d\bar{W}_{2M,0}}{dz}\right|_{-}\\
&+\left.\frac{d\bar{R}_{2M,0}}{dz}\right|_{-}W'_{0-}+\bar{R}_{2M,0-}\left.\frac{\partial W'_0}{\partial z}\right|_{-}+\left.\frac{d\bar{R}_0}{dz}\right|_{-}W'_{2M,0-}+\bar{R}_{0-}\left.\frac{\partial W'_{2M,0}}{\partial z}\right|_{-}=0. \qquad (72)
\end{aligned}
$$

The remaining task is to take the outer limit of (71), by imposing (46) and (47), to get the mass flux of $O(\delta Ma^2)$, on the unburned side, leaving, with the help of (26) and (72),

$$
\begin{aligned}
&R'_{2M,1-}\bar{W}_{0-}+R'_{2M,0-}\bar{W}_{1-}+R'_{1-}\bar{W}_{2M,0-}+R'_{0-}\bar{W}_{2M,1-}+\bar{R}_{2M,1-}\left(W'_{0-}-\frac{\partial F_0}{\partial t}\right)\\
&+\bar{R}_{2M,0-}\left(W'_{1-}-\frac{\partial F_1}{\partial t}\right)+\bar{R}_{1-}\left(W'_{2M,0-}-\frac{\partial F_{2M,0}}{\partial t}\right)+\bar{R}_{0-}\left(W'_{2M,1-}-\frac{\partial F_{2M,1}}{\partial t}\right)\\
&=-\Delta F_{2M,0}-\frac{\gamma-1}{\bar{R}_{0-}}\left(\frac{4}{3}Pr-1\right)\Delta F_0. \qquad (73)
\end{aligned}
$$

This result attains an extension of the Markstein effect to the compressible case. The curvature effect, in combination with the Prandtl number, appears for the mass flux. This implies that the viscosity should be retained when we consider the compressible flow field.

The above results (55), (59), (67) and (73) are summarized as

$$
R_{-}\left(W_{-}-\frac{\partial F}{\partial t}\right)=1-\delta\left(1+Ma^2(\gamma-1)\left(\frac{4}{3}Pr-1\right)\right)\Delta F. \qquad (74)
$$

The flame speed (3) is obtained from (74) by assuming that the flow field is incompressible in the hydrodynamic scale as

$$
R=\begin{cases}1 & (z<F)\\ 1/(1+q) & (z>F)\end{cases}. \qquad (75)
$$

5 Effect of Compressible Markstein Number on DLI

We are now in a position to look into how the compressibility modifies the DLI. As indicated by (3), the compressibility effect is incorporated into the condition of a flame speed, though the flow field is assumed to be incompressible in the hydrodynamic regions. The flame speed S_f is defined as the normal velocity of the fluid relative to that of a flame front, which is evaluated at the edge of the front on the unburned side.

$$S_f = (\vec{V} - \vec{V}_f)|_{z=F_-} \cdot \vec{n} \approx \bar{W}|_{z=F_-} + W'|_{z=F_-} - \frac{\partial F}{\partial t}, \tag{76}$$

where the normal velocity of a flame front and the unit normal vector are given by

$$\vec{V}_f \cdot \vec{n} \approx \frac{\partial F}{\partial t}, \quad \vec{n} \approx \left(-\frac{\partial F}{\partial x}, -\frac{\partial F}{\partial y}, 1\right). \tag{77}$$

At $O(\delta^0 Ma^0)$, we solve the following linearised equations of (5)-(7) for the perturbation form of (13) on the unburned and burned sides of a flame front.

$$\frac{\partial W_0'}{\partial z} + \nabla \cdot \mathbf{V}_0' = 0, \tag{78}$$

$$R\frac{\partial W_0'}{\partial t} + \frac{\partial W_0'}{\partial z} = -\frac{\partial P_{2M,0}'}{\partial z}, \tag{79}$$

$$R\frac{\partial \mathbf{V}_0'}{\partial t} + \frac{\partial \mathbf{V}_0'}{\partial z} = -\nabla P_{2M,0}', \tag{80}$$

where the density R is assumed to be constant as given by (75). The following jump conditions are imposed at a flame front, $z = F_\pm$, at $O(\delta^0 Ma^0)$.

$$[\![R(\vec{V}_0 - \vec{V}_f) \cdot \vec{n}]\!] = 0, \tag{81}$$

$$[\![\vec{V}_0 \times \vec{n}]\!] = \vec{0}, \tag{82}$$

$$[\![P_{2M,0} + R((\vec{V}_0 - \vec{V}_f) \cdot \vec{n})^2]\!] = 0, \tag{83}$$

where, for any quantity ϕ, $[\![\phi]\!] = \phi(z = F_+) - \phi(z = F_-)$ denotes the jump across a flame front in the hydrodynamic zone. In addition to (81)-(83), the condition of a flame speed for perturbations is given by (3), with the help of (76), as

$$W_0'|_{z=F_-} - \frac{\partial F}{\partial t} = -Mr_M \Delta F. \tag{84}$$

Enforcing (81)-(84) on the solutions of (78)-(80), we gain

$$(\sigma+1)\Omega^2 + 2(1 + Mr_M k)\sigma k\Omega - (\sigma - 1 - 2\sigma Mr_M k)\sigma k^2 = 0, \tag{85}$$

where the non-dimensional growth rate Ω is defined by (12) and $\sigma = 1 + q$ is the thermal expansion ratio, with q (> 0) the non-dimensional heat release as defined in Sect. 2. By imposing the condition of $\Omega = 0$, we find the critical wavenumber as

$$k_c = \frac{\sigma - 1}{2\sigma Mr_M}. \tag{86}$$

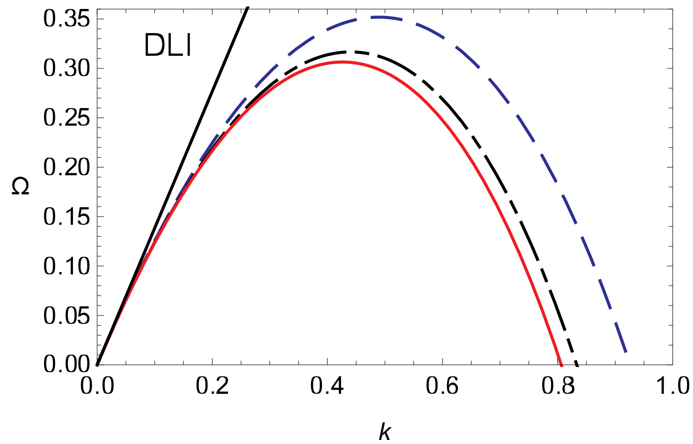

Fig. 2 Growth rate Ω v.s. wavenumber k with $\gamma = 1.4$, $\sigma = 6$, $Ma = 0.5$ and $\delta = 0.5$ for several values of Prandtl number: $Pr = 0$ (dashed), $Pr = 3/4$ (dot-dashed) and $Pr = 1$ (solid). The DLI is also plotted for comparison.

Because positivity of the Markstein number, $Mr_M > 0$, brings the decrease of the growth rate, we need the following requirement for the suppression of the DLI for $k > k_c$.

$$1 + Ma^2(\gamma - 1)\left(\frac{4}{3}Pr - 1\right) > 0. \tag{87}$$

This condition means that if $Pr > 3/4$, the increase of the value of Ma absolutely reinforce the suppression of the DLI. On the other hand, if $0 < Pr < 3/4$, there is a possibility of the enhancement of the DLI, or $Mr_M < 0$, by the compressibility effect. However, under the condition of $Ma^2 \ll 1$, this is unlikely to occur.

Finally, we plot the solution of (85) in Figs. 2 and 3, which is given by

$$\begin{aligned} \Omega = &-\frac{\sigma}{1+\sigma}(1 + Mr_M k)k \\ &+ \frac{1}{1+\sigma}\left\{\sigma^2(1 + Mr_M k)^2 + \sigma(\sigma^2 - 1)\left(1 - 2\frac{\sigma}{\sigma - 1}Mr_M k\right)\right\}^{1/2} k. \end{aligned} \tag{88}$$

We observe from Fig. 2 that the DLI is suppressed at the critical wavenumber given by (86). In the range of $Pr > 3/4$, the reduction of the growth rate is achieved by the increase of the Mach number as shown in Figure 3. Conversely, the rise of the growth rate is caused by the compressibility in the range of $0 < Pr < 3/4$, though such a growth rate is still less than the DLI.

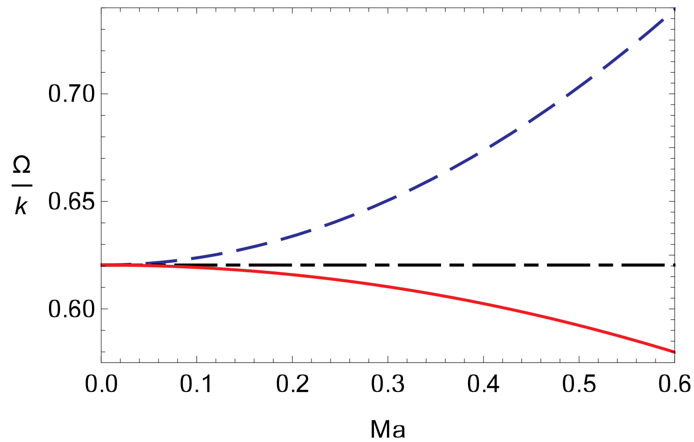

Fig. 3 Growth rate Ω/k v.s. Mach number Ma with the same parameters as Fig. 2 but for $k = 0.5$.

6 Conclusions

We have investigated the effect of the compressibility on the Markstein number by use of the M^2 expansions. Our analysis has been performed on the scale of the preheat zone, represented by δ, by employing the matched asymptotic expansions. The compressibility brings the pressure variation term as a heat source in the heat-conduction equation. The pressure term connects the heat-conduction equation with the Navier-Stokes equation. As a consequence, the viscous effect tales part in the compressibility correction to the Markstein number.

In this investigation, we have appealed to the long-wavelength approximation for the perturbations. The resulting condition of the mass flux (74) implies no perturbation of the mass flux to $O(\delta^0)$. However, at $O(\delta)$, the perturbation of the mass flux is generated due to the curvature effect which is virtually equivalent to the Markstein effect [16]. The term of $O(\delta Ma^2)$ is new, of compressibility origin, which modifies the magnitude of the Markstein number.

The influence of the compressibility on the Darrieus-Landau instability (DLI) is discussed in Sect. 5. Enhancement or reduction of the Markstein effect is sensitive to the value of the Prandtl number Pr. In the range of $0 < Pr < 3/4$, the compressibility leads to the increase of the growth rate of infinitesimal perturbations, though its value is still less than that of the DLI. On the other hand, if $Pr > 3/4$, then the growth rate decreases as the Mach number increases.

The ansatz (24) for the form of infinitesimal perturbation drastically facilitates the integration of the coupled system of the heat-conduction and the Navier-Stokes equations. In a companion paper [23], we tackle with the burning-rate eigenvalue problem in the reaction zone, with allowance for compressibility, and thereby manipulate the laminar flame speed. The present investigation establishes a concise formula (4) for the Markstein number with the compressibility taken into account, though assuming the laminar flame speed to be unity, the first term of (3). These two pieces of papers complements each other. There are a lot to be examined concerning the compressibility effect on combustions, for instance, the inertia and acceleration effects [1, 10] and the deflagration to detonation transition (DDT) [24]. These effects are left for future study.

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