Multi-point maximum principles and eigenvalue estimates

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Abstract Estimates on modulus of continuity, isoperimetric profiles of various kinds, and quantities involving function values at several points have been central in several recent results in geometric analysis. In these lectures I will focus mostly on the applications to partial differential equations, and to estimates on eigenvalues. These lectures were presented at the MATRIX program on "Recent trends on Nonlinear PDE of Elliptic and Parabolic type" at Creswick, November 5-16, 2018.

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Introductory comments

In this article I want to describe some techniques which have been applied with some success recently to a variety of problems, ranging from my proof with Julie Clutterbuck of the sharp lower bound on the fundamental gap for Schrödinger operators [4] to Brendle's proof of the Lawson conjecture [14] and my proof with Haizhong Li of the Pinkall-Sterling conjecture [7]. I will discuss several other interesting applications below. The common thread in these techniques is the application of the maximum principle to functions involving several points or to functions depending on the global structure of solutions. Further related ideas, and more details on some of the methods presented here, can be found in the author's survey article [1].

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Lecture 1: Controlling the modulus of continuity in heat equations

1.1 Moduli of continuity

Today I want to discuss how two-point maximum principles can be used to control the modulus of continuity for solutions of heat equations. This implies a lot of information including sharp gradient estimates. In particular the modulus of continuity estimates imply sharp decay estimates, which are the key to some sharp eigenvalue inequalities, the first of which we will reach by the end of today's lecture.

Recall that for a function f (on a metric space), a function ω of one positive variable is a modulus of continuity for f if $\omega(s)$ bounds the difference in function values $|f(y)-f(x)|$ for all point with separation $d(x,y)=s$.

I will adopt a slightly different definition, for the sake of simplicity further down the track: We say ω is a modulus of continuity for f if

$$\frac{|f(y)-f(x)|}{2} \leq \omega\left(\frac{d(x,y)}{2}\right)$$

for all x and y. This differs from the usual definition by the factors of 2, and the reason for these will become clear in a moment. For a time-dependent function $f(x,t)$, we say that a time-dependent function $\omega(s,t)$ is a modulus of continuity for f if $\omega(.,t)$ is a modulus of continuity for $f(.,t)$ for each t, which means that

$$\frac{|f(y,t)-f(x,t)|}{2} \leq \omega\left(\frac{d(x,y)}{2},t\right)$$

for all x and y and all t. In particular, there is a smallest modulus of continuity (which we will sometimes call 'the modulus of continuity of f') defined by

$$\omega_f(s,t) = \sup\left\{\frac{|f(y)-f(x)|}{2} : \frac{d(x,y)}{2} = s\right\}.$$

The following example is an important one:

Lemma 1. *Suppose that f is a function on the real line which is odd, increasing, and concave on the positive half-line. Then*

$$\omega_f(s) = f(s)$$

for $s > 0$.

Proof. We will show that for any fixed $s>0$, the supremum of $|f(y)-f(x)|$ among points with $|y-x|=2s$ is attained at the points $y=s$, $x=-s$, so that $\omega_f(s) = \frac{f(s)-f(-s)}{2} = f(s)$ since f is odd. First, we can assume that $y>x$ and $f(y)-f(x) = |f(y)-f(x)|$ since f is increasing. Then the function $\eta(x) = f(x+s)-f(x-s)$ is even in x since f is odd, and we have for $x \geq s$ that

$$x-s = \frac{2s}{x+s}(0) + \frac{x-s}{x+s}(x+s); \quad s = \frac{x}{x+s}(0) + \frac{s}{x+s}(x+s),$$

so since f is concave on $[0, x+s]$ and $f(0) = 0$ we have

$$f(x-s) \geq \frac{x-s}{x+s} f(x+s); \quad f(s) \geq \frac{s}{x+s} f(x+s); \quad \Longrightarrow f(x-s) + 2f(s) \geq f(x+s)$$

which is equivalent to

$$\eta(x) - \eta(0) = f(x+s) - f(x-s) - f(s) + f(-s) = f(x+s) - f(x-s) - 2f(s) \leq 0.$$

If $0 < x < s$ then we have

$$\eta(x) - \eta(0) = f(x+s) - f(x-s) - 2f(s) = f(x+s) + f(s-x) - 2f(s) \leq 0$$

since f is concave on the interval $[s-x, s+x] \subset \mathbb{R}_+$. Thus 0 is the global maximum of η, as claimed.

We observe that if f_0 is an odd, increasing function which is concave for positive values, then the same remains true for $f(.,t)$ for each $t > 0$ if f satisfies the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial s^2}. \tag{1}$$

Thus we have the following curious corollary:

Corollary 1. *If f_0 is an odd, increasing function which is concave for positive values, then $\omega_f(s,t) = f(s,t)$ for all $s > 0$ and $t > 0$ if f evolves by* (1). *In particular, the modulus of continuity of f satisfies the one-dimensional heat equation.*

Furthermore, if u is a solution of the heat equation

$$\frac{\partial}{\partial t} u = \Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{2}$$

on $\mathbb{R}^n$ which depends on only one of the spatial variables, so that $u(x_1, \cdots, x_n, t) = f(x_1, t)$, where f is as above, then $\omega_u(s,t) = f(s,t)$, and ω_u is a solution of the one-dimensional heat equation.

Later we will see this as the extreme case of a result for general solutions of heat equations.

1.2 Motivation: Zero counting for equations in one space variable

For equations in one spatial variable, we can use zero-counting methods to get a good understanding of how the modulus of continuity of a solution of the heat equation changes with time. This is based on the fact that the number of zeroes of a solution, or the number of intersections of two solutions, does not increase in time — a result first observed by Sturm in 1836 for solutions of the linear heat equation, and refined into a very general tool more recently, particularly through work of Hiroshi Matano, Sigurd Angenent and others. Roughly speaking, as long as new zeroes (or intersections) are not introduced on the boundary or at infinity, then new ones cannot appear. It is also true that the number strictly decreases whenever a zero (or intersection) becomes degenerate in the sense that the first derivative also vanishes, but we will not need this fact.

Consider a bounded smooth solution $u: \mathbb{R}\times\mathbb{R}_+ \to [-M,M]$ of the heat equation on the real line. We will use the zero-counting argument to compare u with the special solution of (1) with the same range $[-M,M]$ given by $\varphi(x,t) = M\,\mathrm{erf}\left(\frac{x-a}{\sqrt{2t}}\right)$, for any $a \in \mathbb{R}$. Since u is smooth and $\varphi(.,t)$ approaches a Heaviside function as $t \to 0$, for any $\varepsilon > 0$, we have exactly one intersection between $(1+\varepsilon)\varphi(,t)$ and $u(.,t)$ for $t > 0$ sufficiently small. Since the number of intersections does not increase with time (noting that $(1+\varepsilon)\varphi \to (1+\varepsilon)M > u$ as $s \to \infty$ and $(1+\varepsilon)\varphi \to -(1+\varepsilon)M < u$ as $s \to -\infty$, so no new zeroes are produced near infinity), we have at most one intersection between $(1+\varepsilon)\varphi(.,t)$ and $u(.,t)$ for every $t > 0$; on the other hand the asymptotics of φ near $s = \pm\infty$ also imply that there is at least one intersection, by the intermediate value theorem. Therefore we have exactly one intersection between $(1+\varepsilon)\varphi(.,t)$ and $u(.,t)$ for each $t > 0$.

For any given $x \in \mathbb{R}$ and $t > 0$, there is a unique $a \in \mathbb{R}$ such that $(1+\varepsilon)\varphi(x,t) = u(x,t)$. Since there is only one intersection between $u(.,t)$ and $(1+\varepsilon)\varphi(.,t)$, and since $(1+\varepsilon)\varphi(s,t) > u(s,t)$ for large s, we have $u(x+s,t) < (1+\varepsilon)\varphi(x+s,t)$ for $s > 0$, and $u(x+s,t) > (1+\varepsilon)\varphi(x+s,t)$ for $s < 0$. This implies

$$|u(x+s,t) - u(x,t)| \le (1+\varepsilon)|\varphi(x+s,t) - \varphi(x,t)| \le 2(1+\varepsilon)\varphi\left(\frac{s}{2},t\right)$$

by Lemma 1, since φ is odd, increasing, and concave for positive values. We conclude that $\omega_u(s,t) \le (1+\varepsilon)\varphi(s,t)$. Finally, letting $\varepsilon \to 0$ we deduce that $\omega_u \le \varphi$. This gives a universal bound on the modulus of continuity for solutions of the heat equation, depending only on M. Notice that this result is sharp, since equality holds in the particular case where $u = \varphi$.

From the modulus of continuity estimate, we can also deduce a sharp gradient estimate: Taking $y \to x$, we conclude that $|u'(x,t)| \le \varphi'(0,t) = \frac{M}{\sqrt{\pi t}}$. Again, this is sharp because equality holds on the solution φ.

We remark that this argument is very robust, and applies equally well for solutions of other parabolic equations such as the p-Laplacian heat flows and graphical curve shortening flow. In fact, interpreted in the right way, this idea gives sharp estimates for arbitrary parabolic equations in one dimension, by comparison to solutions which approach a Heaviside function at the initial time.

Unfortunately there is no good analogue of the zero-counting argument known in higher dimensions, so we must find other ways to control the modulus of continuity.

In fact the result we just obtained does have a direct analogue in higher dimensions, but to prove it we instead use a two-point maximum principle argument.

1.3 The heat equation on Euclidean space

Now let us consider the case of the heat equation (2) on $\mathbb{R}^n$. For simplicity, we will start by considering solutions which approach a constant at infinity (this assumption can be removed but makes the argument simpler).

The maximum principle is a tool which can be used to show that pointwise inequalities (i.e. some function having a definite sign) can be preserved under a parabolic flow. In order to apply this to control the modulus of continuity, we first observe that the statement "ω is a modulus of continuity for u" is equivalent to the non-positivity of the function Z defined by $Z(x,y,t) = u(y,t) - u(x,t) - 2\omega\left(\frac{|y-x|}{2}, t\right)$. Thus we can try to apply a maximum principle to keep Z non-positive, if it is initially so (that is, if $\omega(.,0)$ is a modulus of continuity for $u(.,0)$). The price we pay for doing this is that Z is now a function of *two* points x and y as well as of t.

Proposition 1. *Suppose that u is a smooth solution to the heat equation* (2) *on $\mathbb{R}^n$ with $u(x,t) \to 0$ as $|x| \to \infty$ and $|u(x,t)| \leq M$ for all (x,t). Suppose that $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies* (1) *with $\omega(0,t) = 0$ and $\lim_{s\to\infty} \omega(s,t) \geq M$, and such that $\omega(.,0)$ is a modulus of continuity for $u(.,0)$. Then $\omega(.,t)$ is a modulus of continuity for $u(.,t)$ for each $t \geq 0$.*

Proof. Fix $\varepsilon > 0$, and let $\tilde{\omega}(s,t) = \omega(s,t) + \varepsilon(1+t)$. We will prove that $Z(x,y,t) = u(y,t) - u(x,t) - 2\tilde{\omega}(\frac{|y-x|}{2}, t)$ is strictly negative. Note that $Z \leq -\varepsilon$ on $\{t = 0\}$ and $\{x = y\}$ and as $|y-x| \to \infty$, and since $u \to 0$ at spatial infinity this also implies that $Z < 0$ for either $|x|$ or $|y|$ large. It follows that Z remains strictly negative unless there is some (x_0, y_0, t_0) with $x_0 \neq y_0$, $t_0 > 0$, and $Z(x_0, y_0, t_0) = 0$ and $Z(x,y,t) \leq 0$ for all x and y and all $t \in [0, t_0]$. We will derive a contradiction by considering this point: Here we have $\frac{\partial Z}{\partial t} \geq 0$, and the spatial derivatives satisfy $DZ = 0$ and $D^2 Z \leq 0$. The first inequality gives

$$0 \leq \frac{\partial Z}{\partial t}\Big|_{(x_0,y_0,t_0)} = \Delta u|_{(y_0,t_0)} - \Delta u|_{(x_0,t_0)} - 2\frac{\partial \tilde{\omega}}{\partial t}\Big|_{(\frac{|y_0-x_0|}{2}, t_0)}, \tag{3}$$

The inequality $D^2 Z \leq 0$ is in the sense of posivite-definiteness of matrices, and this gives a lot of information since it involves a $(2n) \times (2n)$ matrix (with n directions corresponding moving x, and the other n corresponding to moving y). We will only need to compute certain components of this matrix, chosen to extract useful inequalities. In order to do this, we first choose an orthonormal basis for $\mathbb{R}^n$ for which $e_n = \frac{y_0 - x_0}{|y_0 - x_0|}$. Then we consider two special variations:

• Move x and y apart with equal velocities, i.e. $\frac{d}{ds}x = -e_n$, $\frac{d}{ds}y = e_n$. Then we have $\frac{d}{ds}|y-x| = 2$, and $\frac{d^2}{ds^2}|y-x| = 0$, so we obtain

$$0 \geq \frac{d^2}{ds^2}Z = D_nD_nu|_{(y_0,t_0)} - D_nD_nu|_{(x_0,t_0)} - 2\tilde{\omega}''|_{\left(\frac{|y_0-x_0|}{2},t_0\right)}.$$

• Move x and y in parallel in a direction orthogonal to the line between them, i.e. $\frac{d}{ds}x = \frac{d}{ds}y = e_i$ for some $i < n$. Then we have $|y-x|$ constant, and so

$$0 \geq \frac{d^2}{ds^2}Z = D_iD_iu|_{(y_0,t_0)} - D_iD_iu|_{(x_0,t_0)}.$$

Adding these inequalities over all i gives

$$0 \geq \Delta u|_{(y_0,t_0)} - \Delta u|_{(x_0,t_0)} = 2\tilde{\omega}''|_{\left(\frac{|y_0-x_0|}{2},t_0\right)}. \tag{4}$$

Now we combine the inequalities (3) and (4) to give

$$\partial_r \tilde{\omega} \leq \tilde{\omega}''$$

at the point $\left(\frac{|y_0-x_0|}{2}, t_0\right)$. But this is impossible, since

$$\partial_t \tilde{\omega} = \partial_t \omega + \varepsilon = \omega'' + \varepsilon > \omega'' = \tilde{\omega}''.$$

This is a contradiction. Therefore such a point (x_0, y_0, t_0) cannot occur, and Z stays negative. Finally, letting ε approach zero gives the result of the Proposition, that $\omega(.,t)$ is a modulus of continuity for $u(.,t)$ for each t.

It may be useful to make some remarks here about the particular directions which were used to obtain second derivative inequalities: The guide here is the 'equality case', which is when the solution is a 'one-dimensional' solution of the form $u(x,t) = f(x \cdot e_1, t)$, where f is odd, increasing, and concave for positive values. As we saw earlier, in this case the modulus of continuity of u is f, and equality is attained at the points of the form $(x, -x, t)$ for $x > 0$. That is, the equality set in this special case is $\{(x,y,t) : \; y \cdot e_1 = -x \cdot e_1\}$. In the argument we are using the inequality $D^2Z \leq 0$ in some direction, so if we want to get a sharp inequality (and not throw anything away) we can only use those directions for which $D^2Z = 0$ in the equality case. It is easy to see that these are just the ones that we chose in the argument: The directions where equality holds are spanned by those where x and y move in parallel in a direction orthogonal to e_1, and that where x and y move in opposite directions along the line between them. This principle is often a useful guide in constructing these arguments for two-point maximum principles.

As a consequence of the above modulus of continuity estimate, we can deduce the same sharp gradient bound as the zero-counting argument gave us, but now in any dimension:

Corollary 2. *Let u be a smooth solution to the heat equation* (2) *on $\mathbb{R}^n$, with $|u(x,t)| \leq M$ and approaching zero at infinity. Then $|Du(x,t)| \leq \frac{M}{\sqrt{\pi t}}$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.*

Proof. We can apply the Proposition with $\omega(s,t) = M\,\mathrm{erf}\left(\frac{s}{2\sqrt{t+a}}\right)$ for any sufficiently small $a > 0$. Letting $a \to 0$ gives the same bound on the modulus of continuity as for the one-dimensional case, and the same gradient bound.

1.4 The Neumann heat equation on a bounded domain

Next we consider solutions of the heat equation on bounded domains. The simplest case to consider is the Neumann condition, though (non-sharp) results can be obtained for other cases. The main result for the Neumann case is the following:

Proposition 2. *Let Ω be a (smooth) bounded (strictly) convex domain in $\mathbb{R}^n$ with diameter $D = \sup\{|y-x| :\ x, y \in \Omega\}$, and let u be a (smooth) solution of the heat equation* (2) *on $\Omega \times \mathbb{R}_+$, satisfying the Neumann condition $D_\nu u = 0$ on $\partial\Omega \times \mathbb{R}_+$. Suppose that $\omega :\ [0, D/2] \times \mathbb{R}_+ \to \mathbb{R}$ is a solution of the one-dimensional heat equation* (1) *such that $\omega'(s,t) \geq 0$ and $\omega(0,t) = 0$, and such that $\omega(.,0)$ is a modulus of continuity for $u(.,0)$. Then $\omega(.,t)$ is a modulus of continuity for $u(.,t)$ for each $t \geq 0$.*

Proof. The proof is similar to the one we just gave, but we must also deal with the possibility that the maximum occurs on the boundary. As before, we first modify ω in order to obtain strict inequalities: Set $\tilde{\omega}(s,t) = \omega(s,t) + \varepsilon(1+t) + \varepsilon s$. Since Z is continuous on the compact set $\bar{\Omega} \times \bar{\Omega}$, it attains a maximum at each time, and the maximum is a continuous function of time. Therefore if Z does not remain negative, there is a first time $t_0 > 0$ where the maximum of Z reaches zero, and a point (x_0, y_0) in $\bar{\Omega} \times \bar{\Omega}$ where this occurs. Since $Z \leq -\varepsilon$ on $\{x = y\}$, we know that $x_0 \neq y_0$. This leaves two possibilities: Either (x_0, y_0) is in the boundary $\partial\Omega \times \bar{\Omega} \cup \bar{\Omega} \times \partial\Omega$, or both x_0 and y_0 are interior points of Ω. In the latter case, we derive a contradiction exactly as before, since $\partial_t \tilde{\omega} > \tilde{\omega}''$. So we need only consider the case where $x_0 \in \partial\Omega$ (the case $y_0 \in \partial\Omega$ is similar).

We compute the derivative of Z in the direction where $\dot{x} = -\nu$ (where ν is the outward-pointing unit normal to $\partial\Omega$ at x_0), and $\dot{y} = 0$. Since $D_\nu u|_{(x_0,t_0)} = 0$ by the Neumann condition, we have

$$D_{(-\nu,0)} Z|_{(x_0,y_0,t_0)} = -\tilde{\omega}'\big|_{\left(\frac{|y_0-x_0|}{2},t_0\right)} \left\langle \frac{y_0 - x_0}{|y_0 - x_0|}, \nu \right\rangle > 0$$

since $\langle y_0 - x_0, \nu \rangle < 0$ by the strict convexity of Ω, and $\tilde{\omega}' = \omega' + \varepsilon > 0$. This contradicts the assumption that Z attains a maximum at (x_0, y_0) at time t_0, and the proof is complete.

As before, we obtain sharp gradient estimates, determined by the gradient of the solution of the one-dimensional heat equation with Neumann condition and Heaviside initial data on the interval $[-D/2, D/2]$. In fact the modulus of continuity control allows us to prove a famous eigenvalue inequality, the Payne-Weinberger inequality:

1.5 The Payne-Weinberger inequality

Theorem 1 ([30]). *If Ω is a bounded convex domain with diameter D in $\mathbb{R}^n$, then the first Neumann eigenvalue*

$$\lambda_1^N(\Omega) = \inf\left\{\int_\Omega |Du|^2 : \ u \in H^1(\Omega), \int_\Omega u = 0\right\}$$

is no less than $\frac{\pi^2}{D^2}$.

Proof. By approximation, it suffices to consider the case where the boundary of Ω is smooth and strictly convex. Let u_1 be the first eigenfunction of the Neumann Laplacian, so that $\Delta u_1 + \lambda_1^N u_1 = 0$ on Ω, with $D_\nu u_1 = 0$ on $\partial\Omega$. Then define $u(x,t) = \mathrm{e}^{-\lambda_1^N t} u_1(x)$, so that u is a smooth solution of the Neumann heat equation on Ω. We can apply Proposition 2 with $\omega(s,t) = C\mathrm{e}^{-\frac{\pi^2}{D^2}t}\sin\left(\frac{\pi s}{D}\right)$, since this satisfies the one-dimensional heat equation on $[0, D/2]$ and is increasing and positive, provided we choose C sufficiently large to ensure that $\omega(.,0)$ is a modulus of continuity for u_1 (this can always be done since u_1 is smooth).

This implies that for any x and y, $|u(y,t) - u(x,t)| \le \omega(D/2,t) = C\mathrm{e}^{-\frac{\pi^2}{D^2}t}$. In particular this implies

$$\mathrm{osc}\, u_1 \mathrm{e}^{-\lambda_1^N t} \le C\mathrm{e}^{-\frac{\pi^2}{D^2}t}$$

for each $t \ge 0$. Taking $t \to \infty$ implies that $\lambda_1^N \ge \frac{\pi^2}{D^2}$, as claimed.

Notes: The modulus of continuity bounds as presented here were developed in a series of papers with Julie Clutterbuck [2, 3]. The proof of the Payne-Weinberger inequality was included in [4].

Lecture 2: Heat equations on Riemannian manifolds and nonlinear eigenvalues

2.1 Riemannian manifolds: Distance, Curvature and heat equations

In this lecture we will investigate the application of the ideas we developed in the first lecture to a more general context, including equations more general than the heat equation, and domains more general than Euclidean domains. In order to do this we first need to review some Riemannian geometry.

A Riemannian manifold is a smooth manifold M equipped with a Riemannian metric, which is a smoothly varying inner product g_x on each tangent space T_xM. This allows us to make sense of the length of a smooth curve $\sigma : [a,b] \to M$ by setting

$$L[\sigma] = \int_a^b \sqrt{g_{\sigma(s)}(\sigma'(s), \sigma'(s))}\, ds,$$

where $\sigma'(s) \in T_{\sigma(s)}M$ is the tangent vector to the curve. The length then allows us to define a distance function (if M is path-connected), called the *Riemannian distance*, by setting

$$d(x,y) := \inf\{L[\sigma] : \sigma \in C^\infty([0,1], M), \sigma(0) = x, \sigma(1) = y\}.$$

This defines a distance function in the sense of metric spaces. If the manifold is metrically complete, then the Hopf-Rinow theorem tells us that the distance between points is attained by a (possibly non-unique) *geodesic*, which is a locally length-minimizing curve. In the case where the manifold is $\mathbb{R}^n$ and the Riemannian metric is the Euclidean inner product, this is the usual Euclidean distance and the geodesics are straight lines. This is still true if M is a convex subset of $\mathbb{R}^n$, but not for non-convex subsets. In general, the distance function is not smooth (for a simple example, consider the unit circle with arc length parameter s from some given point p, on which the distance from p is non-smooth at the antipodal point $s = \pi$, where it looks like $\pi - |s - \pi|$).

The Riemannian metric also defines a connection, which is a differential operator acting on vector fields. This is called the *Riemannian connection* or *Levi-Civita connection*, and is uniquely determined by the conditions that it be symmetric (so that $\nabla_i \partial_j = \nabla_j \partial_i$ in any local chart) and that it is compatible with the Riemannian metric, so that differentiating an inner product gives the same result as differentiating the vector fields inside the inner product, in the sense that

$$d_w g(U,V) = g(\nabla_w U, V) + g(U, \nabla_w V)$$

for any vector fields U and V and any vector w. The geodesics are then (up to reparametrisation) the curves σ which have parallel tangent vector, so that $\nabla_s \sigma'(s) = 0$ along σ. Accordingly, there is a unique geodesic starting at any point with any given initial tangent vector, and this defines the *exponential map*: This takes any tangent vector v at a point x of M to the endpoint $\exp_x(v)$ of the geodesic $\sigma: [0,1] \to M$ with $\sigma(0) = x$ and $\sigma'(0) = v$. This always exists for small v, but may not exist for all v (the Hopf-Rinow theorem guarantees global existence if M is metrically complete, however).

The *curvature tensor* measures the failure of the commutation of differentiation using the Riemannian connection: For vector fields U, V, W, Z it is defined by

$$R(U,V,W,Z) = g(\nabla_V \nabla_U W - \nabla_U \nabla_V W - \nabla_{[V,U]} W, Z).$$

Given a pair of orthonormal unit vectors e_1 and e_2, the *sectional curvature* of the plane they generate is defined by $R(e_1,e_2,e_1,e_2)$ (this is independent of the choice of orthonormal basis for this plane). Given a unit vector e_1, the *Ricci curvature* in direction e_1 is defined by

$$\mathrm{Rc}(e_1,e_1) = \sum_{j>1} R(e_1,e_j,e_1,e_j),$$

for any orthonormal basis $\{e_i\}$ completing e_1. The Ricci curvature will be particularly important in what follows, as it arises naturally when computing Laplacians of the distance function.

For a smooth function f on M, the Hessian of f is the second derivative computed using the Riemannian connection, so that $\nabla^2 f(u,v) = D_u(D_v f) - D_{\nabla_u v} f$ for any vectors u and v. Equivalently, $\nabla^2 f(e,e) = \frac{d^2}{ds^2} f \circ \sigma(s)\big|_{s=0}$, where σ is the geodesic in M with $\sigma'(0) = e$. The Laplacian Δf of f is the trace of the Hessian with respect to the metric (equivalently, the sum of diagonal elements with respect to any orthonormal basis). This allows us to make sense of the heat equation on a Riemannian manifold. More generally, we will consider quasilinear equations (with cefficients depending on the gradient) which are 'isotropic', in the sense that the diffusion coefficients are unchanged by any orthogonal transformation fixing the gradient vector. This means that the equation has the form

$$\frac{\partial u}{\partial t} = \mathscr{L}[u] := \left[a(|Du|)\frac{u_i u_j}{|Du|^2} + b(|Du|)\left(\delta^{ij} - \frac{u_i u_j}{|Du|^2}\right)\right] \nabla_i \nabla_j u, \tag{5}$$

where a and b are positive functions. Examples include the heat equation with $a = b = 1$, the p-Laplacian heat flows with $a = |Du|^{p-2}$ and $b = (p-1)|Du|^{p-2}$, and the graphical mean curvature flow with $a = \frac{1}{1+|Du|^2}$ and $b = 1$.

2.2 The Ricci non-negative case

In this lecture we will only consider the simplest case, where the Ricci curvature is non-negative on M. We will assume for now that M is compact. We allow the possibility that M has boundary, but in that case we require that the boundary is convex. In this case there is a version of the Hopf-Rinow theorem which says that the distance between any pair of points in M is attained by a geodesic.

The result is as follows:

Proposition 3. *Let M be a compact Riemannian manifold with convex (or empty) boundary, with non-negative Ricci curvature and diameter $D = \sup\{d(x,y):\ x,y \in M\}$. Let u be a smooth solution of equation (5) with Neumann boundary condition $D_\nu u = 0$ on ∂M, where a and b are positive functions. Suppose that ω_0 is a modulus of continuity for $u(.,0)$, which is increasing, and suppose ω satisfies $\partial_t \omega \geq a(\omega')\omega''$ on $[0,D/2]\times[0,T)$, with $\omega(0,t) = 0$ and $\omega'(D/2,t) = 0$. Then $\omega(.,t)$ is a modulus of continuity for $u(.,t)$ for each $t \geq 0$.*

Proof. Assuming a and b are smooth and positive, we can find a sequence ω_ε which strictly decreases uniformly to ω as $\varepsilon \to 0$, such that $\partial_t \omega_\varepsilon > a(\omega_\varepsilon')\omega_\varepsilon''$ and $\omega_\varepsilon'(s,t) > 0$ on $(0,D/2)\times[0,T)$. As before, define $Z:\ M\times M\times[0,T)\to\mathbb{R}$ by $Z(x,y,t) = u(y,t) - u(x,t) - 2\omega_\varepsilon\left(\frac{d(x,y)}{2},t\right)$, where d is the Riemannian distance. By assumption Z is strictly negative at $t=0$, and where $x=y$ for any t. We consider a point (x_0,y_0,t_0) where Z first becomes zero, noting that we have $y_0 \neq x_0$.

Now we must confront the difficulty that Z is in general not a smooth function: Since the Riemannian distance is not smooth (just Lipschitz in general) we cannot just differentiate Z in some direction to deduce inequalities. To get around this problem, we go back to the definition of distance as an infimum of lengths of paths, and extend Z to a function which lives on a space of paths. In doing this we appear to sacrifice something, since we now have to deal with functions defined on an infinite-dimensional space, but we gain smoothness: We define $\tilde{Z}$ to be the function on $C^\infty([0,1],M)\times[0,T)$ defined by

$$\tilde{Z}[\sigma,t] = u(\sigma(1),t) - u(\sigma(0),t) - 2\omega_\varepsilon\left(\frac{L[\sigma]}{2},t\right).$$

This corresponds naturally to Z when σ is a length-minimising geodesic, and since ω_ε is non-decreasing in the first argument we always have the inequality $\tilde{Z}[\sigma,t] \leq Z(\sigma(0),\sigma(1),t)$, with equality if σ is a length-minimising geodesic. Importantly for us, $\tilde{Z}$ is smooth, in the sense that if $\sigma:\ [0,1]\times I\to M$ is a smooth family of paths, then $\tilde{Z}[\sigma(.,r),t]$ is a smooth function of r and t.

It is useful to compute the derivatives of the length under such smooth variations: For the first derivative, we have

$$\frac{d}{dr}L[\sigma(.,r)] = \int_0^1 \frac{g(\sigma_s,\nabla_r\sigma_s)}{|\sigma_s|}\,ds \tag{6}$$

where the subscripts denote derivatives, and s represents the parameter along the curve, while r represents the variation parameter through the family of curves. Using the symmetry of the connection, and the compatibility of the connection with the metric, and assuming we have parametrised at constant speed (so that $|\sigma_s|$ is constant in s) we can re-write this as follows:

$$\begin{aligned}\frac{d}{dr}L[\sigma(.,r)] &= \int_0^1 \frac{g(\sigma_s,\nabla_s\sigma_r)}{|\sigma_s|}\,ds\\ &= g\left(\frac{\sigma_s}{|\sigma_s|},\sigma_r\right)\Big|_0^1 - \int_0^1 \frac{g(\nabla_s\sigma_s,\sigma_r)}{|\sigma_s|}\,ds \end{aligned} \tag{7}$$

In particular if σ is a geodesic then $\nabla_s\sigma_s = 0$, and the second term vanishes.

The second derivative may be computed by differentiating (6) with respect to r. We do this assuming that σ is a geodesic (so that $|\sigma_s| = L$), and obtain the following:

$$\begin{aligned}\frac{d^2}{dr^2}L[\sigma(.,r)]\big|_{r=0} &= \frac{1}{L}\int_0^1 |(\nabla_r\sigma_s)^\perp|^2 + g(\sigma_s,\nabla_r\nabla_r\sigma_s)\,ds\\ &= \frac{1}{L}\int_0^1 \left|(\nabla_r\sigma_s)^\perp\right|^2 + g(\sigma_s,\nabla_s\nabla_r\sigma_r) + R(\sigma_r,\sigma_s,\sigma_r,\sigma_s)\,ds\\ &= \frac{1}{L}\int_0^1 \left|(\nabla_r\sigma_s)^\perp\right|^2 - R(\sigma_r,\sigma_s,\sigma_r,\sigma_s)\,ds + g\left(\frac{\sigma_s}{|\sigma_s|},\nabla_r\sigma_r\right)\Big|_0^1. \end{aligned} \tag{8}$$

Using (7) we can rule out the possibility that x_0 or y_0 is in the boundary, in exactly the same way as in the Euclidean setting: If σ_0 is a minimising geodesic from $x_0 \in \partial M$ to y_0, define a smooth variation $\sigma : [0,1]\times(-\delta,\delta)\to M$ as follows: Set $e = -\nu \in T_xM$, and parallel transport along σ to obtain $e(s) \in T_{\sigma_0(s)}M$ such that $\nabla_s e = 0$. Then define $\sigma(s,r) = \exp_{\sigma_0(s)}(r(1-s)e(s))$. Note that when $r = 0$ this returns $\sigma_0(s)$, and σ exists for small r and is smooth in s and r. We have $\sigma_r(s) = (1-s)e(s)$, so $\sigma_r(0) = -\nu$ and $\sigma_r(1) = 0$. This gives (since $D_\nu u = 0$ by the Neumann condition)

$$\frac{d}{dr}\tilde{Z}[\sigma(.,r),t_0]\big|_{r=0} = -\omega_\varepsilon'\, g\left(\frac{\sigma_s}{|\sigma_s|},\gamma_r\right)\Big|_0^1 = \omega_\varepsilon' g_x\left(\frac{\sigma_0'(0)}{|\sigma_0'(0)|},-\nu\right) > 0$$

since σ_0' points strictly into M at x and $\omega_\varepsilon' > 0$. This contradicts the claim that $Z(.,.,t_0)$ has a maximum at (x_0,y_0) since this would imply that $\tilde{Z}(.,t_0)$ has a maximum at σ_0.

It follows therefore that $Z(.,.,t_0)$ attains a maximum for $x_0 \neq y_0$ both interior points of M, and therefore that $\tilde{Z}(.,t_0)$ attains a maximum at σ_0, where σ_0 is a minimising geodesic from x_0 to y_0. For convenience we choose an orthonormal basis $\{e_i\}$ for $T_{x_0}M$ with $e_n = \frac{\sigma_0'}{|\sigma_0'|}$, and parallel transport along σ to get an orthonormal basis $\{e_i(s)\}$ at each point $\sigma_0(s)$. Since σ_0' is parallel along σ_0 we have that $e_n(s) = \frac{\sigma_0'(s)}{|\sigma_0'(s)|}$ for each s.

The first variation formula (7) yields the following: Since σ_0 is a maximum point of $\tilde{L}$, we have for any smooth variation

$$\begin{aligned}0 &= \frac{d}{dr}L[\sigma(.,r)]\big|_{r=0}\\ &= D_{\sigma_r(1)}u - D_{\sigma_r(0)}u - \omega_\varepsilon'\left(g_{y_0}(e_n(1),\sigma_r(1)) + g_{x_0}(e_n(0),\sigma_r(0))\right),\end{aligned}$$

and therefore since we can choose variations with arbitrary $\sigma_r(0)$ and $\sigma_r(1)$ we must have

$$Du(y_0,t_0) = \omega_\varepsilon' e_n(1); \quad \text{and} \quad Du(x_0,t_0) = \omega_\varepsilon' e_n(0).$$

It follows that the time derivative of $\tilde{Z}$ at (x_0,y_0,t_0) satisfies (since $\tilde{Z}(\sigma_0,t)$ increases to zero as t increases to t_0)

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t}\tilde{Z}[\sigma_0,t_0] &= a(\omega_\varepsilon')\nabla_n\nabla_n u(y_0,t_0) + b(\omega_\varepsilon')\sum_{i<n}\nabla_i\nabla_i u(y_0,t_0) \\ &\quad - a(\omega_\varepsilon')\nabla_n\nabla_n u(x_0,t_0) - b(\omega_\varepsilon')\sum_{i<n}\nabla_i\nabla_i u(x_0,t_0) - 2\frac{\partial\omega_\varepsilon}{\partial t}. \end{aligned} \tag{9}$$

To proceed we need to use the second variation formula (8) to obtain useful inequalities. Recall that in $\mathbb{R}^n$ the variations which were useful were: Moving x and y apart along the line between them; and moving them both in parallel in an orthogonal direction. The first of these has an obvious analogue, in which x and y move apart along the geodesic σ_0: That is, $\sigma(s,r) = \sigma_0(s+(2s-1)r/L)$. In this case, we have $L[\sigma(.,r)] = L+2r$, so we have $\frac{d}{dr}L[\sigma(.,r)] = 2$ and $\frac{d^2}{dr^2}L[\sigma(.,r)] = 0$. Then we have

$$0 \geq \frac{d^2}{dr^2}\tilde{Z}[\sigma(.,r),t_0] = \nabla_n\nabla_n u(y_0,t_0) - \nabla_n\nabla_n u(x_0,t_0) - 2\omega_\varepsilon''\left(\frac{d(x_0,y_0)}{2},t_0\right). \tag{10}$$

The second kind of variation has an analogue in the Riemannian case constructed as follows: For each $i<n$, define $\sigma(s,r) = \exp_{\sigma_0(s)}(re_i(s))$. Thus we have $\sigma_r(s) = e_i(s)$ for each s, and $\nabla_r\sigma_r = 0$ everywhere. The first variation formula (7) gives $\frac{d}{dr}L[\sigma(.,r)]\big|_{r=0} = 0$, and the second variation formula (8) gives

$$\frac{d^2}{dr^2}L[\sigma(.,r)]\Big|_{r=0} = -\frac{1}{L}\int_0^1 R(e_i,Le_n,e_i,Le_n)\,ds = -L\int_0^1 R_{inin}\,ds.$$

This gives

$$0 \geq \frac{d^2}{dr^2}\tilde{Z}[\sigma(.,r),t_0] = \nabla_i\nabla_i u(y_0,t_0) - \nabla_i\nabla_i u(x_0,t_0) + \omega_\varepsilon' L\int_0^1 R_{inin}\,ds.$$

Summing over $i = 1,\cdots,n-1$ gives

$$\begin{aligned} 0 &\geq \sum_{i<n}\nabla_i\nabla_i u(y_0,t_0) - \sum_{i<n}\nabla_i\nabla_i u(x_0,t_0) + \omega_\varepsilon' L\int_0^1 \mathrm{Rc}(e_n,e_n)\,ds \\ &\geq \sum_{i<n}\nabla_i\nabla_i u(y_0,t_0) - \sum_{i<n}\nabla_i\nabla_i u(x_0,t_0), \end{aligned} \tag{11}$$

since M has non-negative Ricci curvature. Combining (10) and (11) gives

$$\begin{aligned} 0 \geq{}& a(\omega_\varepsilon')\left(\nabla_n\nabla_n u(y_0,t_0) - \nabla_n\nabla_n u(x_0,t_0) - 2\omega_\varepsilon''\right) \\ &+ b(\omega_\varepsilon')\left(\nabla_i\nabla_i u(y_0,t_0) - \nabla_i\nabla_i u(x_0,t_0)\right). \end{aligned}$$

Combining this with (9) we deduce that

$$\frac{\partial \omega_\varepsilon}{\partial t} \le a(\omega_\varepsilon')\omega_\varepsilon'',$$

which contradicts our assumption. Therefore we must conclude that Z remains negative, and letting ε approach zero we have the claimed result.

In the case of the heat equation, we can evolve the modulus of continuity by the one-dimensional heat equation (1), and in particular we deduce the same lower bound on the first eigenvalue as was obtained previously in the Euclidean setting:

Theorem 2. *Let (M,g) be a compact Riemannian manifold (with convex or non-empty boundary), with non-negative Ricci curvature and diameter D. Then $\lambda_1^N(M) \ge \frac{\pi^2}{D^2}$.*

Notes: This result was first proved by Zhong and Yang [36] using gradient estimate for the eigenfunction equation, refining previous argument of Li [24] and Li-Yau [25]. The argument given here first appeared in [5], where the argument also produces a sharp estimate on the modulus of continuity (and a sharp lower bound for the first eigenvalue) in terms of diameter, given any lower bound on Ricci curvature — this gives a simple proof of results in [22, 11, 17, 16]. There is also a version which holds on Bakry-Emery spaces, which appeared in [8].

In the proof the solution was assumed to be smooth, but in fact it suffices to consider viscosity solutions. In particular one can apply the result without difficulty for equations such as the p-Laplacian heat flows which are not uniformly parabolic (see [26], [9]).

2.3 Nonlinear eigenvalues

We mention here how the argument can be used to obtain lower bounds on p-eigenvalues, at least in the case $p \le 2$. The first p-eigenvalue is defined by

$$\lambda_{1,p}(M) = \inf\left\{\int_M |Df|^p : \int_M |f|^p = 1, \int_M |f|^{p-2}f = 0\right\}.$$

The minimizer is a solution of the nonlinear eigenvalue problem

$$0 = \Delta_p u + \lambda_{1,p}(M)|u|^{p-2}u, \tag{12}$$

with Neumann boundary condition $D_\nu u = 0$ if the boundary is non-empty, where $\Delta_p f = \nabla_i\left(g^{ij}|Df|^{p-2}D_j f\right)$ is the p-Laplacian. Unfortunately, the obvious corresponding flow, the p-Laplacian heat equation, does not interact well with the p-Laplacian heat equation, and instead we have to consider the more nonlinear heat flow (sometimes called the 'Trudinger equation') given by

$$|u_t|^{p-2}u_t = \Delta_p u,$$

or equivalently

$$u_t = \left|\Delta_p u\right|^{-\frac{p-2}{p-1}} \Delta_p u. \tag{13}$$

This corresponds to the L^p-steepest descent flow for the L^p Dirichlet energy $\int_M |Du|^p$. If f is a solution of (12), then $u(x,t) = \mathrm{e}^{-\lambda^{1/(p-1)}t} f(x)$ is a solution to the flow (13), and in order to find a lower bound on λ it suffices to prove exponential decay of the solution to (13).

The argument to do this is similar to that we used previously: If we set $Z(x,y,t) = u(y,t) - u(x,t) - 2\omega\left(\frac{d(x,y)}{2},t\right)$, then at point where Z first reaches zero (which must be of the form (x_0,y_0,t_0) with x_0 and y_0 interior points of M, and t_0, by the same argument as before if M has convex boundary), then we have the identity $Du(y) = \omega' e_n(1)$ and $Du(x) = \omega' e_n(0)$, and the inequalities (11) and (10), so that we have

$$\Delta_p u(y) - \Delta_p u(x) \leq 2(\omega')^{p-2}\omega''.$$

Note that $\omega'' < 0$, so this can be rephrased as saying that

$$\Delta_p u(x) \geq \Delta_p u(y) + \delta$$

where $\delta = -2(\omega')^{p-2}\omega'' > 0$. Now if $p \leq 2$, then the function $\phi : z \mapsto |z|^{-\frac{p-2}{p-1}} z$ is convex for positive z and odd, so we can conclude that $\phi(b+\delta) - \phi(b) \geq 2\phi(\delta/2)$ for any b. This implies that

$$\phi\left(\Delta_p u(x)\right) \geq \phi\left(\Delta_p u(y)\right)^{1/(p-1)} + 2\phi\left(\frac{(\omega')^{p-2}\omega''}{2}\right),$$

which is equivalent to

$$\partial_t u(y) - \partial_t u(x) \leq 2\phi\left(\frac{(\omega')^{p-2}\omega''}{2}\right),$$

On the other hand since $\partial_t Z \geq 0$ we have

$$\partial_t u(y) - \partial_t u(x) \geq 2\omega_t,$$

and we derive a contradiction as before if $\omega_t > \phi(\Delta_p \omega)$. This allows us to use the one-dimensional p-eigenfunction for ω, yielding the sharp inequality

$$\lambda_{1,p}(M) \geq \lambda_{1,p}[-D/2, D/2].$$

provided the Ricci curvature of M is non-negative. This result was proved previously (for any $p > 1$) by Valtorta [34] using gradient estimates for the eigenfunction equation, and the special case of convex domains in Euclidean space was proved independently in [18]. The result for arbitrary lower bounds on the Ricci curvature was proved by Naber and Valtorta [28], and this can also be recovered using the modulus of continuity argument in the case $p \leq 2$.

Lecture 3: Log-concavity and the fundamental gap

3.1 The fundamental gap conjecture

In this lecture we will discuss the proof of the fundamental gap conjecture, which uses the modulus of continuity argument as well as some other ingredients. The setting is as follows: Given a compact Riemannian manifold M (maybe with boundary), we can consider the sequence of eigenfunctions of the Laplacian (for now, let's say with Dirichlet boundary conditions), and perhaps also incorporating a potential:

$$\begin{aligned} \Delta u_i - V u_i + \lambda_i u_i &= 0 \quad \text{on } M; \\ u &= 0 \quad \text{on } \partial M, \end{aligned}$$

where V is a (smooth) function on M. We have a sequence of eigenfunctions $\{u_i\}$ forming an orthonormal basis for $L^2(M)$, with eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \cdots \to \infty$. The first eigenfunction is unique (up to scaling), and is the unique eigenfunction with a sign (which we can take to be positive). This is called the *ground state*. Since it is unique, there is a strict inequality between the ground state eigenvalue (or ground state energy) λ_0 and the next λ_1 (the energy of the first excited state). This difference is called the *fundamental gap*, and is a reflection of the stability of the ground state. We will denote this by $\Gamma_V(M) = \lambda_1(M) - \lambda_0(M)$.

Note that in the case of Neumann boundary (and $V = 0$) the first eigenfunction is a constant, and $\lambda_0 = 0$, so the fundamental gap coincides with the first non-trivial eigenvalue which we estimated in the first lecture, with the sharp lower bound on a convex Euclidean domain provided by the Payne-Weinberger inequality $\Gamma(\Omega) \geq \frac{\pi^2}{D^2}$, where $D = \mathrm{diam}(\Omega)$.

The corresponding conjecture for the Dirichlet case is the so-called *fundamental gap conjecture*, made independently by van den Berg [12], Yau [32] and Ashbaugh-Benguria [10]:

Conjecture 1. If Ω is a bounded convex domain in $\mathbb{R}^n$, and V is a convex function on Ω, then $\Gamma_V(\Omega) \geq \frac{3\pi^2}{D^2}$, where $D = \mathrm{diam}(\Omega)$.

I proved this with Julie Clutterbuck in 2011 [4]. Today I will explain the ideas of our proof, but first I will digress briefly to discuss the one-dimensional case, which was proved much earlier by Lavine [23].

3.2 The 1D case

In the one-dimensional case, the Sturm comparison theorem implies that each eigenspace of the Laplacian is one-dimensional, so that we have strict inequalities $\lambda_i < \lambda_{i+1}$ for $i = 0, 1, \cdots$. In this situation the eigenfunctions and eigenvalues depend smoothly on the potential V, and we can compute the derivatives of these functions in a standard way: If we consider a smooth variation of the potential, so that $V(s,\varepsilon)$ is smooth on $[-D/2, D/2] \times I$ for some interval I about the origin, write $\dot{V} = \frac{\partial V}{\partial \varepsilon}$ (more generally, we will denote derivatives with respect to ε with dots). Then differentiating the eigenfunction equation $u_i'' - Vu_i + \lambda_i u_i = 0$ gives

$$0 = \dot{u}_i'' - V\dot{u}_i + \lambda_i \dot{u}_i + (\dot{\lambda}_i - \dot{V})u_i. \tag{14}$$

Differentiating the boundary condition also gives $\dot{u}_i = 0$ at the endpoints, if u_i is a Dirichlet eigenfunction. Multiplying (14) by u_i and integrating by parts, and applying the eigenfunction equation, then yields

$$0 = \int_{-D/2}^{D/2} (\dot{\lambda}_i - \dot{V}_i) u_i^2 \, ds,$$

or equivalently

$$\dot{\lambda}_i = \frac{\int \dot{V} u_i^2 \, ds}{\int u_i^2 \, ds}.$$

If we assume that the eigenfunctions form an orthonormal basis in L^2, then we have $\int u_i^2 \, ds = 1$, and the denominator can be removed. This gives the following formula for the change in the gap under a smooth change in potential:

$$\dot{\Gamma} = \int \dot{V}(u_1^2 - u_0^2) \, ds.$$

Lavine's argument is based on the following observation: Given a potential V which is convex on $[-D/2, D/2]$, consider the family of potentials $\{V + as : \ a \in \mathbb{R}\}$. Some simple asymptotics guarantee that the fundamental gap is a proper function of a, so there exists a value $\bar{a}$ where the gap is minimized. We might call the gap $\tilde{\Gamma}_V = \Gamma_{V+\bar{a}s}$ the *linear family gap* of V. Note that at $V + \bar{a}s$ we have

$$0 = \frac{d}{da}\Gamma = \int s(u_1^2 - u_0^2) \, ds.$$

Since the eigenfunctions are normalised, we also have $\int (u_1^2 - u_0^2) \, ds = 0$. Combining these two identities, we discover that the function

$$\psi(s) = \int_{-D/2}^{s} \int_{-D/2}^{s'} \left(u_1(s'')^2 - u_0(s'')^2\right) \, ds'' \, ds' = \int_{-D/2}^{s} (s - s')(u_1(s')^2 - u_0(s')^2) \, ds'$$

satisfies $\psi'' = u_1^2 - u_0^2$, and $\psi(\pm D/2) = \psi'(\pm D/2) = 0$. We also know that $u_1^2 - u_0^2$ has exactly two zeroes (it cannot have more than two since u_1/u_0 is monotone, and it cannot have less than two since it is orthogonal to any linear function. It follows that $u_1^2 - u_0^2$ is positive in a neighbourhood of each endpoint $\pm D/2$, and negative on an interval in between, and this in turn implies that ψ is a positive function.

We can now prove that the linear family gap is monotone with respect to convexity of the potential: That is, if $V_2 - V_1$ is a convex function, then $\tilde{\Gamma}_{V_2} \geq \tilde{\Gamma}_{V_1}$. This follows since $\tilde{\Gamma}_{(1-t)V_1+tV_2} \leq \Gamma_{(1-t)V_1+tV_2+\bar{a}s}$, if $\bar{a}$ is chosen so that equality holds at $t = 0$. Then

$$\begin{aligned}\frac{d}{dt}\Gamma_{(1-t)V_1)+tV_2}|_{t=0} &= \int (V_2 - V_1)(u_1^2 - u_0^2)ds \\ &= \int (V_2 - V_1)\psi'' ds \\ &= \int (V_2 - V_1)''\psi \geq 0\end{aligned}$$

and the inequality is strict unless $V_2 - V_1$ is linear, so the same inequality holds for $\tilde{\Gamma}$.

It follows that if V is convex, then $\Gamma_V \geq \tilde{\Gamma}_V \geq \tilde{\Gamma}_0$. It remains to prove that $\tilde{\Gamma}_0 = \Gamma_0 = \frac{3\pi^2}{D^2}$. That is, the fundamental gap problem for arbitrary convex potentials is reduced to proving that $\Gamma_{ax} \geq \Gamma_0$ for every a, which amounts to some analysis of Airy functions. Lavine proves this using an integral identity which implies that the only critical point of the function $a \mapsto \Gamma_{ax}$ occurs at $a = 0$.

It is interesting to note that the monotonicity of the linear family gap with respect to convexity of the potential also holds for other boundary conditions such as Neumann and Robin conditions. Lavine also proved the last step, that the linear family gap of the zero potential equals the gap of the zero potential, for the Neumann case, and a modification of the argument also gives the corresponding result for Robin problems with positive Robin constant. Therefore, in the one-dimensional case, the natural analogue of the fundamental gap conjecture holds in all of these cases.

3.3 Converting to a Neumann problem

In higher dimensions, we must argue quite differently. Here we will show how the techniques from the last two lectures, controlling the modulus of continuity for heat equations, can be applied here to control the fundamental gap. In those sections, sharp results were obtained for Neumann boundary conditions, and the starting point of our approach is an idea from [32] which converts the Dirichlet boundary condition to a Neumann condition, at the expense of introducing a 'drift' term into the equation:

Suppose that we have two solutions of the Dirichlet heat equation (with potential) on a smoothly bounded domain in $\mathbb{R}^n$:

$$\begin{aligned}\partial_t u_i &= \Delta u_i - V u_i \quad \text{on } \Omega\times\mathbb{R}_+;\\ u_i &= 0 \quad \text{on } \partial\Omega\times\mathbb{R}_+,\end{aligned}$$

for $i=0,1$, where u_0 is positive. Then define $v=\frac{u_1}{u_0}$. Then we have the equation

$$\begin{aligned}\partial_t v &= \Delta v + 2D\log u_0\cdot Dv \quad \text{on } \Omega\times\mathbb{R}_+\\ D_\nu v &= 0 \quad \text{on } \partial\Omega\times\mathbb{R}_+.\end{aligned} \tag{15}$$

To verify the boundary condition, fix any $x\in\partial\Omega$, and choose an orthonormal basis $\{e_k\}$ for $\mathbb{R}^n$ with e_n equal to the inward unit normal to $\partial\Omega$ at x. Since $u_i(x,t)=0$ for all t by the Dirichlet condition, we have $0=\partial_t u_i(x,t)=\Delta u_i(x,t)-V(x)u_i(x,t)=\Delta u_i(x,t)$. Also, differentiating along a geodesic $\gamma_k(s)=x+se_k+\frac{1}{2}s^2h(e_k,e_k)e_n+o(s^2)$ for $k=1,\cdots,n-1$, we have $0=u_i(\gamma_k(s))$ for all s, and so $0=\frac{d}{ds}u_i(\gamma_k(s))\big|_{s=0}=Du_i(\gamma')=D_ku_i$, and $0=\frac{d^2}{ds^2}u_i(\gamma_k(s))\big|_{s=0}=D_kD_ku_i(x)+h_{kk}D_nu_i(x)$. Taking the trace gives $\sum_{k=1}^{n-1}D_kD_ku_i(x)=-H(x)D_nu_i(x)$, and therefore $D_nD_nu_i(x)=\Delta u_i(x)-\sum_{k=1}^{n-1}D_kD_ku_i(x)=HD_nu_i(x)$, where $H(x)$ is the mean curvature of the boundary $\partial\Omega$ at x. This gives

$$u_i(x+se_n)=sD_nu_i(x)+\frac{1}{2}s^2D_nD_nu_i(x)+o(s^2)=D_nu_i(x)(s+\frac{1}{2}Hs^2)+o(s^2).$$

From this we get

$$v(x+se_n)=\frac{D_nu_1(x)}{D_nu_0(x)}+o(s)$$

as $s\to 0$, and therefore $D_\nu v(x)=0$ (note that $D_nu_0(x)>0$ by the Hopf Lemma).

Thus the function v satisfies a Neumann heat equation with drift. What is more, there is a special solution which we can produce by choosing $u_i(x,t)=u_i(x)\mathrm{e}^{-\lambda_i t}$, which gives

$$v(x,t)=\frac{u_1(x)}{u_0(x)}\mathrm{e}^{-(\lambda_1-\lambda_0)t},$$

so that we see the fundamental gap appearing as the exponential decay rate. If we can show that this exponential rate is sufficiently fast (as we did for the Neumann heat equation in the first lecture) then we will have a lower bound for the fundamental gap.

Let's try to do this: Suppose we have a solution of the heat equation with drift (15), and set $Z(x,y,t)=v(y,t)-v(x,t)-2\varphi\left(\frac{|y-x|}{2},t\right)$, in direct analogy to our method for the usual heat equation. We proceed exactly as before, considering points (x_0,y_0,t_0) where Z increases to zero for the first time. The case where x_0 or y_0 is in $\partial\Omega$ can be handled exactly as before, and in the case where both are interior points of Ω, and we choose an orthonormal basis for $\mathbb{R}^n$ with $e_n=\frac{y_0-x_0}{|y_0-x_0|}$, we derive the inequalities

$$D_nD_nv\big|_{(y_0,t_0)}-D_nD_nv\big|_{(x_0,t_0)}\le 2\varphi''$$

and

$$D_iD_iv\big|_{(y_0,t_0)} - D_iD_ni\big|_{(x_0,t_0)} \leq 0$$

for $i = 1,\cdots,n-1$. Adding these gives $\Delta v\big|_{(y_0,t_0)} - \Delta v\big|_{(x_0,t_0)} \leq 2\varphi''$. Also, the first derivative conditions at the maximum give $\nabla v(y_0,t_0) = \nabla v(x_0,t_0) = \varphi' e_n$. Finally, the time derivative gives

$$\begin{aligned}
0 &\leq \partial_t v\big|_{(y_0,t_0)} - \partial_t v\big|_{(x_0,t_0)} - 2\varphi_t \\
&\leq \Delta v\big|_{(y_0,t_0)} + 2D\log u_0(y_0)\cdot Dv(y_0,t_0) - \Delta v\big|_{(x_0,t_0)} - 2D\log u_0(x_0)\cdot Dv(x_0,t_0) - 2\varphi_t \\
&\leq 2\varphi'' + 2\varphi'(D\log u_0(y_0) - D\log u_0(x_0))\cdot\frac{y_0-x_0}{|y_0-x_0|} - 2\varphi_t. \qquad (16)
\end{aligned}$$

This is exactly the same as we have before, except we have replaced u by v and we have the extra term $2(D\log u_0(y_0) - D\log u_0(x_0))\cdot\frac{y_0-x_0}{|y_0-x_0|}$. To proceed, we need to estimate this term from above, and we can do this using the following famous result:

Theorem 3 (Brascamp-Lieb [13]). *If V is convex, and Ω is convex, then $\log u_0$ is concave, where u_0 is the Dirichlet ground state.*

The theorem implies that $2(D\log u_0(y_0) - D\log u_0(x_0))\cdot\frac{y_0-x_0}{|y_0-x_0|} \leq 0$, so we can discard this term and proceed as before, choosing $\varphi(x,t) = Ce^{-\frac{\pi^2}{D^2}t}\sin\left(\frac{\pi x}{D}\right)$. Applying this estimate to the special solution constructed above, we derive the inequality

$$\lambda_1 - \lambda_0 \geq \frac{\pi^2}{D^2}.$$

This is a nice result, proved previously using gradient estimates by Yu and Zhong [35], who improved an earlier result in [32] using essentially the same ideas that Zhong and Yang [36] used for the Neumann problem. However this bound on the gap is not sharp: The term we have thrown away using the result of Brascamp and Lieb is in fact strictly negative, so we are throwing away too much. In order to get a sharp result, we must derive a sharp improvement on the Brascamp-Lieb result.

3.4 Sharp log-concavity

How bad was our mistake in throwing away the negative term $2(D\log u_0(y_0) - D\log u_0(x_0))\cdot\frac{y_0-x_0}{|y_0-x_0|}$? To answer this, we should think about the notional case of equality, which is an interval with zero potential. In this case, we have $u_0(x) = Ce^{-\frac{\pi^2}{D^2}t}\cos\left(\frac{\pi x}{D}\right)$, and so $(\log u_0)'(x) = -\frac{\pi}{D}\tan\left(\frac{\pi x}{D}\right)$. It follows that

$$(D\log u_0(y_0) - D\log u_0(x_0))\cdot\frac{y_0-x_0}{|y_0-x_0|} = (\log u_0)'(y) - (\log u_0)'(x) \leq -\frac{2\pi}{D}\tan\left(\frac{\pi x}{2D}\right),$$

with equality when $y = -x$. This motivates the following sharp log-concavity estimate:

Proposition 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, and V a convex potential on Ω. Let u_0 be the ground state of the Schrödinger operator $\Delta - V$ on Ω with Dirichlet boundary condition. Let $\bar{u}_0$ be the ground state of the Dirichlet Laplacian on the interval $[-D/2, D/2]$ where $D = \mathrm{diam}(\Omega)$. Then*

$$(D\log u_0(y) - D\log u_0(x)) \cdot \frac{y-x}{|y-x|} \leq 2(\log \bar{u}_0)' \left(\frac{|y-x|}{2}\right) \tag{17}$$

for all $y \neq x$ in Ω.

Explicitly, this says that $(D\log u_0(y) - D\log u_0(x)) \cdot \frac{y-x}{|y-x|} \leq -\frac{2\pi}{D}\tan\left(\frac{\pi x}{2D}\right)$, as the calculation above shows.

To prove the proposition, we follow the model of the modulus of continuity bound: Given a positive solution u of the Dirichlet heat equation with potential V, we define a function of two points and time by

$$Q(x,y,t) = (D\log u(y,t) - D\log u(x,t)) \cdot \frac{y-x}{|y-x|} - 2\psi\left(\frac{|y-x|}{2}, t\right),$$

where ψ will be chosen to make $Q < 0$ at $t = 0$ and to evolve according to an equation that we will derive below. In order to compute the time derivative of Q we first compute the evolution equation for $w = \log u$:

$$\partial_t w = \frac{1}{u}(\Delta u - Vu) = \Delta\, w + |Dw|^2 - V.$$

Differentiating, we find

$$\partial_t w_i = \Delta w_i + 2w_k w_{ki} - V_i,$$

where we use subscripts for derivatives for convenience of notation. This gives

$$\begin{aligned}\partial_t Q &= (\Delta w_i(y) - \Delta w_i(x))\frac{y^i - x^i}{|y-x|} \\ &\quad + 2\left(w_k(y)w_{ki}(y) - w_k(x)w_{ki}(x)\right)\frac{y^i - x^i}{|y-x|} \\ &\quad - (V_i(y) - V_i(x))\frac{y^i - x^i}{|y-x|} - 2\psi_t\end{aligned}$$

As usual, we consider a point (x_0, y_0, t_0) where Q first increases to zero. We will ignore the case where x_0 or y_0 is on the boundary, noting that since u_0 approaches zero (and assuming the boundary of Ω is uniformly convex) one can show that Q approaches negative infinity in that case. So we assume that x_0 and y_0 are distinct points in Ω, and choose an orthonormal basis $\{e_i\}$ with $e_n = \frac{y_0 - x_0}{|y_0 - x_0|}$. Then we can derive first the following identities from the fact that (x_0, y_0) is a critical point of Q at time t_0: If $k < n$ then we have

$$0 = \frac{\partial}{\partial x^k} Q = -w_{kn}(x) - (w_k(y) - w_k(x));$$

while if $k = n$ then

$$0 = \frac{\partial}{\partial x^n} Q = -w_{nn}(x) + \psi'.$$

Similarly we have

$$0 = \frac{\partial}{\partial y^k} Q = w_{kn}(y) + (w_k(y) - w_k(x))$$

for $k < n$, and

$$0 = \frac{\partial}{\partial y^n} Q = w_{nn}(y) - \psi'.$$

We also compute the same components of the second derivative as we employed in the modulus of continuity argument: Moving the two points apart gives

$$0 \geq \frac{\partial^2}{\partial s^2} Q(x_0 - se_n, y_0 + se_n, t_0)\big|_{s=0} = w_{nnn}(y) - w_{nnn}(x) - 2\psi'',$$

while moving x and y in parallel in direction e_k with $k < n$ gives

$$0 \geq \frac{\partial^2}{\partial s^2} Q(x_0 + se_k, y_0 + se_k, t_0)\big|_{s=0} = w_{kkn}(y) - w_{kkn}(y).$$

On the other hand we have

$$\begin{aligned}
0 &\leq \frac{\partial}{\partial t} Q|_{(x_0,y_0,t_0)} \\
&= w_{nnn}(y) - w_{nnn}(x) + \sum_{k<n} (w_{kkn}(y) - w_{kkn}(x)) \\
&\quad + 2\sum_k (w_k(y) w_{kn}(y) - w_k(x) w_{kn}(x)) - (V_i(y) - V_i(x)) \frac{y^i - x^i}{|y-x|} - 2\psi_t \\
&\leq 2\psi'' + 2\psi'(w_n(y) - w_n(x)) - 2\sum_{k<n} (w_k(y) - w_k(x))^2 \\
&\leq 2\psi'' + 4\psi'\psi - 2\psi_t.
\end{aligned}$$

We deduce a contradiction, provided ψ is chosen so that $\psi_t > \psi'' + 2\psi\psi'$. In fact it is enough to produce a solution of $\psi_t \geq \psi'' + 2\psi\psi'$, and then add εe^{Ct} for suitably chosen C. To conclude, we show how to produce a solution of this equation for which $Q \leq 0$ for $t = 0$: We observe that if ϕ is a positive solution of the heat equation, then $\psi = (\log\phi)'$ satisfies the required equation. So to produce a suitable solution, we first choose $\psi(.,0)$ suitably large so that $Q \leq 0$ at $t = 0$, and then solve for ϕ by writing $\phi(s,0) = \exp\left(\int_0^s \psi(z,0)\,dz\right)$. Then we solve the heat equation with

Dirichlet boundary conditions, and set $\phi(s,t) = (\log\phi)'(s,t)$. This produces the required barrier. Furthermore, since ϕ is a positive solution of the Dirichlet heat equation on the interval $[-D/2, D/2]$, we know that $\phi \sim C\mathrm{e}^{-\frac{\pi^2}{D^2}t}\cos\left(\frac{\pi s}{D}\right)$, and therefore $\psi \sim -\frac{\pi}{D}\tan\left(\frac{\pi s}{D}\right)$ as $t \to \infty$.

Finally, applying the estimate in the particular case $u = \mathrm{e}^{-\lambda_0 t}u_0(x)$ give the required log-concavity estimate for u_0 as $t \to \infty$.

3.5 The sharp lower bound on the fundamental gap

Now we can return to our modulus of continuity argument, and fix the error we made to get a sharp result: Applying the sharp-log-concavity estimate in the inequality (16), we find that

$$0 \le 2\varphi''(s) - \frac{4\pi}{D}\tan\left(\frac{\pi s}{D}\right)\varphi' - 2\varphi_t.$$

Now we can choose the barrier $\varphi(s,t) = C\mathrm{e}^{-\left(\frac{3\pi^2}{D^2}+\varepsilon\right)t}\sin\left(\frac{\pi s}{D}\right)$ to make the right-hand side negative, producing a contradiction. This gives the estimate $\lambda_1 - \lambda_0 \ge \frac{3\pi^2}{D^2}$, which is the sharp lower bound on the fundamental gap.

3.6 Sharp lower bound in terms of modulus of convexity of the potential

We remark that the proof we outlined also gives sharp results if we know that the potential is 'more convex' than some reference potential, in a certain sense: Suppose we have a function $\tilde{V}$ which is even, and such that

$$(DV(y) - DV(x)) \cdot \frac{y-x}{|y-x|} \ge 2\tilde{V}'\left(\frac{|y-x|}{2}\right). \tag{18}$$

The case where V is convex corresponds to $\tilde{V} = 0$. Then the argument for the concavity of $\log u_0$ has a single extra term, so that the function ψ must satisfy

$$\psi_t = 2\psi'' + 4\psi\psi' - 2\tilde{V}'.$$

We can produce such a barrier ψ by solving the one-dimensional heat equation $\phi_t = \phi'' - \tilde{V}\phi$ with Dirichlet boundary condition, and setting $\psi = (\log\phi)'$. This in turn feeds into the modulus of continuity estimate to give a sharp lower bound on the gap compared to that of the one-dimensional Laplacian with potential $\tilde{V}$:

Theorem 4. *Suppose that V satisfies* (18) *on a convex domain $\Omega \subset \mathbb{R}^n$. Then the ground state u_0 satisfies the sharp log-concavity estimate* (17)*, where $\bar{u}_0$ is the* *ground state of the one-dimensional Dirichlet Laplacian with potential $\tilde{V}$, and the fundamental gap $\Gamma_{V,\Omega} = \lambda_1 - \lambda_0$ of $\Delta - V$ on Ω satisfies*

$$\Gamma_{V,\Omega} \geq \Gamma_{[-D/2,D/2],\tilde{V}}.$$

For example, this implies that for potentials with Hessian bounded below by a multiple κ of the identity, the gap is at least as large as that for a quadratic potential $\tilde{V} = \frac{1}{2}|x|^2$ on the interval. It should be possible to prove this in the one-dimensional case using the methods of Lavine, but to do this we would need to know that the gap on the interval for the potential $\frac{1}{2}\kappa x^2 + ax$ is minimized for $a = 0$, and this seems quite non-trivial. In particular, it would be nice to be able to do this for other boundary conditions (Neumann or Robin) in the one-dimensional case.

It is also interesting to ask whether such sharp estimates on the fundamental gap hold in other geometric situations, such as convex subsets of suitable Riemannian manifolds. In the case of convex subsets of the sphere, substantial progress has been made [31, 21], given a lower bound on the fundamental gap which is sharp in the sense that it at least as large as the lower bound in the Euclidean case, although the result so far falls short of a sharp lower bound for given diameter.

We remark that no sharp results are known about the fundamental gap for Neumann or Robin boundary conditions in higher dimensions if there is a non-trivial potential: Our argument breaks down in the proof of log-concavity of the ground state, and indeed the ground state is in general not log-concave even with zero potential [6].

Lecture 4: Gradient estimates for elliptic equations

In this final lecture I want to consider two-point maximum principles to control solutions of more general elliptic and parabolic equations, including lower order terms. I will concentrate mostly on the elliptic situation, though our methods also work for a class of parabolic equations. Most of the work I will talk about today was carried out together with Changwei Xiong, and further details and extensions of what I talk about here can be found in [9].

4.1 Gradient estimates and P-functions

In order to motivate the results, I will describe a class of well-known gradient estimates for elliptic equations, sometimes called '*P*-function estimates', which were developed by many authors including Laurence Payne (with various co-authors) and presented in detail in the monograph by Rene Sperb [33]. For example [33]*Corollary 5.2 implies the following result:

Theorem 5. *Let Ω be a bounded convex domain in $\mathbb{R}^n$, and u a (smooth) solution of $\Delta u + f(u) = 0$ on Ω with Neumann boundary condition $D_\nu u = 0$. Then $|Du|^2 + 2F(u) \leq \sup\{F(u(z)) : z \in \Omega\}$, where $F' = f$.*

An extension of this by Luciano Modica [27] to bounded entire solutions has been important in work on the de Giorgi conjecture:

Theorem 6 (Modica [27]*Theorem II). *Let $F \in C^2(\mathbb{R})$ be a non-negative function and $u \in C^3(\mathbb{R}^n)$ a bounded entire solution of the equation $\Delta u = f(u)$, where $f = F'$. Then $|Du|^2(x) \leq 2F(u(x))$ for every $x \in \mathbb{R}^n$.*

The P-function method can also be applied to quasilinear equations of divergence form, including the p-Laplacians:

Theorem 7 ([33]*Theorem 7.2,[29]). *If u is a (sufficiently smooth) solution of $D_i(v(|Du|^2 D_i u) + w(|Du|^2) f(u) = 0$ with Neumann boundary condition on a convex domain Ω, where the ellipticity condition $v(q) + 2qv'(q) > 0$ is satisfied, then*

$$P(x) := \int_0^{|Du(x)|^2} \frac{v(s) + 2sv'(s)}{w(s)}\, ds + 2F(u(x)) \leq 2 \sup_{z \in \Omega} F(u(z))$$

where $F(y) = \int_0^y f(s)\, ds$.

An extension to non-compact situations was proved by Caffarelli, Garofalo and Segala [15], and there are also extensions to manifolds [19], and to anisotropic equations [20].

These results have an interpretation in common: In the one-dimensional case, the PDE becomes an ODE, and the quantity $P(x)$ is constant for solutions of this ODE. In particular for any value $P > \sup_{z \in \Omega} F(z)$, we can integrate to find a solution φ of the one-dimensional equation which is increasing and has range containing the range of u.

The results then say precisely that the gradient $|Du(x)|$ is bounded by $\varphi'(z)$, where z is chosen so that $\varphi(z) = u(x)$. That is, the gradient of u is bounded by the gradient of φ at the point with the same height.

In the following we will prove a result of this kind for a much more general class of equations.

4.2 The two-point estimate

We prove the following result:

Theorem 8. *Suppose Ω is a convex domain in $\mathbb{R}^n$, and let u be a (smooth) solution of the boundary value problem*

$$\begin{aligned} a^{ij} D_i D_j u + q(u, |Du|) &= 0 \quad \text{on } \Omega; \\ D_\nu u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{19}$$

where $a^{ij} = a(u,|Du|)\frac{u_iu_j}{|Du|^2} + b(u,|Du|)\left(\delta^{ij} - \frac{u_iu_j}{|Du|^2}\right)$ *for some positive functions a and b. Suppose that* $u(x) \in [m,M]$ *for all* $x \in \Omega$*, and suppose that* $\varphi : [a,b] \to [m,M]$ *is strictly increasing solution of* $a(\varphi,\varphi')\varphi'' + q(\varphi,\varphi') = 0$ *with* $\varphi'(z) \geq \delta > 0$ *for all* $z \in [a,b]$*. Let* $\psi : [m,M] \to [a,b]$ *be the inverse function of* φ*. Then*

$$\psi(u(y)) - \psi(u(x)) \leq |y-x|$$

for all $x,y \in \Omega$*.*

This two-point estimate has an immediate corollary obtained by allowing y and x to approach each other:

Corollary 3. *Under the assumptions above,* $|Du(x)| \leq \varphi' \circ \psi(u(x))$ *for all* $x \in \Omega$*.*

That is, the gradient of u is bounded by that of φ' at the point with the same height. This includes all of the gradient estimates mentioned above, but does not require any special form of the equation.

The proof proceeds as follows: We let $Z(x,y) = \psi(u(y)) - \psi(u(x)) - |y-x|$, and suppose that a positive supremum of Z occurs at some point $(x_0,y_0) \in \bar{\Omega} \times \bar{\Omega}$. Since $Z = 0$ when $x = y$ we know that $y_0 \neq x_0$.

We can rule out the possibility that x_0 or y_0 is on the boundary of Ω: Suppose $x_0 \in \partial\Omega$. Then we compute $\frac{d}{ds}Z(x_0 - s\nu(x_0), y_0)\big|_{s=0} = -\psi' D_\nu u(x_0) - \left\langle \frac{y_0-x_0}{|y_0-x_0|}, \nu(x_0)\right\rangle > 0$, where we used the Neumann boundary condition and the convexity of the domain to get the last inequality. This contradicts the assumption that (x_0,y_0) is a maximum value, so this case cannot occur. The case where $y_0 \in \partial\Omega$ is similar.

This leaves us with the conclusion that x_0 and y_0 are distinct interior points of Ω. We choose an orthonormal basis $\{e_i\}$ such that $e_n = \frac{y_0-x_0}{|y_0-x_0|}$. Then we compute variations of Z in various directions: The first derivatives give

$$\begin{aligned} 0 &= \frac{d}{ds}Z(x_0+se, y_0+s\tilde{e})\big|_{s=0} \\ &= \left\langle e, -\psi'(u(x_0))Du(x_0) + \frac{y_0-x_0}{|y_0-x_0|}\right\rangle + \left\langle \tilde{e}, \psi'(u(y_0)) - \frac{y_0-x_0}{|y_0-x_0|}\right\rangle. \end{aligned}$$

Since $\psi \circ \varphi$ is the identity map, we have $\psi'(u(y)) = \frac{1}{\varphi'(z_y)}$, where $z_y = \psi(u(y))$, so we conclude that $Du(y_0) = \varphi'(z_{y_0})e_n$ and $Du(x_0) = \varphi'(z_{x_0})e_n$.

Next we compute useful parts of the second derivatives: Moving x_0 towards y_0 we have

$$0 \geq \frac{d^2}{ds^2}Z(x_0+se_n, y_0)\big|_{s=0} = -\psi'(u(x_0))D_nD_nu(x_0) - \psi''(u(x_0))(D_nu(x_0))^2;$$

Similarly, moving y_0 towards x_0 gives

$$0 \geq \frac{d^2}{ds^2}Z(x_0, y_0-se_n)\big|_{s=0} = \psi'(u(y_0))D_nD_nu(y_0) + \psi''(u(y_0))(D_nu(y_0))^2.$$

Since $\psi'(\varphi(z))\varphi'(z)=1$, differentiating further gives

$$\psi''(\varphi(z))\varphi'(z)^2+\psi'(\varphi(z))\varphi''(z)=0,$$

so that

$$\psi''(u(x))=-\frac{\varphi''(z_x)}{\varphi'(z_x)^3}.$$

Substituting this above gives

$$D_nD_nu(y_0)\le\varphi''(z_{y_0})$$

and

$$D_nD_nu(x_0)\ge\varphi''(z_{x_0}).$$

Next we move x and y in parallel in a direction orthogonal to the line between them:

$$0\ge\frac{d^2}{ds^2}Z(x_0+se_i,y_0+se_i)\big|_{s=0}=\frac{D_iD_iu(y_0)}{\varphi'(z_{y_0})}-\frac{D_iD_iu(x_0)}{\varphi'(z_{x_0})}. \tag{20}$$

Now we can use the equation:

$$\begin{aligned}0&=a^{ij}(Du(y_0))D_iD_ju(y_0)+q(u(y_0),|Du(y_0)|)\\&=a(\varphi(z_{y_0}),\varphi'(z_{y_0}))D_nD_nu(y_0)+\sum_{i<n}b(\varphi(z_{y_0}),\varphi'(z_{y_0}))D_iD_iu(y_0)+q(\varphi(z_{y_0}),\varphi'(z_{y_0})),\end{aligned}$$

and similarly at x_0. In order to combine these in a way which allows us to use the inequality (20) we write

$$\begin{aligned}0&=\frac{1}{\varphi' b(\varphi,\varphi')}\Big|_{z_{y_0}}\left(a^{ij}D_iD_ju|_{y_0}+q(\varphi,\varphi')|_{z_{y_0}}\right)\\&\quad-\frac{1}{\varphi' b(\varphi,\varphi')}\Big|_{z_{x_0}}\left(a^{ij}D_iD_ju|_{x_0}+q(\varphi,\varphi')|_{z_{x_0}}\right)\\&=\frac{a(\varphi,\varphi')}{\varphi' b(\varphi,\varphi')}\Big|_{z_{y_0}}D_nD_nu(y_0)+\frac{\sum_{i<n}D_iD_iu(y_0)}{\varphi'(z_{y_0})}+\frac{q(\varphi,\varphi')}{\varphi' b(\varphi,\varphi')}\Big|_{z_y}\\&\quad-\frac{a(\varphi,\varphi')}{\varphi' b(\varphi,\varphi')}\Big|_{z_{x_0}}D_nD_nu(x_0)-\frac{\sum_{i<n}D_iD_iu(x_0)}{\varphi'(z_{x_0})}-\frac{q(\varphi,\varphi')}{\varphi' b(\varphi,\varphi')}\Big|_{z_{x_0}}\\&\le\frac{1}{\varphi' b(\varphi,\varphi')}\left(a(\varphi,\varphi')\varphi''+q(\varphi,\varphi')\right)\Big|_{z_{x_0}}^{z_{y_0}}\end{aligned}$$

Now we observe that if φ is a solution of the one-dimensional equation $a\varphi''+q=0$, then the right-hand side vanishes, which does not tell us anything. However, if we start with such a solution (as in the hypotheses of the theorem) then since φ' is bounded away from zero, we can approximate φ by a sequence of increasing functions φ_ε, with range still containing $[m,M]$, and which satisfy

$$a(\varphi_\varepsilon,\varphi_\varepsilon')\varphi_\varepsilon''+q(\varphi_\varepsilon,\varphi_\varepsilon')=-\varepsilon x\varphi_\varepsilon' b(\varphi_\varepsilon,\varphi_\varepsilon').$$

For these choices we conclude that the right-hand side is strictly negative, a contradiction, so we conclude that the estimate of the Theorem holds for φ replaced by φ_ε. Finally, allowing ε to approach zero gives the desired result.

There are many generalisations of this result: The result adapts easily to Dirichlet boundary conditions; It applies with minor modifications on Riemannian manifolds with non-negative Ricci curvature; it also has versions that apply in 'anisotropic' settings such as Finsler manifolds; since the argument does not differentiate the equation, it applies easily to viscosity solutions as well as classical solutions; and because of this, it also applies in non-compact settings, since we can translate (or take a limit of Riemannian manifolds) to make the maximum of Z occur (in a suitable limit) within bounded distance of the origin. For details, see [9].

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