A Note on Liouville type results for a fractional obstacle problem

Jérôme Coville

Abstract This note is a synthesis of my thoughts on some questions that have emerged during the MATRIX event "Recent Trends on Nonlinear PDEs of Elliptic and Parabolic Type" concerning the qualitative properties of solutions to some non local reaction-diffusion equations of the form

$$\mathscr{L}[u](x) + f(u(x)) = 0, \quad \text{for} \quad x \in \overline{\mathbb{R}^n \setminus K},$$

where $K \subset \mathbb{R}^N$ is a bounded smooth compact "obstacle", $\mathscr{L}$ is non local operator and f is a bistable nonlinearity. When K is convex and the nonlocal operator $\mathscr{L}$ is a continuous operator of convolution type then some Liouville-type results for solutions satisfying some asymptotic limiting conditions at infinity have been recently established by Brasseur, Coville, Hamel and Valdinoci [4]. Here, we show that for a bounded smooth convex obstacle K, similar Liouville type results hold true when the operator $\mathscr{L}$ is the regional s-fractional Laplacian.

1 Introduction

A classical topic in applied analysis consists in the study of "diffusive processes" in complex media e.g. media containing obstacles. Roughly speaking, this corresponds to study dispersal processes that follow a random motion in an environment that possess an unreachable region. At the macroscopic level, this problem can be translated into a reaction - diffusion equation that is defined outside a set K, which acts as an impenetrable obstacle.

One of the cornerstones in the study of these processes lies in suitable rigidity results of Liouville-type, which allow the classification of stationary solutions, at least under some geometric assumptions on the obstacle K.

Jérôme Coville
BioSP, INRA, 84914, Avignon, France e-mail: jerome.coville@inra.fr

In this note, we investigate this question for other type of random motion than those described by Brownian motions and provide new Liouville-type result (whose precise statements will be given in Section 2) on the corresponding semi-linear equation .

Concretely, we will suppose that the random motion is modelled by a Lévy flight which at the macroscopic level, leads to consider an integral operator with a singular positive kernel. For such type of processes we will show that the solutions to the stationary equation with a prescribed behaviour at infinity are necessarily constant, at least when the obstacle is convex.

We now provide the detailed mathematical description of the problem that we consider.

1.1 A fractional obstacle problem

Throughout this note, K denotes a smooth compact set of $\mathbb{R}^n$ with $n \geq 2$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $\mathscr{L}$ denotes the regional fractional nonlocal operator ([7, 8]) defined for $s \in (0,1)$ by

$$\mathscr{L}[u](x) := C_{n,s} \lim_{\varepsilon \to 0} \left(\int_{|x-y|>\varepsilon, y \in \mathbb{R}^n \setminus K} \frac{\big(u(y) - u(x)\big)}{|x-y|^{n+2s}} \, \mathrm{d}y \right).$$

We are interested in qualitative properties of smooth bounded solutions to the following non local semilinear equation

$$\mathscr{L}[u](x) + f(u(x)) = 0 \quad \text{ for all } x \in \overline{\mathbb{R}^n \setminus K},$$

where f is a C^1 “bistable” non-linearity and when necessary with the Neumann boundary condition below

$$\nabla u(x) \cdot \nu(x) = 0 \qquad \text{ for all } x \in \partial K,$$

where $\nu(x)$ denotes the outer normal derivative of the set K. The precise assumptions on K, u, f and $\mathscr{L}$ will be given later on. Typically, this homogeneous Neumann boundary condition is required to define properly the regional fractional Laplacian $\mathscr{L}$ on the boundary of the obstacle when $s \in \left(\frac{1}{2}, 1\right)$.

This problem may be thought of as a fractional version of the following problem

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \mathbb{R}^n \setminus K, \\ \qquad \nabla u \cdot \nu = 0 \text{ on } \partial K. \end{cases} \tag{1}$$

For problem (1) with the local diffusion operator Δu, it was shown in [3] that there exist a time-global classical solution $u(t,x)$ to the parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) \text{ in } \mathbb{R}\times\overline{\mathbb{R}^n\setminus K}, \\ \nabla u\cdot\nu = 0 \qquad \text{on } \mathbb{R}\times\partial K \end{cases} \tag{2}$$

satisfying $0 < u(t,x) < 1$ for all $(t,x)\in\mathbb{R}\times\overline{\mathbb{R}^n\setminus K}$, and a classical solution $u_\infty(x)$ to the elliptic problem

$$\begin{cases} \Delta u_\infty + f(u_\infty) = 0 \text{ in } \overline{\mathbb{R}^n\setminus K}, \\ \nabla u_\infty\cdot\nu = 0 \text{ on } \partial K, \\ 0\le u_\infty\le 1 \text{ in } \overline{\mathbb{R}^n\setminus K}, \\ u_\infty(x)\to 1 \text{ as } |x|\to+\infty. \end{cases} \tag{3}$$

The function u_∞ is a stationary solution of (2) and it is actually obtained as the large time limit of $u(t,x)$, in the sense that $u(t,x)\to u_\infty(x)$ as $t\to+\infty$ locally uniformly in $x\in\overline{\mathbb{R}^n\setminus K}$. Under some geometric conditions on K (e.g. if K is starshaped or directionally convex, see [3] for precise assumptions) it is shown in [3, Theorems 6.1 and 6.4] that solutions to (3) are actually identically equal to 1 in the whole set $\overline{\mathbb{R}^n\setminus K}$. This Liouville property shows that the solutions $u(t,x)$ of (2) constructed in [3] then satisfy

$$u(t,x) \underset{t\to+\infty}{\longrightarrow} 1 \qquad \text{locally uniformly in } x\in\overline{\mathbb{R}^n\setminus K}. \tag{4}$$

From an ecological point of view, such a results can be interpreted as follows. Let us consider that $u(t,x)$ represents the density of a population that moves according to a Brownian motion in a environment consisting of the whole space $\mathbb{R}^n$ with a compact obstacle K and that the demography of this population can be described by the nonlinear function f. Then, the equation (2) can be understood as the evolution of this population int the region $\mathbb{R}^n\setminus K$. In this context, (4) means that, at large time, the population tends to occupy the whole space.

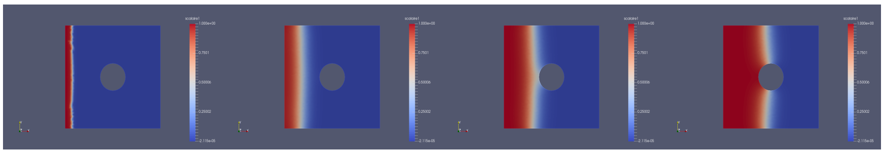
Assuming now that the random movement of the individuals follows, say, a reflected symmetric α-stable Lévy process, then the resulting reaction-diffusion equation will be

$$\frac{\partial u}{\partial t}(t,x) = \mathscr{L}[u](t,x) + f(u(t,x)) \quad \text{for all} \quad t>0, x\in\mathbb{R}^n\setminus K \tag{5}$$
$$+ \text{ A Neumann type boundary condition when necessary}$$

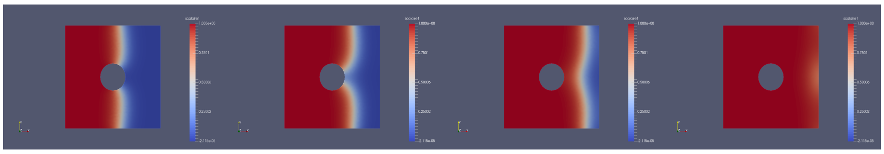
where $\mathscr{L}$ is the regional fractional Laplacian defined above.

The numerical simulations below (see Figure 1) obtained for a non singular version of the fractional operator $\mathscr{L}$ namely when we replace the singular measure $\frac{1}{|z|^{n+2s}}$ by $\frac{1}{\delta+|z|^{n+2s}}$ for $\delta << 1$, suggest that the long time behaviour of solution of positive equation (5) should be identical as those observed in the classical reaction-diffusion equation.

In this note, we deal with qualitative properties of the stationary solutions of equation (5), together with some asymptotic limiting conditions at infinity similar



(a) $t = 0$ (b) $t = 40$ (c) $t = 80$ (d) $t = 120$



(e) $t = 160$ (f) $t = 200$ (g) $t = 240$ (h) $t = 280$

Fig. 1: Simulation of the singular non local evolution problem (5) where the singular Levy kernel has been replaced by a non singular measure $\frac{1}{\delta+|z|^{n+2s}}$ with $\delta = 0.01$, the bistable non-lineartity f is a cubic non-linearity $f(s) = s(s-0.1)(1-s)$ and the initial condition is of Heaviside type. We can see that the density $u(t,x)$ tends to 1 on the all space and the influence of the obstacle on the shape of the transition

to those appearing in (3). Namely, we will be mainly concerned with solutions of

$$\begin{cases} \mathscr{L}[u] + f(u) = 0 \text{ in } \mathbb{R}^n \setminus K, \\ \qquad 0 \le u \le 1 \text{ in } \mathbb{R}^n \setminus K, \\ \qquad u(x) \to 1 \text{ as } |x| \to +\infty. \end{cases} \tag{6}$$

with, when necessary, the additional homogeneous Neumann boundary condition

$$\nabla u \cdot \nu = 0 \qquad \text{on} \qquad \partial K. \tag{7}$$

1.2 General assumptions

Let us now state precisely the assumptions we will use. Along this note, we suppose that the domain K is a smooth (at least $C^{0,1}$) bounded compact domain of $\mathbb{R}^n$, and that f is a smooth bistable non-linearity, that is f will always satisfies

$$f \in C^1([0,1]), \;\; f(0) = 0 = f(1) \ge 0, \;\; f'(1) < 0, \tag{8}$$

$$\begin{cases} \exists \theta \in (0,1), \;\; f(0) = f(\theta) = f(1) = 0, \;\; f < 0 \text{ in } (0,\theta), \;\; f > 0 \text{ in } (\theta,1), \\ \displaystyle\int_0^1 f > 0, \;\; f'(0) < 0, \;\; f'(\theta) > 0. \end{cases} \tag{9}$$

Observe that the assumption on f implies that the associated potential is unbalanced which is a necessary condition to observe the propagation of a front with a positive speed [1, 9]. Thus it seems reasonable to assume such condition in our setting since we expect that the solution u of (6) reflect the outcome of the invasion of the population in the environment $\mathbb{R}^n \setminus K$.

2 Main results

For the local problem (3), the Liouville property obtained in [3] says that $u = 1$ in $\overline{\mathbb{R}^n \setminus K}$ under some geometric conditions on K, in particular when K is convex. A similar Liouville type property was recently obtained for continuous solutions of (6) when the singular kernel $\frac{1}{|z|^{n+2s}}$ is replaced by a non negative integrable kernel J, i.e. $J \in L^1(\mathbb{R})$, see [4]. More precisely, if J is assume to satisfy the assumptions below

$$\begin{cases} J \in L^1(\mathbb{R}^n) \text{ is a non-negative, radially symmetric kernel with unit mass,} \\ \text{there are } 0 \leq r_1 < r_2 \text{ such that } J(x) > 0 \text{ for a.e. } x \text{ with } r_1 < |x| < r_2, \end{cases} \tag{10}$$

and there exists a function $\phi \in C(\mathbb{R})$ satisfying

$$\begin{cases} J_1 * \phi - \phi + f(\phi) \geq 0 \text{ in } \mathbb{R}, \\ \phi \text{ is increasing in } \mathbb{R}, \;\; \phi(-\infty) = 0, \;\; \phi(+\infty) = 1, \end{cases} \tag{11}$$

where $J_1 \in L^1(\mathbb{R})$ is the non-negative even function with unit mass given for a.e. $x \in \mathbb{R}$ by

$$J_1(x) := \int_{\mathbb{R}^{n-1}} J(x, y_2, \cdots, y_n)\, \mathrm{d}y_2 \cdots \mathrm{d}y_n.$$

then in [4] the authors prove the following

Theorem 1 (Brasseur, Coville, Hamel,Valdinoci [4]). *Let* $K \subset \mathbb{R}^n$ *be a compact convex set. Assume that* f *satisfies* (8) *and* (9) *and* J *satisfies* (10) *and* (11) *and let* $u \in C(\overline{\mathbb{R}^n \setminus K}, [0,1])$ *be a function satisfying*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n \setminus K} J(x-y)(u(y) - u(x))\, dy + f(u(x)) \leq 0 \quad \textit{for } x \in \overline{\mathbb{R}^n \setminus K}, \\ u(x) \to 1 \quad \textit{as } |x| \to +\infty. \end{cases} \tag{12}$$

Then, $u = 1$ *in* $\overline{\mathbb{R}^n \setminus K}$.

Observe that the problems (12) and (6) only differ in their formulation by the singularity of the kernels used. In particular, the problem (12) can be reformulated in to the framework of problem (6) since for all $J \in L^1(\mathbb{R}^n)$ and for all $x \in \overline{\mathbb{R}^n \setminus K}$ and $u \in L^\infty(\mathbb{R}^n)$

$$\lim_{\varepsilon\to}\int_{\mathbb{R}^n\setminus K,|x-y|>\varepsilon} J(x-y)(u(y)-u(x))\,dy = \int_{\mathbb{R}^n\setminus K} J(x-y)(u(y)-u(x))\,dy.$$

Therefore, it is expected that (12) and (6) share some common properties. On of the goals of the present note is to extend the results known for (12) to the solutions of (6) and when possible to highlight the role of the singularity of the kernel in this context.

Our main results show that under the right regularity assumptions we can transpose the results of Theorem 1 to solutions to (6). Namely, we first prove that

Theorem 2. *Let $K\subset\mathbb{R}^n$ be a compact smooth convex set ($C^{0,1}$). Assume* (8), (9) *and $s\in(0,\frac{1}{2})$. Let $u\in C^{0,\beta}(\mathbb{R}^n\setminus K,[0,1])$ with $\beta>2s$ be a function satisfying*

$$\begin{cases} \mathscr{L}[u]+f(u)\le 0 & \text{in } \mathbb{R}^n\setminus K,\\ u(x)\to 1 & \text{as } |x|\to+\infty. \end{cases} \tag{13}$$

Then, $u=1$ in $\overline{\mathbb{R}^n\setminus K}$.

Our second result complete the picture, namely we show that

Theorem 3. *Let $K\subset\mathbb{R}^n$ be a compact smooth convex set ($C^{0,1}$). Assume* (8), (9) *and $s\in\left[\frac{1}{2},1\right)$. Let $u\in C^{1,\beta}(\mathbb{R}^n\setminus K,[0,1])$ with $\beta>2s-1$ be a function satisfying*

$$\begin{cases} \mathscr{L}[u]+f(u)\le 0 & \text{in } \mathbb{R}^n\setminus K,\\ \nabla u\cdot\nu=0 & \text{on } \partial K,\\ u(x)\to 1 & \text{as } |x|\to+\infty. \end{cases} \tag{14}$$

Then, $u=1$ in $\overline{\mathbb{R}^n\setminus K}$.

We can already see clearly the effect of the singularity of the kernel. Indeed, unlike the non local operators with integrable kernel the s-fractional Laplacian is well defined in $\overline{\mathbb{R}^n\setminus K}$ only for regular function, i.e. u should be at least $C^{0,\beta}$. In this singular setting, requiring that the super-solution u is solely continuous is not enough.

These results complete our knowledge on the validity of such type of Liouville property for a broad class of reaction diffusion equation. They show some universality of such type of property and prove that such rigidity type result can be viewed as an intrinsic property of the problem which can be related to a generic property of the equation rather than a special property of the diffusion process considered.

2.1 Further comments and strategy of proofs

Prior to proving these results, let us make some comments on our hypotheses and highlight some of the differences that arise when the singular measure $\frac{1}{|z|^{n+2s}}$ is replaced by an integrable kernel J.

First, let us observe that thanks to the regularising property of the regional fractional Laplacian $\mathscr{L}$, see [6, 7, 8] the continuity assumption made on u can be easily weakened when u is assumed to be a solution to (6) instead of a super-solution. Thus, in this situation, the result of Theorems 2 and 3 hold as well for bounded solution u that satisfies the equation (13) respectively (14) in the sense of viscosity solutions. Note that contrary to the regional fractional Laplacian $\mathscr{L}$, the nonlocal operator $\mathscr{M}[u] := \int_{\mathbb{R}^n\setminus K} J(x-y)(u(y)-u(x))\,dy$ has no regularising properties and as a consequence weakening the regularity assumption on the solution u is a hard task which, for the moment, can only been achieved by imposing further restrictions on the data f and J. Nevertheless, in this non regularising context, the regularity of the obstacle K is no more an issue and K can be any arbitrary convex domain. The regularisation effect on the solutions induced by the singularity is in fact the only main distinction between the problem (12) and the singular problems (13) and (14).

This distinction appears also clearly in the set of assumption needed for the existence of monotone travelling front with a positive speed, which is an essentiel key element of the proof of Theorem 1. In particular, as already mentioned at Section 1.2, assumptions (8) and (9) are actually necessary and sufficient for the existence of a travelling wave solution with positive speed c to the one dimensional fractional equation, i.e. a monotone solution to

$$c\,\partial_z\varphi = \partial_z^s\varphi + f(\varphi),$$

with a positive speed $c > 0$. Such assumptions are not any more sufficient in the context of (12), where there exists data f and J that satisfy (8)–(9) for which only discontinuous null speed fronts exist.

Let us also note that, similar condition are also necessary and suffisent for the existence of one dimentional travelling wave with positive speed for the local problem, namely solution of $c\varphi' = \varphi'' + f(\varphi)$ with positive speed (see e.g. [2]). This fact, then suggest a strong connexion between the regularity of the front and the minimal set of assumptions that are required to produced a front of positive speed.

Let us emphasize that the motivation behind these assumptions are that, by analogy with the local problem (2), we expect a solution to (6) to be the large time limit of an entire solution to the evolution problem (5) which behaves like $\varphi(x_1+ct)$ when $t\to-\infty$. For this interpretation to even make sense it is necessary to work in a setting where the function φ exists.

Let us now say a word on our strategy of proofs. The proofs are a rather straightforward adaptation of the arguments developed in [4] for the non local obstacle problem (12). The main idea is to compare by means of adequate sliding methods, a family of planar function of the type $\varphi(x\cdot e-r)$, where $e\in\partial B_1$, $r\in\mathbb{R}$ and $\varphi\in C^{1,1}(\mathbb{R})$ a given monotone function with a given super-solution u. To adapt such technique to our situation, we need first to verify that, as proved for (12) similar comparison principles in half-spaces hold true as well for the fractional equations (13) and (14).

The outline of this note will be as follows. In Section 3 we provide several comparison principles and recall some known results on the 1d travelling fronts for fractional bistable equation. Then we prove in Section 4, following the arguments developed in [4], we prove the Liouville property described in Theorems 2 and 3.

3 Some mathematical background

In this section, we start by collecting some comparison principles that are suitable for our purposes and to shortened the presentation we only fully state the necessary comparison principle for regional fractional Laplacian with $s \in \left[\frac{1}{2}, 1\right)$. Throughout this section, K is any compact subset of $\mathbb{R}^n$, f is any $C^1(\mathbb{R})$ function.

We start with a weak maximum principle

Lemma 1 (Weak maximum principle). *Assume that $s \in \left[\frac{1}{2}, 1\right)$ and*

$$f' \leq -c_1 \text{ in } [1-c_0, +\infty), \text{ for some } c_0 > 0,\ c_1 > 0. \tag{15}$$

Let $H \subset \mathbb{R}^n$ be an open affine half-space such that $K \subset\subset H^c = \mathbb{R}^n \setminus H$. Let $u, v \in L^\infty(\mathbb{R}^n \setminus K) \cap C^{1,\beta}\left(\overline{H}\right)$ for some $\beta > 1-2s$ be such that

$$\begin{cases} \mathscr{L}[u] + f(u) \leq 0 & \text{in } \overline{H}, \\ \mathscr{L}[v] + f(v) \geq 0 & \text{in } \overline{H}. \end{cases}$$

Assume also that

$$u \geq 1 - c_0 \quad \text{in } \overline{H}, \qquad \text{and} \qquad \limsup_{|x| \to +\infty} \left(v(x) - u(x)\right) \leq 0$$

and that

$$v \leq u \quad \text{a.e. in } H^c \setminus K.$$

Then, $v \leq u$ a.e. in $\mathbb{R}^n \setminus K$.

The next lemma is concerned with a strong maximum principle.

Lemma 2 (Strong maximum principle). *Assume that $s \in \left[\frac{1}{2}, 1\right)$ and let $H \subset \mathbb{R}^n$ be an open affine half-space such that $K \subset\subset H^c$. Let $u, v \in L^\infty(\mathbb{R}^n \setminus K) \cap C^{1,\beta}\left(\overline{H}\right)$ for some $\beta > 1-2s$ be such that* (1) *holds true. Assume also that*

$$v \leq u \quad \text{a.e. in } \mathbb{R}^n \setminus K$$

and that there exists $\bar{x} \in \overline{H}$ such that $v(\bar{x}) = u(\bar{x})$. Then,

$$v = u \quad \text{a.e. in } H.$$

These comparison principles are in essence identical to the one derived in [4] and as such we point the interested reader to [4] for a detailed proof of these results.

Remark 1. The above comparison principles have only been stated for regional fractional operators with exponent $s \in \left[\frac{1}{2}, 1\right)$. Identical weak and strong maximum principles can be formulated for the regional fractional Laplacian with exponent $s \in \left(0, \frac{1}{2}\right)$ as soon as we impose the adequate regularity to the functions u and v in order to properly define the regional fractional Laplacian of u and v. In such case, the above statement will holds true if instead of having $u, v \in L^\infty(\mathbb{R}^n \setminus K) \cap C^{1,\beta}\left(\overline{H}\right)$ we assume that $u, v \in L^\infty(\mathbb{R}^n \setminus K) \cap C^{0,\beta}\left(\overline{H}\right)$ with $\beta > 2s$.

Lastly, we recall some known result on the existence and properties of travelling fronts $\varphi(x \cdot e + ct)$, solution of the fractional evolution equation

$$\partial_t u(t,x) = \Delta^s u(t,x) + f(u(t,x)) \quad \text{for} \quad t \in \mathbb{R}, x \in \mathbb{R}^n$$

that is, solution of the following

$$-c\partial_z \varphi(z) + \partial_z^s \varphi(z) + f(\varphi(z)) = 0 \quad \text{for} \quad z \in \mathbb{R} \tag{16}$$

$$\lim_{z \to +\infty} \varphi(z) = 1, \qquad \lim_{z \to -\infty} \varphi(z) = 0 \tag{17}$$

where $\partial_z^s \varphi$ denotes the one dimentional $s-$fractional Laplacian. The existence, uniqueness and some asymptotic properties of such solution φ have been obtained in several context [1, 5, 9, 10]. The next statement is a summary of these results.

Theorem 4 (Fractional Travelling wave [1, 5, 9, 10]). *Assume f is a bistable function that satisfies* (8) *and* (9) *and let $s \in (0,1)$. Then there exists a unique $c \in \mathbb{R}$ and a monotone smooth (at least $C^{1,1}$) increasing function φ such that (c, φ) is a solution to* (16)–(17)*. Moreover, if f is unbalanced with $\int_0^1 f(s)\,ds > 0$, then $c > 0$.*

As a trivial consequence of the existence of a smooth front of positive speed, for any separating open affine half-space $H \subset \mathbb{R}^n$ such that $K \subset\subset H^c$, we can derive a family of function which will be a "sub-solution to the problem (6)" for all $x \in H$. More precisely, let H be an affine subspace of $\mathbb{R}^n$ such that $K \subset H^c$. By definition of the affine space, there exists a unit vector $e \in \partial B_1$ and $x_0 \in \mathbb{R}^n$ such that $H = x_0 + H_e$ with H_e an open-halfspace of direction e, i.e.

$$H_e := \{x \in \mathbb{R}^n | x \cdot e \geq 0\}.$$

For this direction e and for all real $r \in \mathbb{R}$, we can define the family of functions $\phi_{e,r}(x) := \varphi(x \cdot e - r)$ where φ is the smooth increasing profile obtained in Theorem 4. By construction, since φ is monotone increasing we have

$$\forall x \in H, \forall y \in K \quad \phi_{r,e}(y) - \phi_{r,e}(x) \leq 0. \tag{18}$$

In addition, we can check that for all $x \in H$, we have

$$\begin{aligned}\mathscr{L}[\phi_{r,e}](x) &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus K,|x-y|>\varepsilon}\frac{\phi_{r,e}(y)-\phi_{r,e}(x)}{|x-y|^{n+2s}}\,dy\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n,|x-y|>\varepsilon}\frac{\phi_{r,e}(y)-\phi_{r,e}(x)}{|x-y|^{n+2s}}\,dy-\int_{K,|x-y|>\varepsilon}\frac{\phi_{r,e}(y)-\phi_{r,e}(x)}{|x-y|^{n+2s}}\,dy\\ &= \Delta^s\phi_{r,e}(x)-\int_K\frac{\phi_{r,e}(y)-\phi_{r,e}(x)}{|x-y|^{n+2s}}\,dy.\end{aligned}$$

which combined with (18) enforces

$$\mathscr{L}[\phi_{r,e}](x)\geq\Delta^s\phi_{r,e}(x)\quad\text{for}\quad x\in H.$$

Hence, for all $x\in H$, we get

$$\begin{aligned}\mathscr{L}[\phi_{e,r}](x)+f(\phi_{e,r}) &\geq \Delta^s\phi_{e,r}(x)+f(\phi_{e,r})\\ &\geq \partial^s\varphi(x.e-r)+f(\varphi(x.e-r))=c\partial_z\varphi(x\cdot e-r)>0. \quad (19)\end{aligned}$$

4 The case of convex obstacles: proofs of the main Theorem

In this section, based on the arguments introduced in [4] we sketch the proof our main results (Theorems 2 and 3). The proofs of the Theorems 3 and 2 being identical, we only sketch the proof of Theorem 3.

But before we start our discussion,let us first start with the following simple observation.

Lemma 3. *Let $s\in\left[\frac{1}{2},1\right)$, $K\subset\mathbb{R}^n$ be a smooth compact convex set and assume* (8) *and* (9). *Let $u\in C^{1,\beta}(\mathbb{R}^n\setminus K),[0,1])$ with $\beta>1-2s$ be such that*

$$\mathscr{L}[u]+f(u)\leq 0 \ \text{ in } \mathbb{R}^n\setminus K, \quad (20)$$

$$\nabla u\cdot\nu=0 \ \text{ in } \partial K, \quad (21)$$

$$u(x)\to 1 \ \text{ as } |x|\to+\infty. \quad (22)$$

Then there exists $\gamma\in(0,1]$ such that $\gamma\leq u\leq 1$ in $\overline{\mathbb{R}^n\setminus K}$.

The proof of this Lemma being an elementary adaptation of the argument used in [4], we will refer to [4] for its proof.

We now turn to the proof of Theorem 3 .

Proof (Proof of Theorem 3*).*

Let us fix $s\in\left[\frac{1}{2},1\right)$ and let K, f, and u be as in Theorem 3. Let us now follow the argument developed in [4]. Firstly, without loss of generality, one can assume by (8) that f is extended to a $C^1(\mathbb{R})$ function satisfying (15). Secondly, by (13) and the boundedness of K, there exists $R_0>0$ large enough so that $K\subset B_{R_0}$ and $u\geq 1-c_0$ in $\mathbb{R}^n\setminus B_{R_0}$, where $c_0>0$ is given in (15).

We now proceed by contradiction, and suppose that

$$\inf_{\mathbb{R}^n\setminus K} u < 1. \tag{23}$$

From (14) and (23), together with the continuity of u, there exists then $x_0 \in \overline{\mathbb{R}^n \setminus K}$ such that

$$u(x_0) = \min_{\mathbb{R}^n\setminus K} u \in [0,1).$$

We observe that, by Lemma 3, one has $u(x_0) > 0$. In addition, since K is convex, there exists $e \in \partial B_1$ such that $K \subset H_e^c$, where H_e is the open affine half-space defined by

$$H_e := x_0 + \{x \in \mathbb{R}^n;\ x \cdot e > 0\}.$$

As in section 3, let us define for all $r \in \mathbb{R}$, the family of functions

$$\phi_r(x) := \phi_{r,e}(x) = \varphi(x \cdot e - r), \ \ x \in \mathbb{R}^n,$$

where φ is a smooth monotone increasing function given by Theorem 4. Note that by construction, since $K \subset H_e^c$, we can check (as in the section 3) that for any $r \in \mathbb{R}$, ϕ_r satisfies

$$\mathscr{L}[\phi_r](x) + f(\phi_r(x)) > 0 \quad \text{for} \quad x \in \overline{H_e}. \tag{24}$$

First, we claim that

Proposition 1. *There exists $r_0 \in \mathbb{R}$ such that $\phi_{r_0} \le u$ in $\overline{\mathbb{R}^n \setminus K}$.*

Again the proof of this Proposition is an elementary adaptation of a proof done in [4] that we present for the sake of clarity.

Proof. First let us define $H := x_1 + H_e$ with x_1 to be chosen such that $B_{R_0} \subset H^c$. Let us fix x_1 such that $H \subset\subset H_e$. By construction the function φ is monotone increasing and satisfies $\lim_{z\to-\infty} \varphi(z) = 0$. So we can find $r_0 >> 1$ such that $\phi_{r_0}(x) = \varphi(x \cdot e - r_0) \le u(x_0) \le u(x)$ for all $x \in H^c$. Now thanks to our choice of x_1 we have $H \subset\subset H_e$ and from (24) we deduce

$$\begin{cases} \mathscr{L}[u](x) + f(u(x)) \le 0 & \text{for} \quad x \in \overline{H}, \\ \mathscr{L}[\phi_{r_0}](x) + f(\phi_{r_0}(x)) > 0 & \text{for} \quad x \in \overline{H}, \\ u(x) \ge \phi_{r_0}(x) & \text{for} \quad x \in H^c \setminus K, \end{cases}$$

We then get the desired results by applying the weak-maximum principle (Lemma 1).

Equipped with the Proposition 1, we can now define the following quantity

$$r^* := \inf\{r \in \mathbb{R};\ \phi_r \le u \text{ in } \overline{\mathbb{R}^n \setminus K}\}.$$

All the game now is to show that $r^* = -\infty$. So, we claim that

Claim. $r^* = -\infty$.

Assume for the claim is true, then the proof of Theorem 3 is thereby complete. Indeed, from this claim we infer that $\phi_r \le u$ in $\overline{\mathbb{R}^n \setminus K}$ for any $r \in \mathbb{R}$. In particular, recalling that $\varphi(+\infty) = 1$, we get that

$$1 > u(x_0) \ge \lim_{r \to -\infty} \phi_r(x_0) = \lim_{r \to -\infty} \varphi(x_0 \cdot e - r) = 1,$$

a contradiction. Therefore, (23) can not hold. In other words, $\inf_{\overline{\mathbb{R}^n \setminus K}} u = 1$, i.e. $u = 1$ in $\overline{\mathbb{R}^n \setminus K}$ proving thereby Theorem 3.

Let us now conclude our proof by establishing the Claim. Again, the proof of this last Claim is done by a very elementary adaption of the arguments used to prove Theorem 1. As a consequence we will only highlights the main differences.

Proof (Proof of the Claim). The proof is by contradiction. We assume that $r^* \in \mathbb{R}$. Then, there exists a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ of positive real numbers such that $\phi_{r^*+\varepsilon_j}(x) = \varphi(x \cdot e - r^* - \varepsilon_j) \le u(x)$ for all $x \in \overline{\mathbb{R}^n \setminus K}$ and $\varepsilon_j \to 0$ as $j \to +\infty$. Thus, passing to the limit as $j \to +\infty$, we obtain that

$$\phi_{r^*}(x) \le u(x) \quad \text{for all } x \in \overline{\mathbb{R}^n \setminus K}.$$

Let us denote H the open affine half-space

$$H = \left\{ x \in \mathbb{R}^n;\ x \cdot e > R_0 \right\}.$$

Notice that $\overline{H} \cap K = \emptyset$ and that u is well defined and continuous in $\overline{H}$. We also observe that, by construction,

$$\sup_{H^c} \phi_{r^*} < 1. \tag{25}$$

Two cases may occur.

Case 1: $\inf_{H^c \setminus K}(u - \phi_{r^*}) > 0$. In this situation, the argument is identical as in for (12), and we point the reader to [4] for the details.

Case 2: $\inf_{H^c \setminus K}(u - \phi_{r^*}) = 0$. In this situation, by (22) and (25), and by continuity of u and ϕ_{r^*}, there exists a point $\bar{x} \in \overline{H^c \setminus K}$ such that $u(\bar{x}) = \phi_{r^*}(\bar{x})$. Note that $\bar{x} \in \overline{H_e}$, since otherwise $\bar{x} \in \mathbb{R}^n \setminus \overline{H_e}$, namely $\bar{x} \cdot e < x_0 \cdot e$, and the chain of inequalities

$$u(\bar{x}) = \phi_{r^*}(\bar{x}) < \phi_{r^*}(x_0) \le u(x_0) = \min_{\mathbb{R}^n \setminus K} u$$

leads to a contradiction. Therefore, we have $\phi_{r^*} \le u$ in $\overline{\mathbb{R}^n \setminus K}$ with equality at a point $\bar{x} \in \overline{\mathbb{R}^n \setminus K} \cap \overline{H_e}$. Again, two situations can occur either $\bar{x} \in \mathbb{R}^n \setminus K$ or $\bar{x} \in \partial K$. Assume for the moment that the latter situation occurs. Then thanks to convexity of

K, $\nu(\bar{x})$ the outward normal to ∂K at $\bar{x}$ is then the vector e, i.e. $\nu(\bar{x}) = e$ and thanks to (14) $\nabla u \cdot \nu(\bar{x}) = 0$ we deduce

$$0 \leq \nabla(u - \phi_{r^*}) \cdot \nu(\bar{x}) \leq -\nabla(\phi_{r^*}) \cdot \nu(\bar{x}) = -\varphi'(\bar{x} \cdot e - r^*) < 0.$$

This contradiction then rules out this situation. Lastly assume that $\bar{x} \in \mathbb{R}^n \setminus K$, then in this situation $K \subset\subset H_e^c$ and φ_{r^*} and u satisfy respectively

$$\begin{cases} \mathscr{L}[u] + f(u) \leq 0 \ \text{ in } \overline{H_e}, \\ \mathscr{L}[\phi_{r^*}] + f(\phi_{r^*}) > 0 \ \text{ in } \overline{H_e} \ \text{ (by (24))}, \end{cases}.$$

In particular, it follows from the strong maximum principle (Lemma 2) that $\phi_{r^*} = u$ in $\overline{H_e}$. Thus, for any $e^\perp \in \partial B_1$ such that $e^\perp \cdot e = 0$, one infers from (22) and the definition of ϕ_{r^*} that

$$1 = \lim_{t \to +\infty} u(x_0 + t\, e^\perp) = \lim_{t \to +\infty} \phi_{r^*}(x_0 + t\, e^\perp) = \phi_{r^*}(x_0) < 1.$$

This last contradiction then rules out also this situation and therefore rules out Case 2 too.

Remark 2. The proof of Theorem 2 is identical to the one given in [4]. This is due to the fact that the $s-$fractional operator is well defined and continuous up to the boundary of ∂K when $s \in \left(0, \frac{1}{2}\right)$ and as such the strong maximum principle (Lemma 2) holds also true for any half space H such that $K \subset H^c$. In this situation, the case $\bar{x} \in \partial K$ does not need to be analysed separately from the other cases.

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