A rigidity theorem for ideal surfaces with flat boundary

James McCoy and Glen Wheeler

Abstract We are interested in surfaces with boundary satisfying a sixth order non-linear elliptic partial differential equation associated with extremal surfaces of the L^2-norm of the gradient of the mean curvature. We show that such surfaces satisfying so-called 'flat boundary conditions' and small L^2-norm of the second fundamental form are necessarily planar.

1 Introduction

1.1 The energy, notation

We are interested in the geometric energy

$$\mathscr{F}[f] = \int_\Sigma |\nabla H|^2 \, d\mu$$

under the hypothesis

$$\int_\Sigma |A|^2 \, d\mu \le \varepsilon_0 \tag{1}$$

where $\varepsilon_0 > 0$ is a small, universal constant. Here $f : \Sigma \to \mathbb{R}^3$ a smooth immersion of surface Σ with boundary; $d\mu$ is the induced surface area element; $H = \kappa_1 + \kappa_2$ and $|A|^2 = \kappa_1^2 + \kappa_2^2$ are respectively the mean curvature and the norm of the second fundamental form of $f(\Sigma)$ and ∇ is the covariant derivative on $f(\Sigma)$.

James McCoy
School of Mathematical and Physical Sciences, University of Newcastle

Glen Wheeler
School of Mathematics and Applied Statistics, University of Wollongong

1.2 The normal variation

Lemma 1. *Given a smooth normal variation $\phi : \Sigma \to \mathbb{R}^3$ of $f : \Sigma \to \mathbb{R}^3$,*

$$\frac{d}{d\varepsilon}\mathscr{F}[f+\varepsilon\phi]\Big|_{\varepsilon=0} = -2\int_\Sigma \left[\Delta^2 H + |A|^2\Delta H - (A^0)^{ij}\nabla_i H\nabla_j H\right]\langle\phi,\nu\rangle\,d\mu \\ +2\int_{\partial\Sigma}\left\langle\left(\Delta\varphi+|A|^2\varphi\right)\nabla H + \nabla\Delta H\,\varphi - \Delta H\,\nabla\varphi, \eta\right\rangle d\sigma. \quad (2)$$

where $\varphi := \langle\phi,\nu\rangle$.

Above $\left(A^0\right)^{ij} = A^{ij} - \frac{1}{2}H\,g^{ij}$ is the trace-free part of A. The metric is denoted g_{ij}; the entries of its inverse matrix are g^{ij}. The unit normal to $f(\Sigma)$ is denoted ν while the unit conormal to the boundary is denoted η.

Idea of proof: The result follows from the variations

$$\frac{\partial}{\partial\varepsilon}g^\varepsilon_{ij}\Big|_{\varepsilon=0} = -2\varphi A_{ij}, \quad \frac{\partial}{\partial\varepsilon}g^{ij}_\varepsilon\Big|_{\varepsilon=0} = 2\varphi A^{ij},$$

$$\frac{\partial}{\partial\varepsilon}\sqrt{\det(g^\varepsilon_{ij})}\Big|_{\varepsilon=0} = -H\varphi\sqrt{\det(g_{ij})}, \quad \frac{\partial}{\partial\varepsilon}H_\varepsilon\Big|_{\varepsilon=0} = \Delta\varphi+\varphi|A|^2.$$

The boundary terms arise from ‘integration by parts’ on Σ with boundary, ie via the Divergence Theorem

$$\int_\Sigma \mathrm{div}_\Sigma X\,d\mu = \int_{\partial\Sigma}\langle X,\eta\rangle\,d\sigma.$$

□

1.3 The boundary value problem

If $f(\Sigma)$ were closed without boundary, there would be no boundary terms and critical points of $\mathscr{F}[f]$ would satisfy

$$\mathbf{I}[f] := \Delta^2 H + |A|^2\Delta H - \left(A^0\right)^{ij}\nabla_i H\nabla_j H = 0. \quad (3)$$

We will impose *flat boundary conditions* on $\partial\Sigma$:

$$|A| = 0 \text{ and } \nabla_\eta H = \nabla_\eta \Delta H = 0 \quad (4)$$

(defined in terms of limits approaching the boundary). Then the boundary terms in (2) disappear and we are left with (3) for critical points of the energy. We study smooth solutions (3) with boundary conditions (4) and smallness condition (1).

Theorem 1. *Suppose $f:\Sigma\to\mathbb{R}^3$ satisfies* (3) *with boundary conditions* (4)*. If f also satisfies* (1) *for $\varepsilon_0>0$ sufficiently small, then $f(\Sigma)$ is part of a flat plane.*

Remark: In the case of $f:\Sigma\to\mathbb{R}^k$, $k>3$, (3) may be replaced by the condition $\langle \mathbf{I}[f],\mathbf{H}\rangle=0$.

Idea of proof: We establish that surfaces satisfying (4) and (1) also satisfy

$$\int_\Sigma\left(\left|\nabla^2A\right|^2+|A|^2|\nabla A|^2+|A|^4\left|A^0\right|^2\right)\gamma^p\,d\mu \\ \leq c\int_\Sigma \mathbf{I}[f]\,H\,\gamma^p\,d\mu+c\,c_\gamma^4\int_{[\gamma>0]}|A|^2\,d\mu,$$

for appropriately chosen cutoff functions γ (see Definition 1 below). For surfaces additionally satisfying (3), the first term on the right hand side above disappears and the other term is bounded by $\frac{c}{\rho^4}$. Taking $\rho\to\infty$ we see that $f(\Sigma)$ must have $|A|^4\left|A^0\right|^2\equiv 0$. Since $\left|A^0\right|^6\leq|A|^4\left|A^0\right|^2$ we see that $\left|A^0\right|^2\equiv 0$ implying $f(\Sigma)$ is either part of a sphere or part of a plane. The boundary condition (4) implies $f(\Sigma)$ is part of a plane. □

2 Earlier related work

2.1 Previous geometric gap lemmas

Previous higher-order geometric gap lemmas closest in spirit to ours are for Willmore surfaces [2]; for stationary solutions of the surface diffusion flow [7]; for biharmonic surfaces [8]; for some Helfrich surfaces (constrained Willmore surfaces) [5], with some others covered in [1]; for triharmonic surfaces [3] and for polyharmonic surfaces [6]. For several of these there are also versions for surfaces with boundary, with either of two boundary conditions:

1. *umbilic boundary conditions* $\left|\nabla A^0\right|=\left|A^0\right|=0$; or
2. *flat boundary conditions* $|\nabla A|=|A|=0$.

Under suitable smallness conditions, umbilic boundary conditions lead to parts of spheres and planes, while flat boundary conditions allow planes only [9].

Often (including here) results hold for arbitrary codimension, but we will restrict here to codimension 1 for convenience.

2.2 Some tools

Definition 1 (Cut-off functions). For $\tilde{\gamma} \in C_c^2(\mathbb{R}^3)$, $\gamma = \tilde{\gamma} \circ f : \Sigma \to [0,1]$ satisfies

$$\|\nabla\gamma\|_\infty \le c_\gamma, \quad \|\nabla^2\gamma\|_\infty \le c_\gamma(c_\gamma + |A|),$$

for some absolute constant $c_\gamma < \infty$.

Theorem 2 (Michael-Simon Sobolev Inequality with boundary, [4]). *For $f : M^m \to \mathbb{R}^n$ a smooth immersion of M with boundary ∂M into $\mathbb{R}^n$ and any $u \in C^1\left(\overline{M}\right)$,*

$$\left[\int_M |u|^{\frac{m}{m-1}}\, d\mu\right]^{\frac{m-1}{2}} \le \frac{4^{m+1}}{\omega_m^{1/m}} \left[\int_M (|\nabla u| + |H|\,|u|)\, d\mu + \int_{\partial M} |u|\, d\sigma\right]$$

where ω_m is the volume of the unit ball in $\mathbb{R}^m$.

3 Estimates

Using the Divergence Theorem on Σ (integration by parts) we begin with

Lemma 2. *Surfaces satisfying* (4) *also satisfy*

$$\int_\Sigma (\Delta H)^2\,\gamma^p d\mu = \int_\Sigma \boldsymbol{I}[f]\,H\gamma^p d\mu + \int_\Sigma |A|^2\,|\nabla H|^2\,\gamma^p d\mu + \int_\Sigma H\,\nabla^i H \nabla_i\,|A|^2\,\gamma^p d\mu$$
$$+ \int_\Sigma H\left(A^0\right)^{ij}\nabla_i H \nabla_j H\,\gamma^p d\mu + p\int_\Sigma \left[H\,\nabla^i \Delta H + \left(H\,|A|^2 - \Delta H\right)\nabla^i H\right]\nabla_i\gamma \cdot \gamma^{p-1} d\mu$$

Lemma 3. *Surfaces satisfying* (4) *also satisfy*

$$\int_\Sigma \left|\nabla^2 H\right|^2 \gamma^p d\mu \le c\int_\Sigma \boldsymbol{I}[f]\,H\gamma^p d\mu + c\int_\Sigma |A|^2\,|\nabla A|^2\,\gamma^p d\mu + c\,c_\gamma^2 \int_\Sigma \left|\nabla A^0\right|^2 \gamma^{p-2} d\mu$$
$$+ p\int_\Sigma \left[H\nabla^i \Delta H + \left(H\,|A|^2 - \Delta H\right)\nabla^i H\right]\nabla_i\gamma \cdot \gamma^{p-1} d\mu$$

Idea of proof: [9] has for a universal constant c

$$\frac{1}{c}\int_\Sigma \left(\left|\nabla^2 H\right|^2 + H^2\,|\nabla H|^2\right)\gamma^p d\mu$$
$$\le \int_\Sigma \left[(\Delta H)^2 + \left|A^0\right|^2 |\nabla H|^2\right]\gamma^p d\mu + c_\gamma^2 \int_\Sigma \left|\nabla A^0\right|^{p-2} d\mu.$$

Further, we estimate

$$\int_\Sigma H\left(A^0\right)^{ij}\nabla_i H \nabla_j H \gamma^p d\mu \le \frac{1}{2c}\int_\Sigma H^2\,|\nabla H|^2\,\gamma^p d\mu + \frac{c}{2}\int_\Sigma \left|A^0\right|^2 |\nabla H|^2\,\gamma^p d\mu;$$

$$\int_\Sigma H\,\nabla_i\,|A|^2\,\nabla^i H\,\gamma^p\,d\mu \le \tilde{c}\int |A|^2\,|\nabla A|^2\,\gamma^p\,d\mu.$$

Combining these with the previous Lemma yields the result. □

Lemma 4. *Surfaces satisfying* (4) *also satisfy*

$$\begin{aligned}
&\int_\Sigma \left(\left|\nabla^2 H\right|^2 + |A|^4\left|A^0\right|^2\right)\gamma^p\,d\mu \\
&\le c\int_\Sigma \boldsymbol{I}[f]\,H\gamma^p\,d\mu + c\int_\Sigma |A|^2\,|\nabla A|^2\,\gamma^p\,d\mu + c\int_\Sigma \left|A^0\right|^6\gamma^p\,d\mu + c\,c_\gamma^2\int_\Sigma \left|\nabla A^0\right|^2\gamma^{p-2}\,d\mu \\
&\quad + c\,c_\gamma^4\int_\Sigma \left|A^0\right|^2\gamma^{p-4}\,d\mu + p\int_\Sigma \left[H\nabla^i\Delta H + \left(H\,|A|^2 - \Delta H\right)\nabla^i H\right]\nabla_i\gamma\cdot\gamma^{p-1}\,d\mu
\end{aligned}$$

Idea of proof: [9] has

$$\begin{aligned}
&\int_\Sigma \left(H^4\left|A^0\right|^2 + H^2\left|\nabla A^0\right|^2\right)\gamma^p\,d\mu \\
&\qquad \le c\int_\Sigma \left(H^2\,|\nabla H|^2 + \left|A^0\right|^2\left|\nabla A^0\right|^2 + \left|A^0\right|^6\right)\gamma^p\,d\mu + c\,c_\gamma^4\int_\Sigma \left|A^0\right|^2\gamma^{p-4}\,d\mu
\end{aligned}$$

The result follows using Lemma 3 and

$$\int_\Sigma |A|^4\left|A^0\right|^2\gamma^p\,d\mu \le (1+\varepsilon)\int_\Sigma H^4\left|A^0\right|^2\gamma^p\,d\mu + (1+c(\varepsilon))\int_\Sigma \left|A^0\right|^6\gamma^p\,d\mu$$

□

Lemma 5. *Surfaces satisfying* (4) *also satisfy*

$$\begin{aligned}
&\int_\Sigma \left(\left|\nabla^2 A\right|^2 + |A|^2\,|\nabla A|^2 + |A|^4\left|A^0\right|^2\right)\gamma^p\,d\mu \\
&\le c\int_\Sigma \boldsymbol{I}[f]\,H\gamma^p\,d\mu + c\int_\Sigma |A|^2\,|\nabla A|^2\,\gamma^p\,d\mu + c\int_\Sigma \left|A^0\right|^6\gamma^p\,d\mu + c\,c_\gamma^2\int_\Sigma \left|\nabla A^0\right|^2\gamma^{p-2}\,d\mu \\
&\quad + c\,c_\gamma^4\int_\Sigma \left|A^0\right|^2\gamma^{p-4}\,d\mu + p\int_\Sigma \left[H\nabla^i\Delta H + \left(H\,|A|^2 - \Delta H\right)\nabla^i H\right]\nabla_i\gamma\cdot\gamma^{p-1}\,d\mu
\end{aligned}$$

Idea of proof: Simons' identity implies

$$\left(\Delta A^0\right)^2 \le c\left|\nabla^2 H\right|^2 + c\,H^4\left|A^0\right|^2 + c\left|A^0\right|^6;$$

interchange of second covariant derivatives and the Divergence Theorem then shows

$$\begin{aligned}
\int_\Sigma \left|\nabla^2 A^0\right|^2\gamma^p\,d\mu \le 2\int_\Sigma \left(\Delta A^0\right)^2\gamma^p\,d\mu + c\int_\Sigma \left(|A|^2\,|\nabla A|^2 + \left|A^0\right|^6\right)\gamma^p\,d\mu& \\
+ c\,c_\gamma^2\int_\Sigma \left|\nabla A^0\right|^2\gamma^{p-2}\,d\mu&.
\end{aligned}$$

The result then follows using the previous Lemmas. □

Lemma 6. *Surfaces satisfying* (4) *and* (1) *also satisfy*

$$\int_\Sigma \left(\left|\nabla^2 A\right|^2 + |A|^2 |\nabla A|^2 + |A|^4 \left|A^0\right|^2\right) \gamma^p d\mu \le c \int_\Sigma \boldsymbol{I}[f] H \gamma^p d\mu + c\, c_\gamma^4 \int_\Sigma |A|^2 \gamma^{p-4} d\mu$$

Idea of proof: Write $\|A\|^2_{2,[\gamma>0]} = \int_{[\gamma>0]} |A|^2 d\mu$. The idea is to use the smallness condition to estimate the terms on the right hand side of Lemma 4. [9] showed using the Michael-Simon Sobolev inequality

$$\begin{aligned}
&\int_\Sigma \left(\left|A^0\right|^2 |A|^4 + |A|^2 |\nabla A|^2\right) \gamma^p d\mu \\
&\le c\, \|A\|^2_{2,[\gamma>0]} \int_\Sigma \left(\left|\nabla^2 A^0\right|^2 + |A|^2 \left|\nabla A^0\right|^2 + |A|^4 \left|A^0\right|^2\right) \gamma^p d\mu + c\, c_\gamma^4 \|A\|^4_{2,[\gamma>0]},
\end{aligned}$$

so we can absorb the non-c_γ terms on the right hand side of Lemma 4.

We estimate the c_γ terms from Lemma 4 as follows:

$$c\, c_\gamma^4 \int_\Sigma \left|A^0\right|^2 \gamma^{p-4} d\mu \le c\, c_\gamma^4 \int_\Sigma |A|^2 \gamma^{p-4} d\mu;$$

via the Divergence Theorem, Cauchy-Schwarz and Peter-Paul

$$c\, c_\gamma^2 \int_\Sigma \left|\nabla A^0\right|^2 \gamma^{p-2} d\mu \le \varepsilon \int_\Sigma \left|\nabla^2 A\right|^2 \gamma^p d\mu + c(\varepsilon)\, c_\gamma^4 \int_\Sigma |A|^2 \gamma^{p-4} d\mu;$$

with this in turn we estimate

$$\begin{aligned}
p \int_\Sigma \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu &\le c\, c_\gamma \int_\Sigma \left|\nabla^2 A\right| |\nabla H| \gamma^{p-1} d\mu \\
&\le \varepsilon \int_\Sigma \left|\nabla^2 A\right|^2 \gamma^p d\mu + c(\varepsilon)\, c_\gamma^4 \int_\Sigma |A|^2 \gamma^{p-4} d\mu;
\end{aligned}$$

$$p \int_\Sigma H |A|^2 \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu \le \varepsilon \int_\Sigma |A|^2 \left|\nabla A^0\right|^2 \gamma^p d\mu + c(\varepsilon)\, c_\gamma^2 \int_\Sigma H^2 |A|^2 \gamma^{p-2} d\mu.$$

Now by the Michael-Simon Sobolev inequality ($|A| = 0$ on $\partial\Sigma$)

$$\begin{aligned}
&\int_\Sigma H^2 |A|^2 \gamma^{p-2} d\mu \le \int_\Sigma |A|^4 \gamma^{p-2} d\mu \le c \left(\int_\Sigma \left|\nabla |A|^2\right| \gamma^{\frac{p-2}{2}} d\mu\right)^2 + \left(\int_\Sigma |A|^3 \gamma^{\frac{p-2}{2}} d\mu\right)^2 \\
&\le c \left(\int_\Sigma |\nabla A| |A| \gamma^{\frac{p-2}{2}} d\mu\right)^2 + c\, c_\gamma^2 \left(\int_\Sigma |A|^2 \gamma^{\frac{p-4}{2}} d\mu\right)^2 + c\, \|A\|^2_{2,[\gamma>0]} \int_\Sigma |A|^4 \gamma^{p-2} d\mu.
\end{aligned}$$

Absorbing on the left and using the Cauchy-Schwarz inequality we obtain

$$\int_\Sigma |A|^4 \gamma^{p-2} d\mu \le c\, \|A\|^2_{2,[\gamma>0]} \int_\Sigma |\nabla A|^2 \gamma^{p-2} d\mu + c\, c_\gamma^2 \|A\|^4_{2,[\gamma>0]}$$

and so

$$c\, c_\gamma^2 \int_\Sigma H^2 |A|^2 \gamma^{p-2} d\mu \le c\, c_\gamma^2 \|A\|^2_{2,[\gamma>0]} \int_\Sigma |\nabla A|^2 \gamma^{p-2} d\mu + c\, c_\gamma^4 \|A\|^4_{2,[\gamma>0]}.$$

For the remaining term we use the Divergence Theorem ($H=0$ on $\partial\Sigma$)

$$\int_\Sigma H\nabla^i \Delta H \nabla_i\gamma\cdot\gamma^{p-1}d\mu = -\int_\Sigma \Delta H\nabla^i H\nabla_i\gamma\cdot\gamma^{p-1}d\mu - \int_\Sigma H\Delta H\Delta\gamma\cdot\gamma^{p-1}d\mu$$
$$-(p-1)\int_\Sigma H\Delta H\,|\nabla\gamma|^2\gamma^{p-2}d\mu.$$

We now estimate

$$-\int_\Sigma \Delta H\nabla^i H\nabla_i\gamma\cdot\gamma^{p-1}d\mu \le \varepsilon\int_\Sigma (\Delta H)^2\gamma^p d\mu + c(\varepsilon)\,c_\gamma^2\int_\Sigma |\nabla H|^2\gamma^{p-2}d\mu;$$

$$-\int_\Sigma H\Delta H\Delta\gamma\cdot\gamma^{p-1}d\mu \le c\,c_\gamma\int_\Sigma |H|\,|\Delta H|\left(c_\gamma+|A|\right)\gamma^{p-1}d\mu$$
$$\le \varepsilon\int_\Sigma (\Delta H)^2\gamma^p d\mu + c\,c_\gamma^4\int_\Sigma |A|^2\gamma^{p-2}d\mu + c\,c_\gamma^2\int_\Sigma H^2|A|^2\gamma^{p-2}d\mu$$

and

$$-(p-1)\int_\Sigma H\Delta H\,|\nabla\gamma|^2\gamma^{p-2}d\mu \le \varepsilon\int_\Sigma (\Delta H)^2\gamma^p d\mu + c\,c_\gamma^4\int_\Sigma H^2\gamma^{p-4}d\mu$$

Inserting all these estimates and absorbing on the left yields the result. □

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References

1. Y. Bernard, G. Wheeler, V.-M. Wheeler, *Spherocytosis and the Helfrich model*, Interfaces And Free Boundaries, **19** (2017) no. 4, 495–523.
2. E. Kuwert, R. Schätzle, *The Willmore flow with small initial energy*, J. Differential Geom. **57** (2001), 409–441.
3. J. McCoy, S. Parkins, G. Wheeler, *The geometric triharmonic heat flow of immersed surfaces near spheres*, Nonlinear Anal. **161** (2017), 44–86.
4. J. H. Michael, L. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* $\mathbb{R}^n$, Comm. Pure Appl. Math. **26** (1973), 361–379.
5. J. McCoy, G. Wheeler, *A classification theorem for Helfrich surfaces*, Math. Ann. **357** (2013) no. 4, 1485–1508.
6. S. Parkins, *A selection of higher-order parabolic curvature flows*, PhD thesis, Uni. Wollongong, 2017.
7. G. Wheeler, *Surface diffusion flow near spheres*, Calc. Var. **44** (2012), no. 1-2, 131–151.
8. G. Wheeler, *Chen's conjecture and ε-superbiharmonic submanifolds of Riemannian manifolds*, Int. J. Math. **24** (2013) no. 4, 1350028.
9. G. Wheeler, *Gap phenomena for a class of fourth-order geometric differential operators on surfaces with boundary*, Proc. Amer. Math. Soc. **143** (2014) no. 4,