A potential well argument for a semilinear parabolic equation with exponential nonlinearity

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Abstract We consider the Cauchy problem for a two space dimensional parabolic equation with square exponential nonlinearity. More precisely,

$$\begin{cases} \partial_t u = \Delta u - u + \lambda f(u) & \text{in } (0,T) \times \mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases}$$

where $\lambda > 0$, and $f(u) := 2\alpha_0 u e^{\alpha_0 u^2}$, for some $\alpha_0 > 0$. We take into account initial data in the energy space $H^1(\mathbb{R}^2)$, i.e. $u_0 \in H^1(\mathbb{R}^2)$, and in view of the Trudinger-Moser inequality, the nonlinearity f (which has square exponential growth at infinity) is in the energy critical regime.

We look for sufficient conditions in order to predict from the initial data whether the solution blows up in finite time or the solution exists globally in time. Our main tools are energy methods, and the so-called *potential well argument*. If $0 < \lambda < \frac{1}{2\alpha_0}$, we prove that for energies below the ground state level, the dichotomy between blow-up and global existence is determined by the sign of a suitable functional.

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1 Model parabolic problem

We consider the Cauchy problem for a two space dimensional parabolic equation with square exponential nonlinearity, more precisely we focus the attention on the following model problem:

$$\begin{cases} \partial_t u = \Delta u - u + \lambda f(u) & \text{in } (0,T) \times \mathbb{R}^2, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{1}$$

where $\lambda > 0$,

$$f(u) := 2\alpha_0 u e^{\alpha_0 u^2}, \quad \text{for some } \alpha_0 > 0,$$

and we consider initial data in the *energy* space $H^1(\mathbb{R}^2)$, i.e.

$$u_0 \in H^1(\mathbb{R}^2).$$

In this framework, *energy* refers to the functional associated with the stationary problem:

$$I(v) := \frac{1}{2}\|v\|_{H^1}^2 - \lambda \int_{\mathbb{R}^2} F(v)\,dx,$$

where

$$\|v\|_{H^1} := \big(\|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2\big)^{\frac{1}{2}}, \quad \text{and} \quad F(v) := \int_0^v f(\eta)\,d\eta = e^{\alpha_0 v^2} - 1.$$

The above functional is well defined in $H^1(\mathbb{R}^2)$, and the nonlinear term f that we are considering is *critical* in the energy space in view of the Trudinger-Moser embedding [1, 19].

Concerning local existence and uniqueness for (1), Ibrahim, Jrad, Majdoub and Saanouni [5] proved that, for any $u_0 \in H^1(\mathbb{R}^2)$, the Cauchy problem (1) has a local in time solution u up to some finite time $T > 0$ satisfying

$$u \in \mathscr{C}\big([0,T]; H^1(\mathbb{R}^2)\big),$$

and the solution in *unique*. Then the smoothing effect of the heat kernel implies that the local in time solution u found in [5] belongs to the class

$$u \in L^\infty_{loc}\big((0,T]; L^\infty(\mathbb{R}^2)\big) \cap \mathscr{C}^1\big((0,T); L^2(\mathbb{R}^2)\big) \cap \mathscr{C}^{1,2}\big((0,T) \times \mathbb{R}^2\big),$$

see [9, Remark 4.1], and $\Delta u \in \mathscr{C}\big((0,T), L^2(\mathbb{R}^2)\big)$. Therefore, for any $t \in (0,T)$, we have

$$\|\partial_t u(t)\|_{L^2}^2 = -\frac{d}{dt} I\big(u(t)\big), \tag{2}$$

and

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\|u(t)\|_{H^1}^2 + \lambda \int_{\mathbb{R}^2} u(t) f\big(u(t)\big)\,dx. \tag{3}$$

We define the maximal existence time T_* of the solution u as

$$T_* := \sup\Big\{ T>0 : \text{the solution } u \text{ to (1) satisfies } u \in \mathscr{C}\big([0,T]; H^1(\mathbb{R}^2)\big)\Big\} \in (0,+\infty],$$

and the following blow-up alternative holds:

$$\text{if } T_* < +\infty \text{ then } \limsup_{t\to T_*} \|u(t)\|_{L^\infty} = +\infty, \tag{4}$$

see [5, Lemma 4.6].

Our aim is to find sufficient conditions in order to determine from the initial data $u_0 \in H^1(\mathbb{R}^2)$ whether the solution blows up in finite time (i.e. $T_* < +\infty$) or the solution is global in time (i.e. $T_* = +\infty$).

The same problem for nonlinear parabolic equations with polynomial nonlinearities has been widely studied via the potential well argument starting from the seminal papers by Tsutsumi [22], Ishii [10], and Payne and Sattinger [17]. Let us recall the central idea of this method following the presentation given in [18].

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded set with smooth boundary, and let us consider

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{5}$$

with $1 < p \leq 2^* - 1$, with $2^* = \frac{2N}{N-2}$. For any initial data in the energy space $H_0^1(\Omega)$ there exists some finite time $T > 0$ and a local in time solution u belonging to $\mathscr{C}\big([0,T]; H_0^1(\Omega)\big)$ (this is a consequence of the L^{p+1}-existence result in [2] for any $1 < p \leq 2^* - 1$, and of the smoothing effect of the heat kernel). In this case, the energy functional is given by

$$I_p(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$

Let $v \in H_0^1(\Omega)\setminus\{0\}$, and let us analyze the energy of the function σv for any $\sigma \geq 0$. By an easy computation, one can show that

$$I_p(\sigma v) = \frac{\sigma^2}{2}\|\nabla v\|_{L^2}^2 - \frac{\sigma^{p+1}}{p+1}\|v\|_{L^{p+1}}^{p+1}$$

attains its unique maximum at a point $\bar{\sigma} > 0$, and $\bar{v} := \bar{\sigma} v$ satisfies

$$\|\nabla \bar{v}\|_{L^2}^2 - \|\bar{v}\|_{L^{p+1}}^{p+1} = 0.$$

Therefore, the energy $I(\sigma v)$ has the structure of a potential well, and every ray σv, for any $\sigma > 0$ and for $v \in H_0^1(\Omega)\setminus\{0\}$, has a *unique* intersection with the *Nehari manifold*

$$N = \{v \in H_0^1(\Omega)\setminus\{0\} \,:\, \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} = 0\}.$$

The depth of the well is given by the lowest pass over the ridge defined by all possible $I_p(\sigma v)$ as v ranges over $H_0^1(\Omega)\setminus\{0\}$, namely

$$d_p := \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \max_{\sigma\geq 0} I_p(\sigma v).$$

It is well known that d_p can be characterized as

$$d_p = \inf_{v\in N} I_p(v), \quad \text{and also} \quad d_p = \frac{p-1}{2(p+1)}\Lambda^{2(p+1)/(p-1)},$$

where $\Lambda = \Lambda_{p+1}(\Omega)$ is the best constant in the Sobolev embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, i.e.

$$\Lambda = \inf_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{L^2}}{\|v\|_{L^{p+1}}}.$$

If $1 < p < 2^* - 1$ then d_p is the energy level of ground state solutions, i.e.

$$d_p = \inf\Big\{I_p(v) \,:\, v \in H_0^1(\Omega)\setminus\{0\} \text{ satisfies } \langle dI_p(v), \varphi\rangle = 0 \text{ for any } \varphi \in H_0^1(\Omega)\Big\}$$

The potential well associated with the Cauchy problem (5) is the set (*stable* set)

$$W_p := \Big\{v \in H_0^1(\Omega) : I_p(v) < d_p,\, \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} > 0\Big\} \cup \{0\},$$

and the exterior of the potential well (*unstable* set) is

$$V_p := \Big\{v \in H_0^1(\Omega) \,:\, I_p(v) < d_p,\, \|\nabla v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1} < 0\Big\}.$$

The sets V_p and W_p are both invariant under the flow associated with the problem (5). Concerning the stable set if $1 < p < 2^* - 1$, any solution which enters the stable set W_p exists globally in time. This result is a direct consequence of the fact that, in the subcritical case, the time T of local existence of the solution to (5) depends only on the size of the norm of the initial data in $H_0^1(\Omega)$, and for any $v \in W_p$ the $\|\nabla v\|_{L^2}$ is uniformly bounded (see [22]). Similar results have also been proven for $p = 2^* - 1$, where the situation is different because the local existence time of the solution to (5) depends from the specific initial data rather than its *size* (see [10], [8], [11], and [12]). On the other side, if $1 < p \leq 2^* - 1$ then any solution which intersects the unstable set V_p blows up in finite time (see [17] and [10]). Related results can be found in [16, 15, 3, 7]. For the case $p = 2^* - 1$ and $\Omega = \mathbb{R}^N$, $N \geq 3$, we refer to [6] and the references therein.

In the same spirit of these results, we show that for *energies below the ground state level* the dichotomy between blow-up and global existence for the Cauchy problem (1) can be determined by means of a potential well argument.

2 Stable and unstable sets

In analogy with the polynomial case, also the energy I associated with our model problem (1) has a potential well structure. More precisely, for any fixed $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, the function $\sigma \mapsto I(\sigma v)$ has the shape of a potential well. This can be deduced from the study of the sign of the so-called Nehari functional

$$J(v) := \langle dI(v), v \rangle = \|v\|_{H^1}^2 - \lambda \int_{\mathbb{R}^2} v f(v)\, dx,$$

in fact

$$\frac{d}{d\sigma} I(\sigma v) = \frac{1}{\sigma} J(\sigma v).$$

Proposition 1. *Assume that*

$$0 < \lambda < \frac{1}{2\alpha_0}. \tag{6}$$

For any $v \in H^1(\mathbb{R}^2) \setminus \{0\}$, there exists a unique $\overline{\sigma} = \overline{\sigma}(v) > 0$ such that

$$J(\sigma v) \begin{cases} > 0 & \text{if } 0 < \sigma < \overline{\sigma}, \\ = 0 & \text{if } \sigma = \overline{\sigma}, \\ < 0 & \text{if } \sigma > \overline{\sigma}. \end{cases} \tag{7}$$

Moreover,

$$\lim_{\sigma \to +\infty} I(\sigma v) = -\infty, \tag{8}$$

and $\overline{\sigma}$ is the unique maximum point of the function $\sigma \mapsto I(\sigma v)$ on $[0, +\infty)$.

Next, we introduce the depth of the well

$$d := \inf \Big\{ I(v) \,:\, v \in H^1(\mathbb{R}^2) \setminus \{0\},\, J(v) = 0 \Big\},$$

which coincides with the mountain pass level

$$c := \inf_{v \in H^1(\mathbb{R}^2) \setminus \{0\}} \max_{\sigma > 0} I(\sigma v).$$

More precisely,

$$c = d. \tag{9}$$

We recall that the existence of a mountain pass solution for the stationary problem

$$-\Delta v + v = \lambda f(v) \quad \text{in } \mathbb{R}^2 \tag{10}$$

with λ in the range (6) is proved in [20], where it is also shown that

$$0 < c < \frac{2\pi}{\alpha_0}. \tag{11}$$

The potential well argument suggests to consider the splitting of the d-sublevelset of the energy I determined by the Nehari functional J. More precisely, we consider the unstable set V and the stable set W defined respectively by

$$V := \Big\{ v \in H^1(\mathbb{R}^2) \,:\, I(v) < d,\, J(v) < 0 \Big\},$$

and

$$\begin{aligned} W :=& \Big\{ v \in H^1(\mathbb{R}^2) \,:\, I(v) < d,\, J(v) \geq 0 \Big\} \\ =& \Big\{ v \in H^1(\mathbb{R}^2) \,:\, I(v) < d,\, J(v) > 0 \Big\} \cup \{0\}. \end{aligned}$$

Theorem 1. *Let $u \in \mathscr{C}\big([0,T_*);H^1(\mathbb{R}^2)\big)$ be the maximal solution to* (1) *with λ as in* (6)*, and $u_0 \in H^1(\mathbb{R}^2)$.*

i) If $u(t_0) \in V$ for some $t_0 \in [0,T_*)$ then $T_* < +\infty$.
ii) There exists $t_0 \in [0,T_*)$ such that $u(t_0) \in W$ if and only if

$$T_* = +\infty, \quad \text{and} \quad \lim_{t\to+\infty} \|u(t)\|_{H^1} = 0.$$

Theorem 1.i) can be seen as an improvement of the blow-up result obtained in [5] for non-positive energies:

Theorem 2. *[5, Theorem 2.1.3] Let $u \in \mathscr{C}\big([0,T_*);H^1(\mathbb{R}^2)\big)$ be the maximal solution to* (1) *with $0 < \lambda \leq \frac{1}{2\alpha_0}$, and $u_0 \in H^1(\mathbb{R}^2)\setminus\{0\}$. If $I\big(u(t_0)\big) \leq 0$ for some $t_0 \in [0,T_*)$ then $T_* < +\infty$.*

Up to our knowledge, Theorem 1 is a *new* application of the potential well argument to heat equations with *critical* exponential nonlinearities in the 2-dimensional case. The *subcritical* exponential case is studied in [4] and [21].

In the next Section, we will sketch the proof of Theorem 1.i), and the part of the proof of Theorem 1.ii) concerning global existence in W. Complete proofs and detailed explanations can be found in [13].

3 Sketch of the proof of Theorem 1

Blow-up. Thanks to the monotonicity of the energy along the solution and the geometry of the sublevelsets of the energy I, one can show the invariance of the set V with respect to the heat flow associated with (1).

In order to prove that the solutions in V blow up in finite time, we will apply the following blow-up Lemma containing the classical idea of the concavity method due to Levine [14].

Lemma 1. *There exists no non-negative and increasing function $y \in \mathscr{C}^2(\bar{t},+\infty)$, with $\bar{t} \in \mathbb{R}$, such that, for some $\beta > 1$,*

$$y(t)y''(t) \geq \beta[y'(t)]^2 \text{ on } (\bar{t}, +\infty),$$

and

$$\lim_{t\to+\infty} y(t) = +\infty.$$

The concavity method works in our setting due to the fact that the Nehari functional along solutions entering V is bounded away from zero by a strictly negative constant.

Proposition 2. *Let $u \in \mathscr{C}\big([0,T_*); H^1(\mathbb{R}^2)\big)$ be the maximal solution to* (1) *with $u_0 \in H^1(\mathbb{R}^2)$. If $u(t_0) \in V$ for some $t_0 \in [0,T_*)$ then there exists $\varepsilon > 0$ such that $J\big(u(t)\big) < -\varepsilon$ for any $t \in [t_0, T_*)$.*

We argue by contradiction assuming $T_* = +\infty$, and we apply the blow-up Lemma 1 to the non-negative and increasing $\mathscr{C}^2$-function defined by

$$y(t) := \frac{1}{2}\int_{t_0}^{t} \|u(s)\|_{L^2}^2\, ds, \quad t \in [t_0, +\infty).$$

In view of (3), we have

$$y''(t) = \frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 = -J\big(u(t)\big) > \varepsilon, \quad t \in (t_0, +\infty), \tag{12}$$

where $\varepsilon > 0$ is given by Proposition 2. From (12), we deduce that

$$\lim_{t\to+\infty} y'(t) = \lim_{t\to+\infty} y(t) = +\infty. \tag{13}$$

Finally, refining the estimate in (12), it is possible to show that

$$y(t)y''(t) \geq \beta[y'(t)]^2, \quad \text{for any large } t, \qquad \text{for some } \beta > 1.$$

Therefore we are in the framework of the blow-up Lemma 1, and we reach a contradiction.

Global existence. Thanks to the uniqueness of the solution to (1), it is possible to prove that the stable set W is invariant under the flow associated with the problem (1). In order to prove that solutions which intersect the set W at some time $t_0 \in [0, T^*)$ are global in time, we first remark that (3) yields the boundedness in $L^2(\mathbb{R}^2)$ of any solution satisfying $u(t) \in W$ for any $t \in [t_0, T^*)$, more precisely

$$\sup_{t\in[t_0,T_*)} \|u(t)\|_{L^2} < +\infty.$$

Moreover, the following property of W in the energy space holds:

Proposition 3. *For any $v \in W$, we have $\|\nabla v\|_{L^2}^2 < 2d$.*

Therefore, if the solution $u(t) \in W$ for any $t \in [t_0, T^*)$ then there exists $M > 0$ such that

$$\sup_{t\in[t_0,T_*)} \|u(t)\|_{L^2}^2 \leq M, \quad \text{and} \quad \sup_{t\in[t_0,T_*)} \|\nabla u(t)\|_{L^2}^2 \leq 2d < \frac{4\pi}{\alpha_0},$$

where the last inequality is due to (9) and (11). If we knew that the local existence time of the solution to (1) would depend only on the *size* of the initial data, we would be able to conclude in the same way as in the subcritical polynomial case. Unfortunately from the existence result in [5] we cannot prove that the local existence time $T > 0$ is *uniform* with respect to the H^1-norm of the initial data, i.e. $T = T(\|u_0\|_{H^1})$. Nevertheless, if we consider only *small* initial data, we can find a *uniform* local existence time for the solution to (1), and we can *quantify* the smallness condition as follows:

Theorem 3. *Let $0 < m < \frac{4\pi}{\alpha_0}$, and $M > 0$. There exists $T = T(m,M) > 0$ such that for any $u_0 \in H^1(\mathbb{R}^2)$ with $\|\nabla u_0\|_{L^2}^2 \leq m$ and $\|u_0\|_{L^2}^2 \leq M$ then the Cauchy problem* (1) *has a unique solution $u \in \mathscr{C}\big([0,T];H^1(\mathbb{R}^2)\big)$.*

Therefore in a similar way as in the polynomial case, we can conclude that any solution intersecting the stable set exists globally in time.

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