Geometric properties of superlevel sets of semilinear elliptic equations in convex domains

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Abstract In this paper, we report on some recent results dealing with geometrical properties of solutions of some semilinear elliptic equations in bounded smooth convex domains. We investigate the quasiconcavity, i.e. the fact that the superlevel sets of a positive solution are convex or not. We actually construct a counterexample to this fact in two dimensions, showing that the solutions under consideration do not always inherit the convexity of the domain. We report on the results in [23].

1 Introduction and main result

This paper is concerned with some geometrical properties of real-valued solutions of semilinear elliptic equations

$$\Delta u + f(u) = 0 \tag{1}$$

in bounded domains $\Omega \subset \mathbb{R}^N$, in dimension $N = 2$ with Dirichlet-type boundary conditions on $\partial\Omega$. By domains, we mean non-empty open connected subsets of $\mathbb{R}^N$.

The domains Ω are assumed to be convex domains. One is interested in knowing how these geometrical properties of Ω are inherited by the solutions u, under some suitable boundary conditions, that is how the shape of the solutions is influenced

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by the shape of the underlying domains. It is well-known that the convexity or the concavity of the solutions are too strong properties which are not true in general (see e.g. [32]). However, a typical question we address in this paper is the following one: assuming that Ω is convex and that u is a solution of (1) which is positive in Ω and vanishes on $\partial\Omega$, is it true that the superlevel sets

$$\big\{x \in \Omega;\ u(x) > \lambda\big\}$$

of u are all convex? We prove that the answer to this question can be negative, that is we show that the superlevel sets of some solutions u of problems of the type (1) are not all convex. Various examples of solutions with non-convex superlevel sets in convex rings have also been given in [23, 39]. Regarding problem (1) in convex domains, the example given in Theorem 1.1 below is the first one, up to our knowledge.

Let us consider the semilinear elliptic problem

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ \qquad\quad u = 0 \text{ on } \partial\Omega, \\ \qquad\quad u > 0 \text{ in } \Omega. \end{cases} \tag{2}$$

Throughout the paper, the function $f : [0,+\infty) \to \mathbb{R}$ is assumed to be locally Hölder continuous. The domains Ω are always assumed to be of class $C^{2,\alpha}$ (with $\alpha > 0$, we then say that the domains Ω are smooth) and the solutions u are understood in the classical sense $C^2(\overline{\Omega})$. The superlevel set $\big\{x \in \Omega;\ u(x) > 0\big\}$ of a solution u of (2) is equal to the domain Ω, which is convex by assumption. A natural question is to know whether the superlevel sets $\big\{x \in \Omega;\ u(x) > \lambda\big\}$ for $\lambda \geq 0$ are all convex or not. If this is the case, u is called quasiconcave.

In his paper [37] (see Remark 3, page 268), P.-L. Lions writes that, in a convex domain Ω, "[he] believe[s] that [...] for general f, the [super]level sets of any solution u of [(2)] are convex". There is indeed a vast literature containing some proofs of the above statement for various nonlinearities f. We here list some of the most classical references. Firstly, Makar-Limanov [38] proved that, for the two-dimensional torsion problem, that is $f(u) = 1$ with $N = 2$, the solution u is quasiconcave, since $\sqrt{u}$ is actually concave. Brascamp and Lieb [12] showed that, if $f(u) = \lambda u$ (λ is then necessarily the principal eigenvalue of the Laplacian with Dirichlet boundary condition), then the principal eigenfunction u is quasiconcave and more precisely it is log-concave, that is $\log u$ is concave. The proof uses the fact that log-concavity is preserved by the heat equation (but quasiconcavity is not in general, see [24]). When $f(u) = \lambda u^p$ with $0 < p < 1$ and $\lambda > 0$, Keady [29] for $N = 2$ and Kennington [30] for $N \geq 2$ proved that $u^{(1-p)/2}$ is concave, whence u is quasiconcave. Many generalizations under more general assumptions on f and alternate proofs have been given. A possible strategy is to prove that $g(u)$ is concave for some suitable increasing function g, by showing that

$$g(u(tx+(1-t)y)) - tg(u(x)) - (1-t)g(u(y)) \geq 0 \text{ for all } (t,x,y) \in [0,1] \times \Omega \times \Omega$$

and by using the elliptic maximum principle or the preservation of concavity of $g(u)$ by a suitable parabolic equation, see [15, 21, 27, 28, 30, 31, 32, 37]. Other strategies consist in studying the sign of the curvatures of the level sets of u or in proving that the Hessian matrix of $g(u)$ for some suitable increasing g has a constant rank, see [3, 11, 14, 34, 36, 44]. Lastly, we refer to [4, 19] for further references using the quasiconcave envelope and singular perturbations arguments, and to the book of Kawohl [26] for a general overview.

The following result is the first counterexample to the quasiconcavity of solutions u of (2) in convex domains Ω and can be found in [23] . We give here the proof of the following result.

Theorem 1.1 *In dimension $N = 2$, there are some smooth bounded convex domains Ω and some C^∞ functions $f : [0,+\infty) \to \mathbb{R}$ such that*

$$f(s) \geq 1 \ \textit{ for all } s \geq 0$$

and for which problem (2) admits both a quasiconcave solution v and a solution u which is not quasiconcave.

Remark 1.2 *It would be interesting to study the stability of the solution constructed in Theorem 1.1 in connection with results of Cabré and Chanillo [13]. We refer the reader to the remark at this end of this paper.*

2 Counterexamples in convex domains and proof of the theorem

We construct explicit examples of bounded smooth convex two-dimensional domains Ω and of functions f for which problem (2) admits some non-quasiconcave solutions u. The construction is divided into five main steps. Firstly, we define a one-parameter family $(\Omega_a)_{a\geq 1}$ of more and more elongated stadium-like convex domains. Secondly, for each value of the parameter $a \geq 1$, we solve a variational problem in $H^1_0(\Omega_a)$ with a nonlinear constraint, whose solution u_a solves an elliptic equation of the type (2) in Ω_a with some function f_a. Thirdly, we prove some a priori estimates for the superlevel sets of the functions u_a. Next, we compare u_a with a one-dimensional profile in Ω_a when a is large enough. Lastly, we show that the superlevel sets of the functions u_a cannot be all convex when a is large enough.

As a preliminary step, let us fix a C^∞ function $g : \mathbb{R} \to [0,1]$ such that

$$g = 0 \text{ on } (-\infty, 1], \ \ g = 1 \text{ on } [2, +\infty) \ \text{ and } \ g' \geq 0 \text{ on } \mathbb{R}. \tag{3}$$

The function g is fixed throughout the proof.

Step 1: construction of a family of smooth bounded convex domains $(\Omega_a)_{a\geq 1}$.

We first introduce a family of stadium-like smooth convex domains. Let $\varphi:[-1,1]\to\mathbb{R}$ be a fixed continuous nonnegative concave even function such that $\varphi(\pm 1)=0$. For $a\geq 1$, we define

$$\Omega_a=\big\{(x,y)\in\mathbb{R}^2;\ -a-\varphi(y)<x<a+\varphi(y),\ -1<y<1\big\} \tag{4}$$

and we choose φ once for all so that Ω_1 (and then Ω_a for every $a\geq 1$) be of class $C^{2,\alpha}$ with $\alpha>0$ (this means that φ is of class $C^{2,\alpha}_{loc}(-1,1)$ and that φ satisfies some compatibility conditions at ± 1). The $C^{2,\alpha}$ bounded domains Ω_a for $a\geq 1$ are all convex and axisymmetric with respect to both axes $\{x=0\}$ and $\{y=0\}$, see Figure 1.

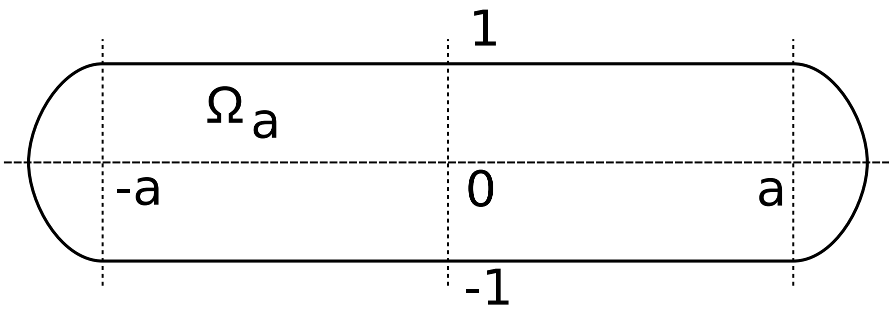

Fig. 1 The convex stadium-like domain Ω_a

Our goal is to show that the conclusion of Theorem 1.1 holds with these convex domains Ω_a and some functions f_a, when a is large enough.

Step 2: a constrained variational problem in Ω_a.

In this step, we fix a parameter $a\geq 1$. We construct a $C^{2,\alpha}(\overline{\Omega_a})$ function u_a as a minimizer of a constrained variational problem in Ω_a.

Let I_a be the functional defined in $H^1_0(\Omega_a)$ by

$$I_a(u)=\frac{1}{2}\int_{\Omega_a}|\nabla u|^2-\int_{\Omega_a}u,\ \ u\in H^1_0(\Omega_a).$$

It is well-known that this functional has a unique minimizer in $H^1_0(\Omega_a)$, which is the classical $C^{2,\alpha}(\overline{\Omega_a})$ solution v_a of the torsion problem

$$\begin{cases}\Delta v_a+1=0 \ \text{ in } \Omega_a,\\ \qquad v_a=0 \ \text{ on } \partial\Omega_a.\end{cases} \tag{5}$$

It follows from the strong maximum principle and the definition of Ω_a that

$$0 < v_a(x,y) < \frac{1-y^2}{2} \text{ for all } (x,y) \in \Omega_a. \tag{6}$$

This function v_a is also known to be quasiconcave in Ω_a, see [38].

We are then going to replace v_a by a function u_a which minimizes the functional I_a over a nonlinear subset of $H_0^1(\Omega_a)$ and which will be our non-quasiconcave candidate for a problem of the type (2).

To do so, let us now define

$$U_a = \Big\{ u \in H_0^1(\Omega_a);\ \int_{\Omega_a} g(u) = 1 \Big\}.$$

Since the Lebesgue measure $|\Omega_a|$ of Ω_a is larger than 1, the set U_a is not empty: for instance, by continuity of the map $\mathbb{R} \ni t \mapsto \int_{\Omega_a} g(tv_a)$, there is a real number $t_a \in (0,+\infty)$ such that $t_a v_a \in U_a$. Furthermore, it is straightforward to check, using Poincaré's inequality together with Rellich's and Lebesgue's theorems, that the minimum of the functional I_a over the set U_a is reached, by a function $u_a \in U_a$, that is

$$I_a(u_a) = \min_{u \in U_a} I_a(u).$$

We observe that $g'(u_a) \in L^\infty(\Omega_a)$ is not the zero function. Otherwise, the gradient $\nabla g(u_a)$ of the $H^1(\Omega_a)$ function $g(u_a)$ would be equal to $\nabla g(u_a) = g'(u_a)\nabla u_a = 0$ a.e. in Ω_a and, by definition of U_a, $g(u_a)$ would then be equal to the positive constant $1/|\Omega_a|$ a.e. in Ω_a. Due to (3), there would then exist $m > 0$ such that $u_a \ge m$ a.e. in Ω_a, contradicting the fact that $u_a \in H_0^1(\Omega_a)$ has a zero trace on $\partial\Omega_a$. Hence, $g'(u_a)$ cannot be the zero function and the differential of the map $H_0^1(\Omega_a) \ni u \mapsto \int_{\Omega_a} g(u)$ is not zero at u_a.

From the Euler-Lagrange formulation and elliptic regularity theory, any such minimizer u_a is then a classical $C^{2,\alpha}(\overline{\Omega_a})$ solution of an equation of the type

$$\begin{cases} \Delta u_a + f_a(u_a) = 0 & \text{in } \Omega_a, \\ u_a = 0 & \text{on } \partial\Omega_a, \end{cases} \tag{7}$$

where

$$f_a(s) = 1 + \mu_a g'(s) \text{ for } s \in \mathbb{R}$$

and $\mu_a \in \mathbb{R}$ is a Lagrange multiplier. Observe that the function f_a is of class $C^\infty(\mathbb{R})$. Furthermore,

$$\Delta(u_a - v_a) = -\mu_a g'(u_a)$$

has a constant sign in Ω_a, since g' is nonnegative. As a consequence of the maximum principle, the function $u_a - v_a$ itself has a constant sign in Ω_a. But

$$\max_{\overline{\Omega_a}} u_a > 1 \tag{8}$$

because of (3) and by definition of U_a. Therefore, from (6), the function v_a cannot majorize u_a. The strong maximum principle finally implies that

$$0 < v_a(x,y) < u_a(x,y) \text{ for all } (x,y) \in \Omega_a. \tag{9}$$

Thus, the function u_a is a classical solution of the problem (2) in Ω_a with the function f_a. Notice also that the sign of $\Delta(u_a - v_a)$ is therefore nonpositive and, since u_a and v_a are not identically equal, one has $\mu_a > 0$. In particular,

$$f_a(s) \geq 1 \text{ for all } s \in \mathbb{R}. \tag{10}$$

On the other hand, since $f_a(s) = 1$ for all $s \geq 2$ because of (3), the maximum principle also yields

$$u_a(x,y) < \frac{1-y^2}{2} + 2 \text{ for all } (x,y) \in \Omega_a. \tag{11}$$

The uniqueness of the minimizer u_a of I_a in the set U_a is not clear, and is anyway not needed in the sequel. However, we point out an important geometrical property fulfilled by u_a, which will be used in the next step. Namely, since Ω_a is convex and symmetric with respect to the axes $\{x = 0\}$ and $\{y = 0\}$, it follows from [18] that u_a is even in x and y and is decreasing with respect to $|x|$ and $|y|$.

In the sequel, we are going to show that, for a large enough, the conclusion of Theorem 1.1 holds with Ω_a, f_a and u_a, that is, the minimizers u_a have some non-convex superlevel sets. Notice that f_a satisfies (10), as stated in Theorem 1.1.

Before going further on, we also point out that the solution v_a of the torsion problem (5) also solves the same equation (2) as u_a, with f_a in Ω_a, because of (6) and the fact that $f_a = 1$ on $[0,1] \supset [0,1/2]$ due to (3). Therefore, problem (2) with f_a in Ω_a admits the solution v_a, which is always quasiconcave by [38] applied to (5), whereas the solutions u_a will be proved to be non-quasiconcave for a large.

Step 3: a priori estimates of the size of a superlevel set of the functions u_a.

In this step, we study the location of the superlevel sets

$$\omega_a = \big\{(x,y) \in \Omega_a;\ u_a(x,y) > 1\big\}$$

of the minimizers u_a of I_a in U_a when a is large. From (8) and the remarks of the previous step, the sets ω_a are non-empty open sets, they are all symmetric with respect to the axes $\{x = 0\}$ and $\{y = 0\}$, and they are convex with respect to both variables x and y.

The key-point in this step is to show a uniform control of the size of the sets ω_a. We first begin with a bound in the x-direction, meaning that the sets ω_a are not too elongated.

Lemma 2.1 *There exists a constant $C_x > 0$ such that*

$$0 \le \sup_{(x,y)\in\omega_a} |x| < C_x \tag{12}$$

for any $a \ge 1$ and for any minimizer u_a of I_a in U_a.

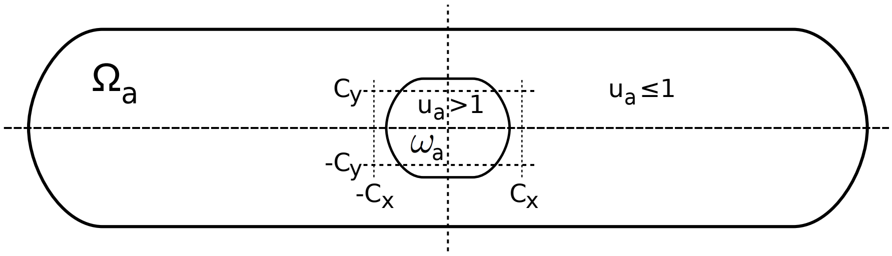

Fig. 2 The set ω_a where $u_a > 1$

Proof. The proof is divided into two main steps. We first estimate from above the quantities $I_a(u_a)$ by introducing a suitable test function in the set U_a, which is not too far from the one-dimensional function $y \mapsto (1-y^2)/2$. Then, we estimate $I_a(u_a)$ from below by observing that if $u_a(x,0)$ is larger than 1 then the contribution of $u_a(x,\cdot)$ to $I_a(u_a)$ in the section $\Omega_a \cap (\{x\} \times \mathbb{R})$ will be uniformly larger than that of the minimizer $y \mapsto (1-y^2)/2$. This eventually provides a control of the size of such points x and then of the size of ω_a, independently of a.

Throughout the proof, one can assume without loss of generality that a is any real number such that $a \ge 2$ (since $\sup_{(x,y)\in\Omega_a} |x| \le a + \|\varphi\|_{L^\infty(-1,1)}$ for all $a \ge 1$ by the definition (4) of Ω_a). We consider any minimizer u_a of the functional I_a in the set U_a and we set

$$x_a = \sup_{(x,y)\in\omega_a} |x|. \tag{13}$$

Let us first bound $I_a(u_a)$ from above by using the minimality of u_a and comparing $I_a(u_a)$ with the value of I_a at some suitably chosen test function. Let w be a fixed $C^\infty(\mathbb{R}^2)$ nonnegative function such that

$$w = 0 \text{ in } \mathbb{R}^2 \backslash (-1,1)^2 \text{ and } w > 0 \text{ in } [-2/3, 2/3]^2.$$

The function w is independent of a. Let ϕ_0 be the $H_0^1(-1,1)$ function defined by

$$\phi_0(y) = \frac{1-y^2}{2} \text{ for all } y \in [-1,1]. \tag{14}$$

We point out that ϕ_0 is the unique minimizer in $H_0^1(-1,1)$ of the functional J defined by

$$J(\phi) = \frac{1}{2}\int_{-1}^{1} \phi'(y)^2 dy - \int_{-1}^{1} \phi(y) dy, \ \ \phi \in H_0^1(-1,1). \tag{15}$$

From Lebesgue's dominated convergence theorem, the function

$$G : t \mapsto \int_{(-1,1)^2} g(\phi_0(y) + t\, w(x,y))\, dx\, dy$$

is continuous in $\mathbb{R}$. Furthermore, $G(0) = 0$ from (3) and (14), and

$$\lim_{t \to +\infty} G(t) = \int_{\{w(x,y)>0\}} dx\, dy \geq \left(\frac{4}{3}\right)^2 > 1.$$

Therefore, there is $t_0 \in (0, +\infty)$, independent of a, such that

$$G(t_0) = \int_{(-1,1)^2} g(\phi_0(y) + t_0 w(x,y))\, dx\, dy = 1.$$

Let us now consider the test function w_a defined in $\overline{\Omega_a}$ by

$$w_a(x,y) = \phi_0(y)\chi_a(x) + t_0 w(x,y),$$

where $\chi_a : \mathbb{R} \to [0,1]$ is even and defined in $[0, +\infty)$ by

$$\chi_a(x) = \begin{cases} 1 & \text{if } x \in [0, a-1], \\ a - x & \text{if } x \in (a-1, a), \\ 0 & \text{if } x \geq a. \end{cases}$$

The function w_a belongs to $H_0^1(\Omega_a)$. Furthermore, since $a \geq 2$, one has

$$w_a(x,y) = \phi_0(y) + t_0 w(x,y) \ \text{ for all } (x,y) \in (-1,1)^2,$$

while

$$w_a(x,y) = \phi_0(y)\chi_a(x) \leq \phi_0(y) < 1 \ \text{ for all } (x,y) \in \Omega_a \backslash (-1,1)^2.$$

Therefore,

$$\int_{\Omega_a} g(w_a) = \int_{(-1,1)^2} g(w_a) = \int_{(-1,1)^2} g(\phi_0(y) + t_0 w(x,y))\, dx\, dy = G(t_0) = 1.$$

In other words, $w_a \in U_a$. By definition of u_a, one infers that

$$I_a(u_a) \leq I_a(w_a). \tag{16}$$

Let us now estimate $I_a(w_a)$ from above. By using the facts that the domain Ω_a is symmetric in x and that the function χ_a is even in x and by decomposing the integral $I_a(w_a)$ into three subdomains, one gets that

$$
\begin{aligned}
&I_a(w_a) \qquad (17)\\
=&\int_{(-1,1)^2} \frac{|\nabla(\phi_0(y)+t_0 w(x,y))|^2}{2}\,dx\,dy - \int_{(-1,1)^2} (\phi_0(y)+t_0 w(x,y))\,dx\,dy\\
&+2\int_{(1,a-1)\times(-1,1)} \frac{|\nabla\phi_0(y)|^2}{2}\,dx\,dy - 2\int_{(1,a-1)\times(-1,1)} \phi_0(y)\,dx\,dy\\
&+2\int_{(a-1,a)\times(-1,1)} \frac{|\nabla(\phi_0(y)\chi_a(x))|^2}{2}\,dx\,dy - 2\int_{(a-1,a)\times(-1,1)} \phi_0(y)\chi_a(x)\,dx\,dy\\
=&2(a-2)J(\phi_0)+\beta,
\end{aligned}
$$

where β is a real number which does not depend on a (it is indeed immediate to see by setting $x = x' + a$ in the last two integrals of (17) that these quantities do not depend on a). Finally, it follows from (16) and (17) that

$$
I_a(u_a) \le 2(a-2)J(\phi_0)+\beta. \qquad (18)
$$

In the second step, we bound $I_a(u_a)$ from below. On the set $\Omega_a\backslash(-a,a)\times(-1,1)$, one simply uses the fact that

$$
\int_{\Omega_a\backslash(-a,a)\times(-1,1)} \Big(\frac{|\nabla u_a|^2}{2} - u_a\Big) \ge -\int_{\Omega_a\backslash(-a,a)\times(-1,1)} \frac{5}{2} \ge -10\|\varphi\|_{L^\infty(-1,1)}
$$

from (11) and from the definition (4) of Ω_a. Therefore,

$$
\begin{aligned}
I_a(u_a) &\ge \int_{(-a,a)\times(-1,1)} \Big(\frac{|\nabla u_a|^2}{2} - u_a\Big) - 10\|\varphi\|_{L^\infty(-1,1)}\\
&\ge \int_{-a}^{a} J(u_a(x,\cdot))\,dx - 10\|\varphi\|_{L^\infty(-1,1)},
\end{aligned} \qquad (19)
$$

where the functional J has been defined in (15) and where we have used the fact that $u_a(x,\cdot)$ belongs to $H^1_0(-1,1)$ for all $x \in (-a,a)$. Remember that ϕ_0 is the (unique) minimizer of J. As a consequence,

$$
J(u_a(x,\cdot)) \ge J(\phi_0) \text{ for all } x \in (-a,a). \qquad (20)
$$

On the other hand, by definition of x_a in (13) and by convexity and symmetry of ω_a with respect to both variables x and y, it follows that $(x,0) \in \omega_a$ for all $x \in (-x_a, x_a)$, whence

$$
u_a(x,0) > 1 > \phi_0(0) \;\text{ for all } x \in (-x_a, x_a).
$$

Hence, there is a positive real number $\gamma > 0$, independent of a, such that

$$
\|u_a(x,\cdot) - \phi_0\|_{H^1(-1,1)} \ge \gamma > 0 \;\text{ for all } x \in (-x_a, x_a).
$$

By definition of ϕ_0 and from the coercivity of the functional J, one infers the existence of a positive constant $\delta > 0$, independent of a, such that

$$J(u_a(x,\cdot)) \geq J(\phi_0) + \delta \ \text{ for all } x \in (-x_a, x_a).$$

From (19) and (20), one then gets that

$$I_a(u_a) \geq 2\delta \min(x_a, a) + 2aJ(\phi_0) - 10\|\varphi\|_{L^\infty(-1,1)}. \tag{21}$$

Putting together (18) and (21) with the inequality $x_a - \|\varphi\|_{L^\infty(-1,1)} \leq \min(x_a, a)$ yields

$$2\delta(x_a - \|\varphi\|_{L^\infty(-1,1)}) + 2aJ(\phi_0) - 10\|\varphi\|_{L^\infty(-1,1)} \leq 2(a-2)J(\phi_0) + \beta,$$

where $\beta > 0$ and $\delta > 0$ are independent of a. Hence, there exists a constant $C_x > 0$, independent of a, such that $0 \leq x_a < C_x$, that is (12). The proof of Lemma 2.1 is thereby complete. □

The second lemma gives a bound from below of the "vertical" size of the sets ω_a, meaning that the sets ω_a are not too thin. We just state the lemma, which is an immediate consequence of Lemma 2.1 and the definition of U_a.

Lemma 2.2 *There exists a constant $C_y > 0$ such that*

$$0 < C_y < \sup_{(x,y)\in\omega_a} |y| \tag{22}$$

for any $a \geq 1$ and for any minimizer u_a of I_a in U_a.

Step 4: comparison of $u_a(x,y)$ with $\phi_0(y)$ when a is large.

In this step, we prove that the minimizers u_a of I_a in U_a are close to the one-dimensional profile $\phi_0(y) = (1-y^2)/2$ far away from the origin and far away from the leftmost and rightmost points of $\overline{\Omega_a}$ in the direction x.

Lemma 2.3 *For all $\varepsilon > 0$, there exist $A \geq 1$ and $M \in [0, A/2]$ such that*

$$\begin{aligned} |u_a(x,y) - \phi_0(y)| = &\Big|u_a(x,y) - \frac{1-y^2}{2}\Big| \\ &\leq \varepsilon \ \text{in} \ \big([-a+M, -M] \cup [M, a-M]\big) \times [-1,1] \ (\subset \overline{\Omega_a}), \end{aligned}$$

for all $a \geq A$ and for any minimizer u_a of I_a in U_a.

Proof. Assume that the conclusion does not hold for some $\varepsilon > 0$. Then there are some sequences $(a_n)_{n\in\mathbb{N}}$ and $(x_n, y_n)_{n\in\mathbb{N}}$ of real numbers and points in $\mathbb{R}^2$ such that

$$a_n \geq n,\ \frac{n}{2} \leq |x_n| \leq a_n - \frac{n}{2},\ |y_n| \leq 1,\ |u_{a_n}(x_n,y_n) - \phi_0(y_n)| > \varepsilon \ \text{ for all } n \in \mathbb{N}, \tag{23}$$

where u_{a_n} is a minimizer of the functional I_{a_n} in the set U_{a_n}. For each $n \in \mathbb{N}$, define

$$\overline{u}_n(x,y) = u_{a_n}(x+x_n,y) \text{ for all } (x,y) \in \overline{\Omega_{a_n}} - (x_n,0) = \big\{(x,y) \in \mathbb{R}^2;\ (x+x_n,y) \in \overline{\Omega_{a_n}}\big\}.$$

Each function $\overline{u}_n$ satisfies a semilinear elliptic equation of the type (7) in $\overline{\Omega_{a_n}} - (x_n,0)$ with a nonlinearity $f_{a_n} = 1 + \mu_{a_n} g'$ for some $\mu_{a_n} \in \mathbb{R}$. Lemma 2.1 and (9) imply that

$$0 < u_{a_n}(x,y) \le 1 \text{ for all } n \in \mathbb{N} \text{ and } (x,y) \in \Omega_{a_n} \backslash (-C_x,C_x) \times (-1,1).$$

Hence, because of (3) and (23), for every fixed $C \ge 0$, there holds

$$0 \le \overline{u}_n(x,y) \le 1 \text{ and } \Delta\overline{u}_n(x,y) + 1 = 0 \text{ for all } (x,y) \in [-C,C] \times [-1,1],$$

for all n large enough. From standard elliptic estimates up to the boundary, it follows that, up to extraction of a subsequence, the functions $\overline{u}_n$ converge in $C^2_{loc}(\mathbb{R} \times [-1,1])$ to a classical solution $\overline{u}_\infty$ of

$$\begin{cases} \Delta\overline{u}_\infty + 1 = 0 \text{ in } \mathbb{R} \times [-1,1], \\ 0 \le \overline{u}_\infty \le 1 \text{ in } \mathbb{R} \times [-1,1], \\ \overline{u}_\infty = 0 \text{ on } \mathbb{R} \times \{\pm 1\}. \end{cases}$$

Without loss of generality, one can also assume that $y_n \to y_\infty \in [-1,1]$ as $n \to +\infty$, whence

$$|\overline{u}_\infty(0,y_\infty) - \phi_0(y_\infty)| \ge \varepsilon \tag{24}$$

from (23).

On the other hand, a standard Liouville-type result implies that $\overline{u}_\infty$ is necessarily identically equal to the one-dimensional profile $\phi_0(y)$ in $\mathbb{R} \times [-1,1]$. Indeed, the function $h(x,y) = \overline{u}_\infty(x,y) - \phi_0(y)$ is bounded and harmonic in $\mathbb{R} \times [-1,1]$, and it vanishes on $\mathbb{R} \times \{\pm 1\}$. The maximum principle implies that

$$|h(x,y)| \le \eta \cos\left(\frac{\pi y}{4}\right) \cosh\left(\frac{\pi x}{4}\right)$$

for all $(x,y) \in \mathbb{R} \times [-1,1]$ and for all $\eta > 0$ (otherwise, the same inequality would hold in $\mathbb{R} \times [-1,1]$ for some $\eta^* > 0$, with equality at some point in $\mathbb{R} \times (-1,1)$, contradicting the strong maximum principle). Thus, since $\eta > 0$ can be arbitrarily small, one gets that $h(x,y) = 0$ for all $(x,y) \in \mathbb{R} \times [-1,1]$. In other words,

$$\overline{u}_\infty(x,y) = \phi_0(y) \text{ for all } (x,y) \in \mathbb{R} \times [-1,1].$$

This is in contradiction with (24) and the proof of Lemma 2.3 is thereby complete. □

Step 5: the superlevel sets of the minimizers u_a cannot be all convex when a is large enough.

In this last step, we complete the proof of Theorem 1.1. Actually, Lemma 2.2 and the one-dimensional convergence given in Lemma 2.3 will prevent any minimizer u_a of I_a in U_a from being quasiconcave when a is large enough.

Given $C_y > 0$ as in Lemma 2.2, let P, Q_a and R_a be the points of $\mathbb{R}^2$ whose coordinates are given by

$$P = (0, C_y),\ Q_a = \Big(\frac{a}{4}, \frac{C_y}{2}\Big) \text{ and } R_a = \Big(\frac{a}{2}, 0\Big)$$

for all $a \geq 1$, see Figure 3. From Lemma 2.2 and the convexity and symmetry of ω_a with respect to x and y, there holds $P \in \omega_a$, that is,

$$u_a(P) > 1$$

for any minimizer u_a of I_a in U_a. On the other hand, the point R_a belongs to Ω_a for all $a \geq 1$ by definition (4) of Ω_a and the point Q_a is at the middle of the segment $[P, R_a]$ and is thus in Ω_a too by convexity of Ω_a.

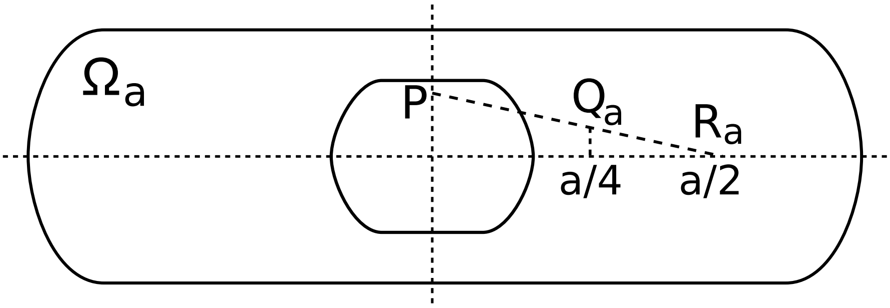

Fig. 3 The aligned points P, Q_a and R_a

Furthermore, Lemma 2.3 implies that

$$u_a(Q_a) \longrightarrow \frac{1-(C_y/2)^2}{2} = \frac{1}{2} - \frac{C_y^2}{8} \text{ and } u_a(R_a) \longrightarrow \frac{1}{2} \text{ as } a \to +\infty,$$

for any minimizer u_a of I_a in U_a. As a consequence, given any real number λ such that

$$\frac{1}{2} - \frac{C_y^2}{8} < \lambda < \frac{1}{2},$$

one has

$$u_a(Q_a) < \lambda < u_a(R_a) < 1 < u_a(P) \text{ for all } a \text{ large enough}$$

and for any minimizer u_a of I_a in U_a. Since the point Q_a belongs to the segment $[P,R_a]$, it follows that, for a large enough, the superlevel set

$$\big\{(x,y)\in\Omega_a;\ u_a(x,y)>\lambda\big\} \tag{25}$$

of any minimizer u_a of I_a in U_a is not convex, whence u_a is not quasiconcave. The proof of Theorem 1.1 is thereby complete. □

Remark 2.4 By replacing Q_a by $\widetilde{Q}_a=(\varepsilon a,(1-2\varepsilon)C_y)$ and by choosing $\varepsilon\in(0,1/2)$ arbitrarily small, it follows from the above arguments that, given any real number λ such that

$$\frac{1-C_y^2}{2}<\lambda<\frac{1}{2},$$

the superlevel set (25) of any minimizer u_a of I_a in U_a is not convex when a is large enough.

Remark 2.5 We comment on the stability properties of the solution u_a previously constructed. The functions u_a minimize the functional

$$\int_{\Omega_a}\frac{1}{2}|\nabla\phi|^2-\phi-\mu_a g(\phi)$$

in the set of $H_0^1(\Omega_a)$ functions ϕ such that

$$\int_{\Omega_a} g(\phi)=1.$$

So the space associated to the negative eigenvalues of the linearized equation

$$-\Delta\psi-f_a'(u)\psi=-\Delta\psi-\mu_a g''(u)\psi$$

is at most one-dimensional.

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