Liouville-type theorems for nonlinear elliptic and parabolic problems

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Abstract We give a survey of Liouville-type theorems and their applications for various classes of semilinear elliptic and parabolic equations and systems.

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1 Introduction

1.1 Motivation and classical results: Fujita, Gidas-Spruck, Liouville

The past few decades have seen intensive development of Liouville type nonexistence theorems for elliptic and parabolic problems (equations and systems). At the same time, these have emerged as a fundamental tool for many applications to the qualitative properties of solutions of these problems. The aim of these notes is to summarize some of the main results and their applications. We shall also emphasize a number of methods for the derivation of Liouville type theorems (sometimes with only a sketch of proof, though). In view of the huge existing literature and the large variety of problems treated, we stress that no attempt to exhaustivity is made. We refer to, e.g., [31], [32] for further references.

In all this article, p is a real number with $p>1$. Consider the semilinear parabolic equation

$$u_t - \Delta u = u^p. \tag{1}$$

The following two results are classical and fundamental. The first one is essentially due to Fujita [10], except for the critical case (see [15], [41], [31] and the references therein). The so-called Fujita exponent is defined by

$$p_F = 1 + 2/n.$$

Theorem 1.1 *Equation* (1) *does not admit any positive global classical solution in* $\mathbb{R}^n \times (0,\infty)$ *if and only if* $p \le p_F$.

Theorem 1.1 remains even valid for distributional solutions (see e.g. [31]). The second result, which concerns the corresponding stationary equation

$$-\Delta u = u^p, \tag{2}$$

is the celebrated elliptic Liouville-type theorem of Gidas and Spruck [12] (see also [3], [5]). We recall that the Sobolev exponent is given by

$$p_S := \begin{cases} \infty, & \text{if } n \le 2, \\ (n+2)/(n-2), & \text{if } n > 2. \end{cases}$$

Theorem 1.2 *Equation* (2) *does not admit any positive classical solutions in* $\mathbb{R}^n$ *if and only if* $p < p_S$.

Extensions and applications of both results have received considerable attention in the last 30 years. Although a natural question, parabolic Liouville-type theorems for equation (2) have not been as intensively studied until recently and are up to now not yet fully understood. More precisely, the question is the following:

If one now considers positive (classical) solutions of $u_t - \Delta u = u^p$ that are global for both positive and negative time, i.e. solutions on the whole space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, can one prove nonexistence for a larger range of p's than in the Fujita problem ?

The exponent p_S is a natural candidate for the dividing line between existence and nonexistence. On the other hand, like for Fujita-type and elliptic Liouville-type results, it is also useful to consider the same question on a half-space. As it will turn out, we shall see in Section 3 that such results have interesting applications in the study of a priori estimates and blow-up singularities of solutions.

1.2 Equations vs. inequalities – a first method: rescaled test-functions

The Fujita result remains true for *supersolutions* (see e.g. [21], [31]), namely:

Theorem 1.3 *The inequality*

$$u_t - \Delta u \geq u^p, \quad x \in \mathbb{R}^n,\ t > 0$$

does not admit any positive classical solutions if and only if $p \leq p_F$.

In this respect it can be considered as the parabolic analogue of the following well-known elliptic property, due to Gidas [11]. To this end we introduce the so-called Serrin's exponent:

$$p_{sg} := \begin{cases} \infty, & \text{if } n \leq 2, \\ n/(n-2), & \text{if } n > 2, \end{cases}$$

which is critical for the existence of radial *singular* solutions of the form $cr^{-2/(p-1)}$.

Theorem 1.4 *The inequality*

$$-\Delta u \geq u^p, \quad x \in \mathbb{R}^n$$

does not admit any positive classical solutions if and only if $p \leq p_{sg}$.

Both the Fujita and the Gidas result, namely Theorems 1.3 and 1.4, can (nowadays) be proved by a rather simple technique of rescaled test-functions (see e.g. [21], [31]). Namely, one tests the equation with functions of the form

$$\phi(x/R) \quad \text{or} \quad \psi(t/R^2)\phi(x/R),$$

where ϕ, ψ are suitable compactly supported smooth functions. Then, after integration by parts and use of Hölder's inequality, one obtains that

$$\int_{\mathbb{R}^n} u^p\,dx = 0 \quad \text{or} \quad \int_0^\infty \int_{\mathbb{R}^n} u^p\,dxdt = 0$$

by letting $R \to \infty$ (the critical case $p = p_{sg}$ or $p = p_F$ requires a slightly more delicate additional argument).

The full Gidas-Spruck theorem is considerably more difficult (in the complementary range $(p_{sg},\, p_S)$). It can be proved either by Bochner formula and hard integral estimates (original proof of [12], see also [3]) or by Kelvin transform and moving planes [5]. See also [36] and [4], for alternative proofs based on moving-spheres. We shall see that the parabolic Liouville case is equally or even more delicate.

2 Liouville-type theorems for the nonlinear heat equation

2.1 Results and conjectures

Let us first consider the case of radial solutions, for which we have the following result in the optimal range [26].

Theorem 2.1 *Let $1 < p < p_S$. Then the equation*

$$u_t - \Delta u = u^p, \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R} \tag{3}$$

has no positive, radial, bounded classical solution.

Theorem 2.1 is optimal in view of the existence of positive radial stationary solutions for $n \geq 3$ and $p \geq p_S$. Moreover, the boundedness assumption can be removed (see [28] and cf. Section 3 below). It is very likely that Theorem 2.1 should hold without the radial symmetry assumption, but this has not been proved so far. However, if $n \leq 2$ or under the stronger restriction $p < p_B$ if $n \geq 3$, where

$$p_B := \frac{n(n+2)}{(n-1)^2},$$

we have the following Liouville-type theorem in the general (nonradial) case. We note that $p_F < \frac{n}{n-2} < p_B < p_S$ (for $n \geq 3$). The first case is from [29] and the second case is a consequence of [2].

Theorem 2.2 *Let $p > 1$ and assume either $n \leq 2$ or $p < p_B$. Then equation* (3) *has no positive solution.*

The proofs of Theorem 2.1 and of each case of Theorem 2.2 are completely different:

- radial case: intersection-comparison with steady-states;
- case $n \leq 2$: similarity variables and rescaled energy arguments. This technique actually works for all $p < n/(n-2)_+$ ($< p_B$ if $n \geq 3$);

- case $p < p_B$: Bochner formula and hard integral estimates.

The last two techniques can be modified to apply to more general problems, including certain classes of parabolic systems. We shall now give the first two proofs. As for the third proof, an application of it to certain parabolic systems will be sketched in Section 5.3.

2.2 Radial case: proof based on zero-number

For the proof of Theorem 2.1, we need some simple preliminary observations concerning radial steady states. Let ψ_1 be the solution of the equation

$$\psi'' + \frac{n-1}{r}\psi' + |\psi|^{p-1}\psi = 0, \quad r > 0,$$

satisfying $\psi(0) = 1$, $\psi'(0) = 0$. Obviously $\psi_1''(0) < 0$. It is known that the solution is defined on some interval and it changes sign due to $p < p_S$ (cf. [12]). We denote by $r_1 > 0$ its first zero. By uniqueness for the initial-value problem, $\psi_1'(r_1) < 0$. We thus have

$$\psi_1(r) > 0 \text{ in } [0,r_1) \text{ and } \psi_1(r_1) = 0 > \psi_1'(r_1).$$

Clearly, $\psi_\alpha(r) := \alpha\psi_1(\alpha^{\frac{p-1}{2}} r)$ is the solution of (2.2) with $\psi(0) = \alpha$, $\psi'(0) = 0$, and with the first positive zero $r_\alpha = \alpha^{-\frac{p-1}{2}} r_1$. As an elementary consequence of the properties of ψ_1 we obtain the following

Lemma 2.3 *Given any $m > 0$, we have*

$$\lim_{\alpha\to\infty} \bigl(\sup\{\psi_\alpha'(r) : r \in [0,r_\alpha] \textit{ is such that } \psi_\alpha(r) \le m\}\bigr) = -\infty.$$

We shall use the well-known properties of the zero-number of the difference of two solutions, in particular the nonincreasing property (see e.g. [31]).

Proof of Theorem 2.1. The proof is by contradiction. Assume that u is a positive, bounded classical solution of (3), $u(x,t) = U(r,t)$, where $r = |x|$. By the boundedness assumption and parabolic estimates, U and U_r are bounded on $[0,\infty) \times \mathbb{R}$. It follows from Lemma 2.3 that if α is sufficiently large then $U(\cdot,t) - \psi_\alpha$ has exactly one zero in $[0,r_\alpha]$ for any t and the zero is simple.

We next claim that

$$z_\alpha(t) := z_{[0,r_\alpha]}(U(\cdot,t) - \psi_\alpha) \ge 1, \qquad t \le 0,\ \alpha > 0, \tag{4}$$

where $z_{[0,r_\alpha]}(w)$ denotes the zero number of the function w in the interval $[0,r_\alpha]$. Indeed, if not then $U(\cdot,t_0) > \psi_\alpha$ in $[0,r_\alpha]$ for some t_0. We know (see e.g. [31]) that each solution of the Dirichlet problem

$$\left.\begin{array}{ll} \bar{u}_t - \Delta \bar{u} = \bar{u}^p, & |x| < r_\alpha,\ t > t_0, \\ \bar{u} = 0, & |x| = r_\alpha,\ t > t_0, \\ \bar{u}(x,t_0) = U_0(|x|), & |x| < r_\alpha, \end{array}\right\}$$

blows up in finite time whenever $U_0 > \psi_\alpha$ in $[0, r_\alpha)$. Choosing the initial function U_0 between ψ_α and $U(\cdot,t_0)$ we conclude, by comparison, that $\bar{u}$ and u both blow up in finite time, in contradiction to the global existence assumption on u. This proves the claim.

Set

$$\alpha_0 := \inf\{\beta > 0 : z_\alpha(t) = 1 \text{ for all } t \le 0 \text{ and } \alpha \ge \beta\}.$$

In view of the above remark on large α, we have $\alpha_0 < \infty$. Also $\alpha_0 > 0$. Indeed, for small $\alpha > 0$ we have $\psi_\alpha(0) < U(0,t)$ for $t = 0$ and for $t \approx 0$. By the properties of the zero number, we can choose $t \approx 0$, $t < 0$, such that $\psi_\alpha(0) - U(\cdot,t)$ has only simple zeros and then, by (4), $z_\alpha(t) \ge 2$.

By definition of α_0 (and (4)), there are sequences $\alpha_k \to \alpha_0^-$ and $t_k \le 0$ such that

$$z_{[0,r_{\alpha_k}]}(U(\cdot,t_k) - \psi_{\alpha_k}) \ge 2, \quad k = 1,2,\ldots.$$

We get

$$z_{[0,r_{\alpha_k}]}(U(\cdot,t_k+t) - \psi_{\alpha_k}) \ge 2, \quad t \le 0,\ k = 1,2,\ldots. \tag{5}$$

This in particular allows us to assume, choosing different t_k if necessary, that $t_k \to -\infty$. By the boundedness assumption and parabolic estimates, passing to a subsequence, we may further assume that

$$u(x,t_k+t) \to v(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

with convergence in $C^{2,1}_{loc}(\mathbb{R}^n \times \mathbb{R})$. Clearly then, there is $\delta > 0$ such that for each fixed t,

$$U(\cdot,t_k+t) - \psi_{\alpha_k} \to V(\cdot,t) - \psi_{\alpha_0}$$

in $C^1[0, r_{\alpha_0} + \delta]$, where $v(x,t) = V(|x|,t)$. This and (5) imply that for each $t \le 0$, $V(\cdot,t) - \psi_{\alpha_0}$ has at least two zeros or a multiple zero in $[0, r_{\alpha_0})$. By the properties of the zero number, we may choose $t < 0$ so that $V(\cdot,t) - \psi_{\alpha_0}$ has only simple zeros (and, hence at least two of them). Since $U(\cdot,t_k+t) - \psi_{\alpha_0}$ is close to $V(\cdot,t) - \psi_{\alpha_0}$ in $C^1[0, r_{\alpha_0}]$, if k is large, it has at least two simple zeros in $[0, r_{\alpha_0})$ as well. But then, for $\alpha > \alpha_0$, $\alpha \approx \alpha_0$, the function $U(\cdot,t_k+t) - \psi_\alpha$ has at least two zeros in $[0, r_\alpha)$, contradicting the definition of α_0.

We have thus shown that the assumption $u \not\equiv 0$ leads to a contradiction, which proves the theorem. □

2.3 Nonradial case: proof based on similarity variables and energy estimates

We will now prove Theorem 2.2 for all $p < n/(n-2)_+$ in the case of bounded solutions. We shall see in Section 3.1 that the boundedness assumption can be removed as a consequence of a general principle based on a "rescaling-doubling" procedure.

The proof consists of 4 steps:

(i) First rescaling by similarity variables along a sequence of final times $T = k$
(ii) Energy estimates
(iii) Second rescaling according to the maximum points
(iv) Contradiction with the nonexistence of steady states.

Proof of Theorem 2.2 for $p < n/(n-2)_+$ in the case of bounded solutions. Assume on the contrary that there exists a positive bounded solution u of (3). Replacing u by $\tilde{u}(x,t) := \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t)$ with $\lambda = (\sup u)^{-(p-1)/2}$ we may assume

$$u(x,t) \le 1 \qquad \text{for all } x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

Denote $c_0 := u(0,0)$. For $k = 1,2,\ldots$, we rescale equation (1) by similarity variables about $T = k$ and $a = 0$ by setting $\beta = 1/(p-1)$, $s = -\log(k-t)$ for $t < k$ and

$$w_k(y,s) := (k-t)^\beta u(y\sqrt{k-t},t) = e^{-\beta s}u(e^{-s/2}y, k-e^{-s}).$$

By direct computation, we see that $w = w_k$ satisfies

$$w_s - \Delta w + \frac{y}{2}\cdot\nabla w = w^p - \beta w, \qquad y \in \mathbb{R}^n,\ s \in \mathbb{R}. \tag{6}$$

Then, setting also $s_k := -\log k$, we have

$$w_k(0,s_k) = k^\beta c_0$$

and

$$\|w_k(\cdot,s)\|_\infty \le e^{2\beta}k^\beta \qquad \text{for } s \in [s_k-2,\infty). \tag{7}$$

Define the weighted energy functional

$$\mathscr{E}(w) := \int_{\mathbb{R}^n}\Big(\frac{1}{2}|\nabla w|^2 + \frac{\beta}{2}w^2 - \frac{1}{p+1}|w|^{p+1}\Big)\rho\,dy, \qquad \rho(y) := e^{-|y|^2/4},$$

and set $\mathscr{E}_k(s) := \mathscr{E}(w_k(s))$. By direct computation, we have

$$\frac{d}{ds}\mathscr{E}\big(w(s)\big) = -\int_{\mathbb{R}^n} w_s^2\rho\,dy \le 0, \tag{8}$$

and

$$\frac{1}{2}\frac{d}{ds}\int_{\mathbb{R}^n} w^2\rho\,dy = -2\mathscr{E}\big(w(s)\big) + \frac{p-1}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}\rho\,dy$$
$$\geq -2\mathscr{E}\big(w(s)\big) + c\Big(\int_{\mathbb{R}^n} w^2\rho\,dy\Big)^{(p+1)/2}. \tag{9}$$

This implies

$$\mathscr{E}_k(s) \geq 0, \quad s \in \mathbb{R} \tag{10}$$

(since otherwise $\mathscr{E}_k(s) \geq 0$ would be negative for all large s and $\int_{\mathbb{R}^n} w^2\rho\,dy$ would blow up in finite time). Multiplying (6) with $w = w_k$ by ρ, integrating over $y \in \mathbb{R}^n$ and using Jensen's inequality yields

$$\frac{d}{ds}\int_{\mathbb{R}^n} w_k(y,s)\rho(y)\,dy + \beta\int_{\mathbb{R}^n} w_k(y,s)\rho(y)\,dy = \int_{\mathbb{R}^n} w_k^p(y,s)\rho(y)\,dy$$
$$\geq C_{n,p}\Big(\int_{\mathbb{R}^n} w_k(y,s)\rho(y)\,dy\Big)^p,$$

where $C_{n,p} := (4\pi)^{-n(p-1)/2}$. It follows that

$$\int_{\mathbb{R}^n} w_k(y,s)\rho(y)\,dy \leq \tilde{C}_{n,p} \tag{11}$$

(since otherwise $\int_{\mathbb{R}^n} w\rho\,dy$ would blow up in finite time), hence

$$\int_\sigma^{s_k}\int_{\mathbb{R}^n} w_k^p(y,s)\rho(y)\,dy\,ds \leq \tilde{C}_{n,p}(1+\beta(s_k-\sigma)), \quad \sigma < s_k, \tag{12}$$

where $\tilde{C}_{n,p} = (\beta/C_{n,p})^\beta$. Now (7), (8), (9), (11) and (12) guarantee

$$\begin{aligned}
2\mathscr{E}_k(s_k-1) &\leq 2\int_{s_k-2}^{s_k-1}\mathscr{E}_k(s)\,ds \leq 2\int_{s_k-2}^{s_k}\mathscr{E}_k(s)\,ds\\
&\leq \frac{1}{2}\int_{\mathbb{R}^n} w_k^2(y,s_k-2)\rho(y)\,dy + \frac{p-1}{p+1}\int_{s_k-2}^{s_k}\int_{\mathbb{R}^n} w_k^{p+1}(y,s)\rho(y)\,dy\,ds\\
&\leq e^{2\beta}k^\beta\Big(\int_{\mathbb{R}^n} w_k(y,s_k-2)\rho(y)\,dy + \int_{s_k-2}^{s_k}\int_{\mathbb{R}^n} w_k^p(y,s)\rho(y)\,dy\,ds\Big)\\
&\leq 2C(n,p)k^\beta,
\end{aligned}$$

where $C(n,p) := e^{2\beta}\tilde{C}_{n,p}(1+\beta)$, hence $\mathscr{E}_k(s_k-1) \leq C(n,p)k^\beta$. This estimate, (8) and (10) guarantee

$$\int_{s_k-1}^{s_k}\int_{\mathbb{R}^n}\Big|\frac{\partial w_k}{\partial s}(y,s)\Big|^2\rho(y)\,dy\,ds = \mathscr{E}(s_k-1) - \mathscr{E}(s_k) \leq C(n,p)k^\beta. \tag{13}$$

Next denote $\lambda_k := k^{-1/2}$ and set

$$v_k(z,\tau) := \lambda_k^{2/(p-1)} w_k(\lambda_k z, \lambda_k^2\tau + s_k), \qquad z \in \mathbb{R}^n,\ -k \leq \tau \leq 0.$$

Then $0 < v_k \le e^{2\beta}$, $v_k(0,0) = c_0$,

$$\frac{\partial v_k}{\partial \tau} - \Delta v_k - v_k^p = -\lambda_k^2 \Big(\frac{1}{2} z \cdot \nabla v_k + \beta v_k\Big)$$

and, denoting $\alpha := -n + 2 + 4/(p-1)$ and using (13) we also have

$$\int_{-k}^0 \int_{|z|<\sqrt{k}} \Big|\frac{\partial v_k}{\partial \tau}(z,\tau)\Big|^2 dz\,d\tau = \lambda_k^\alpha \int_{s_k-1}^{s_k} \int_{|y|<1} \Big|\frac{\partial w_k}{\partial s}(y,s)\Big|^2 dy\,ds$$
$$\le C(n,p) e^{1/4} k^{-\alpha/2+\beta} \to 0 \quad \text{as } k \to \infty.$$

Now, by using parabolic estimates, one can show that v_k converges (up to a subsequence) to a positive solution $v = v(z)$ of the problem $\Delta v + v^p = 0$ in $\mathbb{R}^n$, which contradicts Theorem 1.2. □

Remark 2.1 *Notice that the explicit formula*

$$v_k(z,\tau) = e^{-\beta\tau/k} u(e^{-\tau/2k} z, k(1 - e^{-\tau/k}))$$

guarantees $v_k \to u$.

3 Applications of parabolic Liouville theorems

3.1 Results

Liouville type theorems such as Theorem 2.2 have many applications. Since its optimal range of validity is presently unknown, let us thus assume that:

Bounded parabolic Liouville theorem is true for a given $p \in (1, p_S)$
(i.e. there exist no bounded positive solution of $u_t - \Delta u = u^p$ in $\mathbb{R}^{n+1}$). (14)

We have the following result from [28], which reduces a number of a priori (universal) bounds and singularity estimates to a Liouville theorem.

Theorem 3.1 *Assume* (14) *and set* $\beta := \frac{1}{p-1}$. *Then we have:*
(i) Blow-up rate estimates (final and initial), and with universal constants

$$u \ge 0 \text{ solution of (1) in } \mathbb{R}^n \times (0,T) \implies u \le C(n,p)\big[t^{-\beta} + (T-t)^{-\beta}\big].$$

(ii) The Liouville property in (14) *is true without boundedness assumption.*
(iii) Blow-up rate estimates (final and initial) in any smooth domain

$$u \ge 0 \text{ solution of (1) in } \Omega \times (0,T) \text{ with zero B.C.} \implies u \le C(p,\Omega)\big[1 + t^{-\beta} + (T-t)^{-\beta}\big].$$

(iv) Universal bounds away from $t = 0$ *for global solutions in any smooth domain*

$$u \geq 0 \text{ solution of } (1) \text{ in } \Omega \times (0,T) \text{ with zero B.C.} \implies u \leq C(p,\Omega)\big[1+t^{-\beta}\big].$$

(v) Decay estimates for all global solutions in $\mathbb{R}^n$

$$u \geq 0 \text{ solution of } (1) \text{ in } \mathbb{R}^n \times (0,\infty) \implies u \leq C(n,p)t^{-\beta}.$$

(vi) Local universal estimate in space and time

$$u \geq 0 \text{ solution of } (1) \text{ in } \Omega \times (0,T) \implies u \leq C(n,p)\big[t^{-\beta}+(T-t)^{-\beta}+(\mathrm{dist}(x,\partial\Omega))^{-2\beta}\big].$$

Remark 3.1 *(i) Similar singularity estimates in space for the elliptic problem can be deduced from the elliptic Liouville Theorem 1.2 (see [27]).*

(ii) Similar results can be obtained for parabolic systems (see [29], [25]), as well as for degenerate parabolic and elliptic problems (porous medium, p-Laplacian – see e.g. [27]).

(iii) We stress that the reduction principle given by Theorem 3.1 works at a purely local level (cf. (vi) and also (i), (v)). Moreover, its proof does not involve any energy argument (so that the principle is applicable to problems without variational structure, as well as more general nonlinearities).

(iv) Earlier a priori estimates at global level, deduced from rescaling arguments (without doubling), appeared in [13] for elliptic boundary value problems and in [14] for global solutions of the parabolic initial-boundary value problem. The latter was based, in addition to rescaling, on energy arguments and reduction to the elliptic Liouville theorem 1.2, leading to nonuniversal estimates.

(v) Such Liouville type theorems have other applications, such as: existence of periodic solutions (for associated periodic problems), existence of optimal controls (for associated control problems), nonuniqueness for singular initial data, existence of nontrivial equilibria by dynamical methods. See [31] and the references therein for more details.

(vi) A different parabolic Liouville-type theorem was obtained in [19] for $1 < p < p_S$. *It asserts that all entire bounded solutions of the rescaled equation* (6) *are spatially homogeneous. This has an equivalent formulation in terms of ancient solutions of the original equation* (1)*, and can be used to obtain refined blow-up estimates solutions of* (1) *near the blow-up time.*

3.2 Sketch of proof of Theorem 3.1(i) (initial-final blow-up estimate in $\mathbb{R}^n$)

The proof is based on a rescaling procedure (similar to those in [13], [14]), combined with the following doubling lemma from [27]:

Lemma 3.2 *Let* (E,d) *be a complete metric space and let*

$$\emptyset \neq D \subset \Sigma \subset E, \quad \textit{with } \Sigma \textit{ closed.}$$

Let $M : D \to (0,\infty)$ be bounded on compact subsets of D. Set $\Gamma = \Sigma \setminus D$, fix a real $k > 0$ and assume that $y \in D$ satisfies

$$M(y)\,\mathrm{dist}(y,\Gamma) > 2k.$$

Then there exists $x \in D$ such that

$$M(x)\,\mathrm{dist}(x,\Gamma) > 2k, \qquad M(x) \geq M(y),$$

and

$$M(z) \leq 2M(x) \quad \textit{for all } z \in D \cap \overline{B}_E\big(x, kM^{-1}(x)\big).$$

The proof of the doubling lemma (see [27]) is by contradiction and induction (in the spirit of the proof of Baire's lemma).

Sketch of proof of Theorem 3.1(i). Denote $X = (x,t)$, $Y = (y,s)$ and consider the parabolic distance

$$d_P(X,Y) = |x-y| + |t-s|^{1/2}.$$

The result will follow from more general estimate for solutions u on domains $D \subset \mathbb{R}^{n+1}$:

$$u(x,t) \leq C(n,p) d_P^{-2/(p-1)}((x,t),\partial D), \quad (x,t) \in D. \tag{15}$$

Indeed, choosing $D = (0,T) \times B_R$, (15) will imply

$$u(x,t) \leq C(n,p)\big[t^{-1/(p-1)} + (T-t)^{-1/(p-1)} + (R-|x|)^{-2/(p-1)}\big],$$

hence the desired estimate by letting $R \to \infty$.

Assume (15) fails. Then there exist sequences D_k, u_k, $Y_k \in D_k$ s.t.

$$M_k := u_k^{(p-1)/2} \quad \text{satisfy} \quad M_k(Y_k) > 2k\, d_P{}^{-1}(Y_k, \partial D_k).$$

By the Doubling Lemma with $E = \mathbb{R}^{n+1}$, applied with $\Sigma = \Sigma_k = \overline{D}_k$, $D = D_k$ and $\Gamma = \partial D_k$, there exists $X_k = (x_k, t_k) \in D_k$ such that

$$M_k(X_k) > 2k\, d_P{}^{-1}(X_k, \partial D_k),$$

and

$$M_k(X) \leq 2M_k(X_k) \quad \text{in } \big\{X;\ d_P(X,X_k) \leq \underbrace{kM_k^{-1}(X_k)}_{\leq \frac{1}{2} d_P(X_k,\partial D_k)}\big\}. \tag{16}$$

Now set $\lambda_k = M_k^{-1}(X_k)$ and rescale u_k as

$$v_k(y,s) := \lambda_k^{2/(p-1)} u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s),$$

which solves the same eqn. with $v_k(0,0) = 1$. Moreover, (16) implies

$$v_k^{(p-1)/2}(y,s) \le 2 \quad \text{for } |y| + \sqrt{|s|} \le k.$$

Local parabolic estimates guarantee that (up to a subsequence), v_k converges to a nontrivial bounded solution v of (1) on $\mathbb{R}^{n+1}$, contradicting the assumed Liouville property (14). □

Remark 3.2 *The Dirichlet case can be treated by a modification of the above argument provided we also have the Liouville property in the half-space $\mathbb{R}^n_+ \times \mathbb{R}$. The latter (for a given $p > 1$) is a consequence of the Liouville theorem in $\mathbb{R}^n \times \mathbb{R}$ and a moving planes argument (see [28] for details).*

4 Elliptic systems

Many Liouville type results are available for elliptic systems. We shall present some of them and illustrate different methods.

4.1 Elliptic systems I: Lane-Emden

Let us consider the Lane-Emden system:

$$\begin{cases} -\Delta u = v^p, & x \in \mathbb{R}^n \\ -\Delta v = u^q, & x \in \mathbb{R}^n \end{cases} \tag{17}$$

where $p, q > 0$. The so-called Sobolev hyperbola is defined by

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}.$$

The following result [20] shows that the Sobolev hyperbola is the sharp dividing line for the existence of positive solutions in the radial case.

Theorem 4.1 *System* (17) *admits positive radial solutions if and only if*

$$\frac{1}{p+1} + \frac{1}{q+1} \le \frac{n-2}{n}.$$

.

It is conjectured that the Liouville property should be true without radial restriction. It has been known so far only in dimensions $n \le 4$ ([38]):

Theorem 4.2 *Assume*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}.$$

If $n \le 4$, then (17) *admits no nontrivial nonnegative classical solution.*

Remark 4.1 *(Previous and other results)*

(i) The Liouville property was proved before in [37] for $n = 3$ *and polynomially bounded solutions.*

(ii) The assumption of polynomial bound for $n = 3$ *was removed in [27] (consequence of [37] and of doubling argument).*

(iii) For $n \geq 5$*, only partial results are available. See for instance [17] (biharmonic case (*$p = 1$*,* $q < (n+4)/(n-4)$*), and also [9], [36], [4], [38], [16].*

Ideas of proof of Theorem 4.2. By a doubling argument, it is enough to consider bounded solutions. The proof is done in two steps:

Step 1. *Basic a priori bounds.* Denote by $\alpha = 2(p+1)/(pq-1)$, $\beta = 2(q+1)/(pq-1)$ the scaling exponents of system (17). By the rescaled test-functions method (cf. Section 1.2), we obtain

$$\int_{B_R} u^q \leq CR^{n-q\alpha} \quad \text{and} \quad \int_{B_R} v^p \leq CR^{n-p\beta}, \qquad R > 0.$$

Step 2. *Maximum principle argument.* Assume $p \geq q$ without loss of generality. Then, by a suitable maximum principle argument, one can show that

$$v^{p+1} \leq \frac{p+1}{q+1} u^{q+1}, \quad x \in \mathbb{R}^n.$$

Step 3. *Pohozaev-type identity.* Let us write $u(x) = u(r,\theta)$ in spherical coordinates. By a Pohozaev-type multiplier argument, one obtains the identity

$$\begin{aligned}
&\Big(\frac{n}{p+1} - a_1\Big) \int_{B_R} v^{p+1} + \Big(\frac{n}{q+1} - a_2\Big) \int_{B_R} u^{q+1} \\
&= R^n \int_{S^{n-1}} \Big[\frac{v^{p+1}(R,\theta)}{p+1} + \frac{u^{q+1}(R,\theta)}{q+1}\Big]\, d\theta \\
&\quad + R^{n-1} \int_{S^{n-1}} \big[a_1 u_r v + a_2 u v_r\big](R,\theta)\, d\theta \\
&\quad + R^n \int_{S^{n-1}} \big[u_r v_r - R^{-2} \nabla_\theta u \cdot \nabla_\theta v\big](R,\theta)\, d\theta
\end{aligned}$$

for any $a_1, a_2 \in \mathbb{R}$ with $a_1 + a_2 = n-2$. Moreover, one can choose a_1, a_2 such that $\frac{n}{p+1} - a_1 > 0$ and $\frac{n}{q+1} - a_2 > 0$ whenever (p,q) is below Sobolev hyperbola. From this identity, defining the volume and surface terms:

$$F(R) := \int_{B_R} u^{q+1},$$

$$G_1(R) = R^n \int_{S^{n-1}} u^{q+1}(R,\theta)\, d\theta, \quad G_2(R) = R^n \int_{S^{n-1}} \Big(|D_x u| + \frac{u}{R}\Big)\Big(|D_x v| + \frac{v}{R}\Big)\, d\theta,$$

we have

$$F(R) \leq CG_1(R) + CG_2(R).$$

Step 4. *Feedback argument*

The idea is to estimate the surface terms by combining:
• Basic a priori estimates above
• Sobolev imbeddings and interpolation inequalities on S^{n-1}
• Elliptic estimates in B_R
• Averaging in r and measure argument.

In this way, one can prove that

$$F(R) \leq CG_1(R) + CG_2(R) \leq CR^{-a}F^b(4R), \quad \text{along some sequence } R = R_i \to \infty,$$

for some powers a, b, which satisfy $a > 0$ and $b < 1$ whenever the pair (p,q) is below the Sobolev hyperbola and satisfies an additional condition (which is always true if $n \leq 4$).

Taking a suitable subsequence and using the boundedness of u, if follows that $u \equiv 0$, hence $v \equiv 0$. □

Remark 4.2 *A heuristic explanation of the dimension restriction $n \leq 4$ (say for $p = q$) can be given as follows. First recall that, due to the standard elliptic theory, bootstap/interpolation from L^r estimate is possible provided $r > r_c := d(p-1)/2$, where d is the underlying space dimension. Here our basic a priori estimate (cf. Step 1) is in L^p (on n dimensional balls). But by means of the Pohozaev-type identity, this estimate can be "projected" onto the unit sphere, whose dimension is $d = n-1$. This allows for a crucial gain, since*

$$p > (n-1)(p-1)/2 \Longleftrightarrow p < (n-1)/(n-3) \quad (= p_{sg}(n-1))$$

and

$$p < (n+2)/(n-2) \leq (n-1)/(n-3) \Longleftrightarrow n \leq 4.$$

4.2 Elliptic systems II: positive self-interaction

We now turn to the following class of Schrödinger-type systems:

$$-\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_i^q u_j^{q+1}, \tag{18}$$

where $B = (\beta_{ij})$ is a real $m \times m$ symmetric matrix with positive diagonal entries, $m \geq 2$, $q > 0$. We denote the total degree by $p := 2q + 1$.

We begin with the cooperative case, with the following result in the optimal range [36]:

Theorem 4.3 *Assume $\beta_{ii} > 0$, $\beta_{ij} \geq 0$ and $p < p_S$. Then* (18) *has no positive classical solution.*

Method of proof: moving spheres. □

We now consider the case where some off-diagonal coefficients may be negative. The following matrix property plays an important role.

Definition. B is *strictly copositive* if

$$\sum_{1\le i,j\le m} \beta_{ij} z_i z_j > 0, \quad \text{for all } z \in [0,\infty)^m,\ z \neq 0.$$

We have the following necessary condition [40] for the Liouville property to hold.

Theorem 4.4 *Assume $p < p_S$. If B is not strictly copositive, then* (18) *has a nontrivial nonnegative bounded solution.*

Method of proof: Construction of a periodic solution by variational techniques. □

The following result (cf. [35], [39]) shows that the copositivity condition is (necessary and) sufficient under suitable assumption on p.

Theorem 4.5 *Let B be strictly copositive. Assume in addition that either*

$$p < p_S, \quad n \le 4, \quad m = 2$$

or

$$p < n/(n-2)_+, \quad m \ge 3.$$

Then (18) *has no positive bounded classical solution.*

Ideas of proof. It is based on modifications of the ideas in the proof of Theorem 4.2. For $m \ge 3$, the above ideas are combined with a device from [40], which uses a test-function of the form u_i^{-q}. □

Remark 4.3 *(i) The problem remains open for $m,n \ge 3$ in the range $n/(n-2) \le p < p_S$.*

(ii) The boundedness assumption can be partially relaxed

(iii) Earlier results were obtained in [7], [40].

4.3 Elliptic systems III: negative self-interaction

We now consider the system

$$\begin{cases} -\Delta u = u^q v^m [a v^r - c u^r], & x \in \mathbb{R}^n \\ -\Delta v = v^q u^m [b u^r - d v^r], & x \in \mathbb{R}^n \end{cases} \tag{19}$$

where $m,q \geq 0$, $r > 0$, $a,b,c,d \geq 0$, with total degree $p := q+m+r$. Such systems enter in models of problems with negative self-interaction, and are thus in a sense opposite to the case studied in Section 4.2. The typical cases are the following:

• **Schrödinger:** $m = 0$, $r = q+1$

$$-\Delta u_i = \sum_{j=1}^{2} \beta_{ij} u_i^q u_j^{q+1}$$

• **Lotka-Volterra:** $m = 0$, $q = r = 1$

$$\begin{cases} -\Delta u = u(av - cu), & x \in \mathbb{R}^n \\ -\Delta v = v(bu - dv), & x \in \mathbb{R}^n \end{cases}$$

• **Reversible chemical reactions:** $m = q = r = 1$

$$\begin{cases} -\Delta u = uv(av - cu), & x \in \mathbb{R}^n \\ -\Delta v = uv(bu - dv), & x \in \mathbb{R}^n \end{cases}$$

These are reactions of the form

$$A + 2B \underset{k_2}{\overset{k_1}{\rightleftarrows}} 2A + B.$$

We here consider the approach based on the reduction to a scalar Liouville-type theorem, by showing the proportionality of components (or synchronisation). The following result is due to [23].

Theorem 4.6 *Assume* $r \geq |q-m|$,

$$ab > cd \qquad \text{and} \qquad q \leq n/(n-2)_+.$$

(i) Then any positive bounded solution of (19) *satisfies* $u/v = Const$.
(ii) If also $p < p_S$, *then* (19) *has no positive bounded solution.*

Remark 4.4 *(i) Theorem 4.6(i) applies to (some) critical and supercritical cases. Also the boundedness assumption can be partially relaxed.*

(ii) One can show that the proportionality constant is unique.

(iii) The condition $q \leq n/(n-2)_+$ *is optimal (cf. [34], [32]). On the other hand, it can be replaced by* $m \leq 2/(n-2)_+$ *if* $c,d > 0$.

(iv) Other related results showing proportionality of components of various elliptic systems can be found in [18], [34], [1], [6], [8], [22]. See Section 5.4 for a result of this type for parabolic systems.

Sketch of proof of Theorem 4.6. **Step 1.** *Key "dissipativity" property.* We show that there exists a unique constant $K > 0$ (independent of the solution) such that

$$\forall u, v > 0, \ \ [f(u,v) - Kg(u,v)](u - Kv) \leq 0.$$

Moreover, we have $aK^r > c$.

Step 2. *Auxiliary functions.* Set

$$W = |u - Kv| \geq 0,$$

$$Z = \min(u, Kv) > 0.$$

One can show that (W,Z) is a weak solution of the *auxiliary system*

$$\begin{cases} \Delta W \geq 0 \\ -\Delta Z \geq cW^\alpha Z^q \end{cases} \qquad \text{with } \alpha = \max(m+r, 1).$$

Step 3. *Extension of Gidas' Liouville theorem for inequalities.* One can prove the following:

Lemma 4.7 *Let $0 < q \leq n/(n-2)_+$ and $V \in C(\mathbb{R}^n)$, $V \geq 0$, satisfy*

$$\liminf_{R\to\infty} R^{-n} \int_{B_{2R}\setminus B_R} V(x)\,dx > 0.$$

If $U \geq 0$ and

$$-\Delta U \geq V(x) U^q, \quad x \in \mathbb{R}^n,$$

then $U \equiv 0$.

Step 4. *Contradiction argument to prove (i).* Assume $W \not\equiv 0$. Since $\Delta W \geq 0$, it is well known that the average $\tilde{W}(R)$ of W on the sphere of radius R is nondecreasing in R. Consequently,

$$\frac{1}{|B_R|}\int_{B_R} W(x)\,dx \leq \frac{n}{R}\int_0^R \tilde{W}(r)\,dr \leq \frac{2n}{R}\int_{R/2}^R \tilde{W}(r)\,dr \leq C(n) R^{-n}\int_{B_{2R}\setminus B_R} W(x)\,dx.$$

Since $W \not\equiv 0$, it follows easily from the mean-value inequality that

$$\liminf_{R\to\infty} R^{-n}\int_{B_{2R}\setminus B_R} W(x)\,dx > 0, \quad \text{hence} \ \liminf_{R\to\infty} R^{-n}\int_{B_{2R}\setminus B_R} W^\alpha(x)\,dx > 0$$

by Jensen's inequality. Since $-\Delta Z \geq cW^\alpha Z^q$, it suffices to apply Lemma 4.7 with $V = cW^\alpha$.

Step 5. *Proof of (ii).* It suffices to note that $v = Ku$ and $A := aK^r - c > 0$ imply $-\Delta u = Au^p$. $\square$

5 Liouville for parabolic systems

We consider parabolic systems that can be written in the general vector form

$$\partial_t U - \Delta U = F(U), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{20}$$

where $U = (u_1, \dots, u_m),\ F = (F_1, \dots, F_m)$.

As mentioned in the previous section, many results are known in the elliptic case. In comparison, only few results are available in the parabolic case. We will review some them in the next subsections.

5.1 Low values of p

A first basic approach is to rely on Fujita type results (nonexistence of global solutions in $(0,\infty) \times \mathbb{R}^n$) which of course guarantee the Liouville property, but usually in a quite nonoptimal way in terms of exponent range.

Proposition 5.1 *Assume:*

$$F \text{ is } p\text{-coercive:} \quad \exists \xi > 0,\ \ \xi \cdot F(U) \geq c|U|^p,$$

$$1 < p \leq p_F := \frac{n+2}{n}.$$

Then system (20) *has no nontrivial entire solutions* $U \geq 0$.

Idea of proof. Apply the classical scalar Fujita result Theorem 1.1 to $z := \xi \cdot U$. □

5.2 Gradient structure-homogeneous case

The following result is due to [29].

Theorem 5.2 *Let* $G \in C^{2+\alpha}$ *for some* $\alpha > 0$ *and* $G(U) > G(0)$ *for all* $U \in [0,\infty)^m \setminus \{0\}$. *Assume*

$$\begin{aligned} &F = \nabla G, \\ &F \ \ p\text{-coercive}, \\ &F \ \text{is } p\text{-homogeneous}, \\ &1 < p < n/(n-2)_+. \end{aligned}$$

Then system (20) *has no nontrivial entire solutions* $U \geq 0$.

Idea of proof. Similar to the proof of Theorem 2.2 for $p < n/(n-2)_+$, based on a combination of similarity variables, weighted energy and rescaling. □

Remark 5.1 *Theorem 5.2 is true in the full range* $1 < p < p_S$ *if* U *is radial. The proof is based on the 1d Liouville result, combined with doubling and energy arguments, so as to reduce the parabolic Liouville property to an elliptic one (see [29] and cf. also [33]). Note that zero-number is not available for systems. On the other hand, partial related results were previously obtained in [24] for* $n = 1$ *or radial solutions.*

5.3 Gross-Pitaevskii case

We consider system (20) with nonlinearities of the form

$$f_i(U) = u_i^r \sum_{j=1}^{m} \beta_{ij} u_j^{r+1}$$

where $\beta = (\beta_{ij})$ is a symmetric matrix and $r > 0$. This system enjoys a gradient structure and is p-homogeneous with $p = 2r + 1$ The classical cubic case corresponds to $r = 1$:

$$f_i(U) = u_i \sum_{j=1}^{m} \beta_{ij} u_j^2.$$

In the case of nonnegative coefficients, the following result from [25] improves the range of p with respect to Theorem 5.2 for $n \geq 3$.

Theorem 5.3 *Assume*

$$\beta_{ii} > 0, \quad \beta_{ij} \geq 0,$$

$$1 < p < p_B := \frac{n(n+2)}{(n-1)^2}.$$

Then system (20) *has no positive (component-wise) entire solutions. In particular this is true for* $p = n = 3$.

Sketch of proof of Theorem 5.3. It is based on modifications of ideas from [2] in the scalar case, which was a parabolic modification of the elliptic proof from [12] (also [3]).

Step 1. *Basic functionals and* 1*-parameter family of inequalities* (No PDE involved !) Let

$$I(u) = \int \frac{|\nabla u|^4}{u^2} \varphi, \qquad J(u) = \int \frac{|\nabla u|^2}{u} (-\Delta u) \varphi, \qquad K(u) = \int (\Delta u)^2 \varphi$$

where $\int \equiv \int\int_Q dxdt$, $Q = B_1 \times (-1,1)$, $\varphi \in C_0^\infty(Q)$. We have, the following lemma, where "L.O.T." means that the total number of derivatives of u is less than 4, e.g. $\int |\nabla u|^2 \Delta\varphi$, $\cdots$.

Lemma 5.4 *Let* $0 < u \in C^{1,2}(Q)$ *(real valued),* $0 \le \varphi \in C_0^\infty(Q)$ *and* $\alpha \in \mathbb{R}$. *Then we have*

$$\alpha J(u) - K(u) + A(\alpha) I(u) \le L.O.T.,$$

where $A(\alpha) = \dfrac{n}{n+2}\alpha\Big(1 - \dfrac{(n-1)^2}{n(n+2)}\alpha\Big)$.

Proof (Sketch of proof of Lemma 5.4). It is based on the following three ingredients:

(i) the Bochner formula

$$\frac{1}{2}\Delta|\nabla v|^2 = \nabla v \cdot \nabla(\Delta v) + |D^2 v|^2;$$

(ii) testing with φv^m and integration by parts;
(iii) the substitution $v = u^k$ (for suitable choices of m,k in terms of α). □

Step 2. *Transformation of J and K for solutions of* (20). We let

$$I := \sum_i I(u_i), \quad J := \sum_i J(u_i), \quad K := \sum_i K(u_i), \quad L := \int \sum_i (f_i(U))^2 \varphi.$$

Lemma 5.5 *Let* $U > 0$ *be a solution of* (20) *in* Q. *Then*

$$K = L + L.O.T.,$$
$$L \le pJ + L.O.T.$$

Ideas of proof of Lemma 5.5. It is done in two steps:
(i) First write $(\Delta u_i)^2 = (f_i(U) - \partial_t u_i)^2$ and transform ∂_t terms to L.O.T. by using a localized energy.
(ii) Then integrate by parts $\int |\nabla u_i|^2 u_i^{r-1} u_j^{r+1} \varphi$. □

Step 3. *Conclusion of sketch of proof of Theorem 5.3.* Combining Lemmas 5.4 and 5.5, we obtain:

$$\alpha J - K + A(\alpha) I \le L.O.T. \qquad \text{with } A(\alpha) = \frac{n}{n+2}\alpha\Big(1 - \frac{(n-1)^2}{n(n+2)}\alpha\Big),$$
$$L \le pJ + L.O.T, \qquad K = L + L.O.T.$$

It then follows that

$$\underbrace{\Big(\frac{\alpha}{p} - 1\Big)}_{>0} L + \underbrace{A(\alpha)}_{>0} I \le L.O.T. \qquad \text{if} \quad p < \alpha < \frac{n(n+2)}{(n-1)^2}.$$

Choosing suitable cut-off, we can absorb L.O.T. in the left-hand side, to get

$$\int_{-1/2}^{1/2}\int_{B_{1/2}} |U|^{2p} \le cL \le C(n,p). \tag{21}$$

If $U > 0$ is an entire solution, we then rescale as:

$$U_\lambda(x,t) = \lambda^{2/(p-1)} U(\lambda x, \lambda^2 t).$$

Applying (21) to U_λ and letting $\lambda \to \infty$, we get $\int_{\mathbb{R}^{n+1}} |U|^{2p} = 0$: a contradiction. □

5.4 Lotka-Volterra case

We consider the system

$$\begin{cases} u_t - \Delta u = u^q[av^r - cu^r], & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ v_t - \Delta v = v^q[bu^r - dv^r], & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \end{cases} \tag{22}$$

with $q \ge 0$, $r > 0$, $a,b,c,d \ge 0$. We denote the total degree by $p := q + r$. Note that, unlike (20), system (22) has no variational structure in general. The following result is due to [30].

Theorem 5.6 *Let $q,a,b,c,d > 0$ with $q + r > 1$. Assume*

$$ab > cd \qquad \textit{and} \qquad r \ge q.$$

(i) Then any positive solution of (22) *satisfies $u/v = Const$.*
(ii) If also $n = 2$ or $p < p_B$, then (22) *has no positive (component-wise) solution.*

Method of proof: It is a parabolic modification of the proof of Theorem 4.6, based on suitable maximum principle arguments. □

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