

# Decay estimates in time for classical and anomalous diffusion

Elisa Affili, Serena Dipierro and Enrico Valdinoci

**Abstract** We present a series of results focused on the decay in time of solutions of classical and anomalous diffusive equations in a bounded domain. The size of the solution is measured in a Lebesgue space, and the setting comprises time-fractional and space-fractional equations and operators of nonlinear type. We also discuss how fractional operators may affect long-time asymptotics.

## 1 Decay estimates, methods, results and perspectives

In this note we present some results, recently obtained in [2, 19], focused on the long-time behavior of solutions of evolution equations which may exhibit anomalous diffusion, caused by either time-fractional or space-fractional effects (or both). The case of several nonlinear operators will be also taken into account (and indeed some of the results that we present are new also for classical diffusion run by nonlinear operators).

The results that we establish give quantitative bounds on the decay in time of smooth solutions, confined in a smooth bounded set with Dirichlet data. The size of the solution will be measured in classical Lebesgue spaces, and we will detect different types of decays according to the different cases that we take into consid-

---

Elisa Affili

Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milan, Italy, and Centre d'Analyse et de Mathématique Sociales, École des Hautes Études en Sciences Sociales, 54 Boulevard Raspail, 75006 Paris, France, e-mail: `elisa.affili@unimi.it`

Serena Dipierro

Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley WA 6009, Australia, e-mail: `serena.dipierro@uwa.edu.au`

Enrico Valdinoci

Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley WA 6009, Australia, e-mail: `enrico.valdinoci@uwa.edu.au`

eration (the main order of decay being affected by the structure of the diffusion in time and by the possible nonlinear character of the spatial operator).

The evolution equation that we take into account is very general, and it can be written as an initial datum problem with homogeneous external Dirichlet condition of the type

$$\begin{cases} \lambda_1 \partial_t^\alpha u + \lambda_2 \partial_t u + \mathcal{N}[u] = 0 & \text{in } \Omega \times (0, +\infty), \\ u = 0 & \text{in } (\mathbb{R}^n \setminus \Omega) \times (0, +\infty), \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega. \end{cases} \quad (1)$$

In this setting,  $u = u(x, t)$  is a smooth solution of (1),  $\Omega$  is a bounded set of  $\mathbb{R}^n$  with smooth boundary (and we are not trying here to optimize the smoothness assumptions on the solution or on the domain), the convex parameters  $\lambda_1, \lambda_2 \in [0, 1]$  are such that  $\lambda_1 + \lambda_2 = 1$ , the (possibly nonlinear) operator  $\mathcal{N}$  acts on the space variable  $x$ , and the time-fractional parameter  $\alpha$  lies in  $(0, 1)$ .

Also, in our setting, the symbol  $\partial_t^\alpha$  stands for the so-called Caputo time-fractional derivative, defined, up to normalizing constants that we omit for simplicity, by

$$\partial_t^\alpha v(t) := \frac{d}{dt} \int_0^t \frac{v(\tau) - v(0)}{(t - \tau)^\alpha} d\tau.$$

Such a time-fractional derivative naturally arises in many context, including geophysics [14], neurology [18, 36] (see also [29] and the references therein) and viscoelasticity [5], and can be seen as a natural consequence of classical models of diffusion in highly ramified media such as combs [4]. In addition, from the mathematical point of view, equations involving the Caputo derivatives can be framed into the broad line of research devoted to Volterra type integrodifferential operators, see [28, 40].

The operator  $\mathcal{N}$  in (1) takes into account the diffusion in the space variable, and can be either of classical or of fractional type, and concrete choices will be made in what follows. More precisely, our setting always comprises, as particular situations, the cases of diffusion driven by the Laplacian or by the fractional Laplacian, defined by

$$(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad \text{with } s \in (0, 1)$$

where the integral is taken in the principal value sense (to allow cancellations near the singularity).

In our framework, we also deal with the case in which  $\mathcal{N}$  is nonlinear, studying the cases of the classical  $p$ -Laplacian and porous media diffusion (see [17, 39])

$$\Delta_p u^m := \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m), \quad \text{with } p \in (1, +\infty) \text{ and } m \in (0, +\infty),$$

the case of graphical mean curvature, given in formula (13.1) of [23],

$$\operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right),$$

the case of the fractional  $p$ -Laplacian (see e.g. [10])

$$(-\Delta)_p^s u(x) := \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{n+sp}} dy,$$

with  $p \in (1, +\infty)$  and  $s \in (0, 1)$ ,

and possibly even the sum of different nonlinear operator of this type, with coefficients  $\beta_j > 0$ ,

$$\sum_{j=1}^N \beta_j (-\Delta)_{p_j}^{s_j} u, \quad \text{with } p_j \in (1, +\infty) \text{ and } s_j \in (0, 1),$$

the case of the anisotropic fractional Laplacian, that is the sum of fractional directional derivatives in the directions of the space  $e_j$ , given by

$$(-\Delta_\beta)^\sigma u(x) = \sum_{j=1}^n \beta_j (-\partial_{x_j}^2)^{\sigma_j} u(x)$$

for  $\beta_j > 0$ ,  $\beta = (\beta_1, \dots, \beta_n)$  and  $\sigma = (\sigma_1, \dots, \sigma_n)$ , where

$$(-\partial_{x_j}^2)^{\sigma_j} u(x) = \int_{\mathbb{R}} \frac{u(x) - u(x + \rho e_j)}{\rho^{1+2\sigma_j}} d\rho,$$

considered for example in [20]. The list of possible diffusion operators continues with two fractional porous media operators (see [13, 33])

$$\mathcal{P}_{1,s}(u) := (-\Delta)^s u^m \quad \text{with } s \in (0, 1) \text{ and } m \in (0, +\infty),$$

$$\text{and } \mathcal{P}_{2,s}(u) := -\operatorname{div}(u \nabla \mathcal{R}(u)), \quad \text{where } \mathcal{R}(u)(x) := \int_{\mathbb{R}^n} \frac{u(y)}{|x - y|^{n-2s}} dy$$

and  $s \in (0, 1)$ ,

the graphical fractional mean curvature operator (see [6])

$$\mathcal{H}^s(u)(x) := \int_{\mathbb{R}^n} F \left( \frac{u(x) - u(x+y)}{|y|} \right) \frac{dy}{|y|^{n+s}},$$

with  $s \in (0, 1)$  and  $F(r) := \int_0^r \frac{d\tau}{(1 + \tau^2)^{\frac{n+1+s}{2}}}$ ,

the classical Kirchhoff operator for vibrating strings

$$\mathcal{K}(u)(x) := -M \left( \|\nabla u\|_{L^2(\Omega)}^2 \right) \Delta u(x),$$

and the fractional Kirchhoff operator (see [21])

$$\mathcal{K}_s(u)(x) := M \left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(y) - u(Y)|^2}{|y - Y|^{n+2s}} dy dY \right) (-\Delta)^s u(x),$$

with  $M : [0, +\infty) \rightarrow [0, +\infty)$  nondecreasing and  $s \in (0, 1)$ .

The case of complex valued operators is also considered, in view of a classical (see [26]) and fractional (see [32]) magnetic settings, in which we took into account the operators

$$\begin{aligned} \mathcal{M}u &:= -(\nabla - iA)^2 u, \\ \text{and } \mathcal{M}_s u(x) &:= \int_{\mathbb{R}^n} \frac{u(x) - e^{i(x-y)A\left(\frac{x+y}{2}\right)} u(y)}{|x-y|^{n+2s}} dy, \quad \text{with } s \in (0, 1), \end{aligned}$$

where  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  represents the magnetic field.

For further motivations and additional details on these operators, we refer to [2, 19]: here we just mention that, given the general assumptions that we take, the operator  $\mathcal{N}$  in (1) comprises many cases of interest in both pure and applied mathematics, with applications in several disciplines, see for instance [11, 27, 30] for detailed discussions on anomalous diffusion with several applications in different contexts.

In our setting, we will obtain decay estimates in suitable Lebesgue spaces  $L^\ell(\Omega)$ , for some appropriate exponent  $\ell \geq 1$ . The typical estimate that we establish is that all solutions  $u$  of (1) satisfy

$$\|u(\cdot, t)\|_{L^\ell(\Omega)} \leq C^* \Theta(t) \quad \text{for all } t \geq 1, \quad (2)$$

where  $C^* > 0$  depends on the structural assumptions of the problem (namely on  $\Omega$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $\alpha$ ,  $\mathcal{N}$ ,  $u_0$  and  $\ell$ ), and  $\Theta : [1, +\infty) \rightarrow (0, +\infty)$  is an appropriate decay function, described here below in concrete situations, possibly depending on another constant  $C > 0$ . The proof of the decay in (2) relies on energy estimates, which are in turn based on suitable Sobolev embeddings that employ the ‘‘parabolic’’ structure of the problem, leading to an appropriate ordinary differential inequality (if  $\lambda_1$  in (1) is equal to zero), or an appropriate integral inequality (if  $\lambda_1 = 1$ ), or a mixed differential/integrodifferential inequality (if  $\lambda_1 \in (0, 1)$ ), for the norm-map  $t \mapsto \|u(\cdot, t)\|_{L^\ell(\Omega)}$ . The solutions of the equations related to those inequalities are used as barriers and compared to the function  $\|u(\cdot, t)\|_{L^\ell(\Omega)}$ , as presented in Theorem 1.1 of [19] and Theorems 1.1 and 1.2 of [2].

More precisely, the ‘‘elliptic’’ character of the spatial diffusive operator is encoded in an inequality of the type

$$\|u(\cdot, t)\|_{L^\ell(\Omega)}^{\ell-1+\gamma} \leq C \int_{\Omega} |u(x, t)|^{\ell-2} \Re \left( \bar{u}(x, t) \mathcal{N} u(x, t) \right) dx, \quad (3)$$

where  $\gamma$  and  $C$  are positive structural constants, “ $\Re$ ” denotes the real part and  $\bar{u}$  is the complex conjugate of  $u$  (in case the problem is set in the reals, the inequality in (3) obviously simplifies).

We observe that (3) becomes more transparent when  $\ell = 2$  and  $\mathcal{N}u = -\Delta u$ , with  $u$  real valued: in such a case, after an integration by parts which takes into account the Dirichlet datum of  $u$ , the inequality in (3) boils down to the classical Sobolev-Poincaré inequality with  $\gamma = 1$ .

Once the inequality in (3) is established for the operator  $\mathcal{N}$  under consideration, one obtains a bound in terms of an ordinary differential equation, or more generally of a nonlinear integral equation on the variable  $t$ : depending on  $\gamma$  and on the type of time-derivative, this provides an estimate on the decay in  $t$  of the norm-map  $t \mapsto \|u(\cdot, t)\|_{L^\ell(\Omega)}$ , which can be either polynomial or exponential (in particular, different operators  $\mathcal{N}$  can lead to different values of  $\gamma$  and therefore to different asymptotics in time for the solution  $u$ ).

This strategy, suitably adapted to the different situations, applies to many operators: the concrete cases that we comprise are listed explicitly in Tables 1 and 2, which presents the main results achieved in [2, 19]. For the first table, the theorems cited in the last column are the ones proving (3) and a decay estimate in the case  $\lambda_1 = 1$  and  $\lambda_2 = 0$  for the operators in their row. Then, combining these results with Theorem 1.1 and 1.2 of [2], the declared estimates trivially follow. However, in Table 1 for the first time we apply the estimates for the case  $\lambda_2 = 1$  of [2] to the operators analyzed in [19], stating the expected decays in a quantitative way.

Table 1: Results from [19].

	Operator $\mathcal{N}$	Values of $\lambda_1, \lambda_2$	Range of $\ell$	Decay $\Theta(t)$	Reference
<b>Nonlinear classical diffusion</b>	$\Delta_p u^m$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{m(p-1)}}$	Thm 1.2 [19]
<b>Nonlinear classical diffusion</b>	$\Delta_p u^m$ with $(m, p) \neq (1, 2)$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{t^{\frac{1}{m(p-1)-1}}$	Thm 1.2 [19]
<b>Bi-Laplacian</b>	$\Delta_2 u$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.2 [19]
<b>Graphical mean curvature</b>	$\operatorname{div} \left( \frac{\nabla u}{\sqrt{1+ \nabla u ^2}} \right)$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^\alpha}$	Thm 1.5 [19]
<b>Graphical mean curvature</b>	$\operatorname{div} \left( \frac{\nabla u}{\sqrt{1+ \nabla u ^2}} \right)$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.5 [19]
<b>Fractional <math>p</math>-Laplacian</b>	$(-\Delta)_p^s u$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{p-1}}$	Thm 1.6 [19]

*Continued on next page*

Table 1 – Continued from previous page

	Operator $\mathcal{N}$	Values of $\lambda_1, \lambda_2$	Range of $\ell$	Decay rate $\Theta$	Reference
<b>Fractional <math>p</math>-Laplacian</b>	$(-\Delta)_p^s u, p > 2$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{t^{\frac{1}{p-2}}}$	Thm 1.6 [19]
<b>Fractional <math>p</math>-Laplacian</b>	$(-\Delta)_p^s u, p \leq 2$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.6 [19]
<b>Superposition of fractional <math>p</math>-Laplacians</b>	$\sum_{j=1}^N \beta_j (-\Delta)_{p_j}^{s_j} u,$ with $\beta_j > 0$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{p_{\max}-1}}}$	Thm 1.7 [19]
<b>Superposition of fractional <math>p</math>-Laplacians</b>	$\sum_{j=1}^N \beta_j (-\Delta)_{p_j}^{s_j} u,$ with $\beta_j > 0$ and $p_{\max} > 2$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{t^{\frac{1}{p_{\max}-2}}}$	Thm 1.7 [19]
<b>Superposition of fractional <math>p</math>-Laplacians</b>	$\sum_{j=1}^N \beta_j (-\Delta)_{p_j}^{s_j} u,$ with $\beta_j > 0$ and $p_{\max} \leq 2$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.7 [19]
<b>Superposition of anisotropic fractional Laplacians</b>	$\sum_{j=1}^N \beta_j (-\partial_{x_j}^2)^{s_j} u$ with $\beta_j > 0$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\alpha}}$	Thm 1.8 [19]
<b>Superposition of anisotropic fractional Laplacians</b>	$\sum_{j=1}^N \beta_j (-\partial_{x_j}^2)^{s_j} u,$ with $\beta_j > 0$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.8 [19]
<b>Fractional porous media I</b>	$\mathcal{P}_{1,s}(u)$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{m}}}$	Thm 1.9 [19]
<b>Fractional porous media I</b>	$\mathcal{P}_{1,s}(u), m > 1$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{t^{\frac{1}{m-1}}}$	Thm 1.9 [19]
<b>Fractional porous media I</b>	$\mathcal{P}_{1,s}(u), m \leq 1$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.9 [19]
<b>Fractional graphical mean curvature</b>	$\mathcal{H}^s(u)$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\alpha}}$	Thm 1.10 [19]
<b>Fractional graphical mean curvature</b>	$\mathcal{H}^s(u)$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.10 [19]

Table 2: Results from [2].

	<b>Operator <math>\mathcal{N}</math></b>	<b>Values of <math>\lambda_1, \lambda_2</math></b>	<b>Range of <math>\ell</math></b>	<b>Decay rate <math>\Theta</math></b>	<b>Reference</b>
<b>Fractional porous media II</b>	$\mathcal{P}_{2,s}$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{2}}}$	Thm 1.3 [2]
<b>Fractional porous media II</b>	$\mathcal{P}_{2,s}$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{t}$	Thm 1.3 [2]
<b>Classical Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(0) > 0$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^\alpha}$	Thm 1.4 [2]
<b>Classical Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(t) = bt$ , $b > 0$ and $n \leq 4$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{3}}}$	Thm 1.4 [2]
<b>Classical Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(t) = bt$ , $b > 0$ and $n \geq 5$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, \frac{2n}{n-4})$	$\frac{1}{t^{\frac{\alpha}{3}}}$	Thm 1.4 [2]
<b>Classical Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(0) > 0$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.4 [2]
<b>Classical Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(t) = bt$ , $b > 0$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{\sqrt{t}}$	Thm 1.4 [2]
<b>Fractional Kirchhoff operator</b>	$\mathcal{H}_s(u)$ with $M(0) > 0$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^\alpha}$	Thm 1.5 [2]
<b>Fractional Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(t) = bt$ , $b > 0$ and $n \leq 4s$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^{\frac{\alpha}{3}}}$	Thm 1.5 [2]
<b>Fractional Kirchhoff operator</b>	$\mathcal{H}(u)$ with $M(t) = bt$ , $b > 0$ and $n > 4s$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, \frac{2n}{n-4s})$	$\frac{1}{t^{\frac{\alpha}{3}}}$	Thm 1.5 [2]
<b>Fractional Kirchhoff operator</b>	$\mathcal{H}_s(u)$ with $M(0) > 0$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.5 [2]
<b>Fractional Kirchhoff operator</b>	$\mathcal{H}_s(u)$ with $M(t) = bt$ , $b > 0$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$\frac{1}{\sqrt{t}}$	Thm 1.5 [2]

Continued on next page

Table 2 – Continued from previous page

	Operator $\mathcal{N}$	Values of $\lambda_1, \lambda_2$	Range of $\ell$	Decay $\Theta(t)$	Reference
<b>Classical magnetic operator</b>	$\mathcal{M}(u)$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^\alpha}$	Thm 1.6 [2]
<b>Classical magnetic operator</b>	$\mathcal{M}(u)$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.6 [2]
<b>Fractional magnetic operator</b>	$\mathcal{M}_s(u)$	$\lambda_1 \in (0, 1]$ $\lambda_2 \in [0, 1)$	$[1, +\infty)$	$\frac{1}{t^\alpha}$	Thm 1.7 [2]
<b>Fractional magnetic operator</b>	$\mathcal{M}_s(u)$	$\lambda_1 = 0$ $\lambda_2 = 1$	$[1, +\infty)$	$e^{-\frac{t}{c}}$	Thm 1.7 [2]

It would be interesting to detect the optimality of the estimates listed in Tables 1 and 2, and to investigate other cases of interest as well. For related decay estimates, see [7, 22, 28]. As a matter of fact, decay estimates for evolutionary problems are a classical topic of research that has produced a very abundant, and extremely interesting, literature. Without aiming at providing an exhaustive list of all the important contributions on this topic, we mention that:

- the classical doubly-nonlinear operator  $\Delta_p u^m$  with  $\lambda_2 = 1$  has been addressed in [9],
- the classical 1-Laplace operator has been dealt with in [3, 24, 25],
- decay estimates for the fractional  $p$ -Laplacian  $(-\Delta_p)^s$  with  $s \in (0, 1)$  and  $p > 1$  with  $\lambda_2 := 1$  have been first established in Section 6 of [16] (see also [15]),
- the case of the fractional 1-Laplacian, namely  $(-\Delta_p)^s$  with  $s \in (0, 1)$  and  $p := 1$  has been treated in [25],
- the porous medium equation for  $\lambda_2 = 1$  has been deeply analyzed in [7],
- some interesting estimates for the Kirchoff equation are given in [22],
- see also [35], where several decay estimates have been obtained by using integral inequalities.

We also remark that the interplay between time derivatives and fractional diffusion produces interesting decay patterns also in nonlinear equations, see e.g. [34]. Furthermore, in general, the fractional aspect of the problem can cause significant differences, as can be observed also from the time decays of Table 1. For instance, one may notice that the decay switch from polynomial to exponential in the Kirchoff equations when the time-diffusion changes from fractional to classical, and this independently on the fact that the space diffusion is of classical or fractional type (roughly speaking, in this context, it is just the character of the diffusion in time which detects the time decay, regardless the character of the diffusion in space).

We also point out that classical and fractional operators share several common properties, but they also exhibit structural differences at a fundamental level. For



instance, to exhibit an elementary but very interesting feature in which long-time behaviors are affected by fractional environments, we recall that fractional diffusion in space, as modeled by the fractional Laplacian  $(-\Delta)^s$  with  $s \in (0, 1)$ , is related to Lévy-type and  $2s$ -stable stochastic processes, and in such case the long jumps of the underlying random walk causes significant differences with respect to the classical Brownian motion. In particular, fractional processes are typically recurrent only in dimension 1 and for values of  $s$  greater or equal to  $1/2$  (being transient in dimension 2 and higher, and also in dimension 1 for values of  $s$  smaller than  $1/2$ ), and this is an important difference with respect to the case of Gaussian processes, which are recurrent in dimensions 1 and 2 (and transient in dimension 3 and higher). See Example 3.5 in [37] and the references therein for a detailed treatment of recurrence and transiency for Lévy-type processes. In this note, in § 2, we present a very simple, and somewhat heuristic, discussion of the recurrence and transiency properties related to the long jump random walks, based on PDE methods and completely accessible to a broad audience.

For a detailed list of other elementary structural differences between classical and fractional diffusion see also § 2.1 in [1].

## 2 Recurrence and transiency of long jump random processes

In this section, we discuss a simple PDE approach to the recurrence of the long jump random walk related to  $(-\Delta)^s$  in dimension 1 and for values of  $s$  greater or equal to  $1/2$  and to its transiency in dimension 2 and higher, and also in dimension 1 for values of  $s$  smaller than  $1/2$ . The treatment will comprise the classical case  $s = 1$  as well, showing how the structural differences between the different regimes naturally arise from a PDE analysis.

To this end, for  $s \in (0, 1]$ , we denote by  $\mathcal{G}_s(x, t)$  the solution of the (possibly fractional) heat equation with initial datum given by the Dirac's Delta, namely

$$\begin{cases} \partial_t \mathcal{G}_s + (-\Delta)^s \mathcal{G}_s = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ \mathcal{G}_s(x, 0) = \delta_0(x). \end{cases}$$

When  $s = 1$ , we have that such function reduces to the classical Gauss kernel for the heat flow, namely

$$\mathcal{G}_1(x, t) = \frac{1}{(4t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

In general, when  $s \in (0, 1)$ , the expression of  $\mathcal{G}_s$  is less explicit, except when  $s = 1/2$ ; in the latter case, it holds that

$$\mathcal{G}_{1/2}(x, t) = \frac{ct}{(t^2 + |x|^2)^{\frac{n+1}{2}}},$$

where  $c > 0$  is a normalizing constant – the need of which lying in the general mass conservation law

$$\int_{\mathbb{R}^n} \mathcal{G}_s(x, t) dx = 1, \quad (4)$$

see also formula (2.29) in [1] and page 1363 in [12]. Furthermore, see again page 1363 in [12], we have that

$$\mathcal{G}_s(x, t) > 0 \quad \text{for all } (x, t) \in \mathbb{R}^n \times (0, +\infty), \quad (5)$$

and it enjoys the natural scaling property

$$\mathcal{G}_s(x, t) = \frac{1}{t^{\frac{n}{2s}}} \mathcal{G}_s\left(\frac{x}{t^{\frac{1}{2s}}}, 1\right). \quad (6)$$

See [8], [12], formulas (2.41)–(2.45) in [1] and the references therein for a discussion about the fractional heat kernel and its differences with the classical case.

For every  $k \in \{1, 2, 3, \dots\}$  and  $\rho > 0$ , we define

$$q_k(s, \rho) := \int_{\mathbb{R}^n \setminus B_\rho} \mathcal{G}_s(x, k) dx. \quad (7)$$

We observe that

$$0 \leq q_k(s, \rho) \leq \int_{\mathbb{R}^n} \mathcal{G}_s(x, k) dx = 1, \quad (8)$$

thanks to (4). Let also

$$q(s, \rho) := \prod_{k=1}^{+\infty} q_k(s, \rho) \in [0, 1] \quad (9)$$

$$\text{and } q(s) := \lim_{\rho \rightarrow 0} q(s, \rho).$$

In view of (8), we can consider  $q(s)$  as related to the probability of the stochastic process associated with the operator  $(-\Delta)^s$  of “drifting away without coming back”. Namely (see [38]), we know that

$$\int_A \mathcal{G}_s(x, t) dx$$

represents the probability that a particle starting at the origin at time 0 and following the stochastic process producing  $(-\Delta)^s$  ends up in the region  $A \subseteq \mathbb{R}^n$  at time  $t$ . In this sense, the quantity  $q_k(s, \rho)$  in (7) represents the probability that this particle lies outside  $B_\rho$  at time  $k$ .

Roughly speaking, for small  $\rho$ , a natural Ansatz is to assume these events to be more or less independent from each other: indeed, in view of (6), using the substitution  $y := x/\rho$  we have that

$$\begin{aligned}
q_k(s, \rho) &= \frac{1}{k^{\frac{n}{2s}}} \int_{\mathbb{R}^n \setminus B_\rho} \mathcal{G}_s \left( \frac{x}{k^{\frac{1}{2s}}}, 1 \right) dx = \frac{\rho^n}{k^{\frac{n}{2s}}} \int_{\mathbb{R}^n \setminus B_1} \mathcal{G}_s \left( \frac{\rho y}{k^{\frac{1}{2s}}}, 1 \right) dy \\
&= \frac{1}{(k/\rho^{2s})^{\frac{n}{2s}}} \int_{\mathbb{R}^n \setminus B_1} \mathcal{G}_s \left( \frac{y}{(k/\rho^{2s})^{\frac{1}{2s}}}, 1 \right) dy = \int_{\mathbb{R}^n \setminus B_1} \mathcal{G}_s \left( y, \frac{k}{\rho^{2s}} \right) dy,
\end{aligned}$$

representing the probability of a particle to lie outside  $B_1$  at time  $k/\rho^{2s}$ . In view of this, since the time steps  $k/\rho^{2s}$  are very separated from each other when  $\rho$  is small, we may think that the quantity  $q(s, \rho)$  in (9) is a good approximation of the probability that the particle does not lie in  $B_1$  in all the time steps  $k/\rho^{2s}$ , as well as a good approximation of the probability that the particle does not lie in  $B_\rho$  in all the time steps  $k \in \{1, 2, 3, \dots\}$ . In this heuristics, the case in which the quantity  $q(s)$  in (9) is equal to 0 indicates that the particle will come back infinitely often to its original position at the origin in integer times (with probability 1); conversely, the case in which the quantity  $q(s)$  in (9) is equal to 1 indicates that the particle will return to its original position at the origin in integer times only with probability zero.

In this sense, computing  $q(s)$  gives an interesting indication of the recurrence properties of the associated stochastic process, and, in our case, this calculation can be performed as follows. First of all, we notice that

$$\inf_{x \in B_1} \mathcal{G}_s(x, 1) := \iota_s > 0,$$

thanks to (5), and

$$\sup_{x \in \mathbb{R}^n} \mathcal{G}_s(x, 1) := \mu_s < +\infty.$$

As a consequence, recalling (4) and (6), if  $\rho \in (0, 1]$

$$\begin{aligned}
p_k(s, \rho) &:= 1 - q_k(s, \rho) = \int_{B_\rho} \mathcal{G}_s(x, k) dx \\
&= \frac{1}{k^{\frac{n}{2s}}} \int_{B_\rho} \mathcal{G}_s \left( \frac{x}{k^{\frac{1}{2s}}}, 1 \right) dx \in \left[ \frac{\iota_s |B_\rho|}{k^{\frac{n}{2s}}}, \frac{\mu_s |B_\rho|}{k^{\frac{n}{2s}}} \right].
\end{aligned}$$

This gives that

$$q_k(s, \rho) = 1 - p_k(s, \rho) \in \left[ 1 - \frac{C\rho^n}{k^{\frac{n}{2s}}}, 1 - \frac{c\rho^n}{k^{\frac{n}{2s}}} \right],$$

for some  $C > c > 0$ , depending only on  $n$  and  $s$ , and accordingly

$$\begin{aligned}
\log q(s, \rho) &= \log \left( \prod_{k=1}^{+\infty} q_k(s, \rho) \right) \\
&= \sum_{k=1}^{+\infty} \log q_k(s, \rho) \in \left[ \sum_{k=1}^{+\infty} \log \left( 1 - \frac{C\rho^n}{k^{\frac{n}{2s}}} \right), \sum_{k=1}^{+\infty} \log \left( 1 - \frac{c\rho^n}{k^{\frac{n}{2s}}} \right) \right]. \tag{10}
\end{aligned}$$

Also, for a fixed  $C_0 \in (0, +\infty)$ , the convergence of the series

$$\sum_{k=1}^{+\infty} \log \left( 1 - \frac{C_0 \rho^n}{k^{\frac{n}{2s}}} \right)$$

can be reduced to that of the series

$$- \sum_{k=1}^{+\infty} \frac{C_0 \rho^n}{k^{\frac{n}{2s}}},$$

and consequently

$$\sum_{k=1}^{+\infty} \log \left( 1 - \frac{C_0 \rho^n}{k^{\frac{n}{2s}}} \right) = \begin{cases} -C_1 \rho^n & \text{if } n > 2s, \\ -\infty & \text{if } n \leq 2s, \end{cases}$$

for some  $C_1 > 0$  depending only on  $n$ ,  $s$  and  $C_0$ . This and (10) lead to

$$\begin{aligned} \log q(s, \rho) &= -\infty & \text{if } n \leq 2s, \\ \log q(s, \rho) &\in [-C_2 \rho^n, -C_3 \rho^n] & \text{if } n > 2s, \end{aligned}$$

for some  $C_2 > C_3 > 0$  depending only on  $n$  and  $s$ .

Therefore,

$$\begin{aligned} q(s, \rho) &= 0 & \text{if } n \leq 2s, \\ q(s, \rho) &\in [e^{-C_2 \rho^n}, e^{-C_3 \rho^n}] & \text{if } n > 2s. \end{aligned}$$

Taking the limit as  $\rho \rightarrow 0$ , we thereby find that

$$\begin{aligned} q(s) &= 0 & \text{if } n \leq 2s, \\ q(s) &= 1 & \text{if } n > 2s. \end{aligned} \tag{11}$$

When  $s = 1$ , we can write (11) as

$$\begin{aligned} q(s) &= 0 & \text{if } n \in \{1, 2\}, \\ q(s) &= 1 & \text{if } n \geq 3, \end{aligned}$$

that is, in our framework, the classical random walk “comes back to the initial” point in dimensions 1 and 2, and it “drifts away forever” in dimension 3 and higher.

When  $s \in (0, 1)$ , the situation is different, since (11) produces the alternative

$$\begin{aligned} q(s) &= 0 & \text{if } n = 1 \text{ and } s \in [1/2, 1), \\ q(s) &= 1 & \text{if } n \geq 2, \text{ and also if } n = 1 \text{ and } s \in (0, 1/2). \end{aligned}$$

That is, in our setting, the fractional random walk “comes back to the initial” point only in dimensions 1 and only if the fractional parameter is above a certain threshold (namely  $s \geq 1/2$ ). Conversely, the fractional random walk “drifts away for ever”

already in dimension 2, and even in dimension 1 if the fractional parameter is too small (namely  $s < 1/2$ ).

See e.g. [31] and the references therein for a comprehensive treatment of recurrence and transiency of general stochastic processes.

**Acknowledgements** This work has been supported by the Australian Research Council Discovery Project 170104880 NEW “Nonlocal Equations at Work”. Part of this work was carried out on the occasion of a very pleasant visit of the first author to the University of Western Australia, which we thank for the warm hospitality. The authors are members of INdAM/GNAMPA.



## References

- [1] Nicola Abatangelo and Enrico Valdinoci, *Getting acquainted with the fractional Laplacian*, Contemporary Research in Elliptic PDEs and Related Topics, 2019, pp. 1–105, DOI 10.1007/978-3-030-18921-1.
- [2] Elisa Affili and Enrico Valdinoci, *Decay estimates for evolution equations with classical and fractional time-derivatives*, J. Differential Equations **266** (2019), no. 7, 4027–4060, DOI 10.1016/j.jde.2018.09.031. MR3912710
- [3] Fuensanta Andreu-Vailló, Vicent Caselles, and José M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, vol. 223, Birkhäuser Verlag, Basel, 2004. MR2033382
- [4] V. E. Arkhincheev and É. M. Baskin, *Anomalous diffusion and drift in a comb model of percolation clusters*, J. Exp. Theor. Phys. **73** (1991), 161–165.
- [5] Ron Bagley, *On the equivalence of the Riemann-Liouville and the Caputo fractional order derivatives in modeling of linear viscoelastic materials*, Fract. Calc. Appl. Anal. **10** (2007), no. 2, 123–126. MR2351653
- [6] Begoña Barrios, Alessio Figalli, and Enrico Valdinoci, *Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **13** (2014), no. 3, 609–639. MR3331523
- [7] Piotr Biler, Cyril Imbert, and Grzegorz Karch, *The nonlocal porous medium equation: Barenblatt profiles and other weak solutions*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 497–529, DOI 10.1007/s00205-014-0786-1. MR3294409
- [8] Krzysztof Bogdan and Tomasz Jakubowski, *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators*, Comm. Math. Phys. **271** (2007), no. 1, 179–198, DOI 10.1007/s00220-006-0178-y. MR2283957
- [9] Matteo Bonforte and Gabriele Grillo, *Super and ultracontractive bounds for doubly nonlinear evolution equations*, Rev. Mat. Iberoam. **22** (2006), no. 1, 111–129. MR2268115
- [10] Lorenzo Brasco, Erik Lindgren, and Armin Schikorra, *Higher Hölder regularity for the fractional  $p$ -Laplacian in the superquadratic case*, Adv. Math. **338** (2018), 782–846, DOI 10.1016/j.aim.2018.09.009. MR3861716
- [11] Claudia Bucur and Enrico Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, vol. 20, Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016. MR3469920
- [12] Xavier Cabré and Jean-Michel Roquejoffre, *Propagation de fronts dans les équations de Fisher-KPP avec diffusion fractionnaire*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 23-24, 1361–1366, DOI 10.1016/j.crma.2009.10.012 (French, with English and French summaries). MR2588782
- [13] Luis A. Caffarelli and Juan Luis Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, Discrete Contin. Dyn. Syst. **29** (2011), no. 4, 1393–1404, DOI 10.3934/dcds.2011.29.1393. MR2773189

- [14] Michele Caputo, *Linear models of dissipation whose  $Q$  is almost frequency independent. II*, *Fract. Calc. Appl. Anal.* **11** (2008), no. 1, 4–14. Reprinted from *Geophys. J. R. Astr. Soc.* **13** (1967), no. 5, 529–539. MR2379269
- [15] Thierry Coulhon and Daniel Hauer, *Regularisation effects of nonlinear semigroups*, arXiv e-prints (2016), available at [1604.08737](https://arxiv.org/abs/1604.08737).
- [16] ———, *Regularisation effects of nonlinear semigroups – theory and applications*, Springer-Briefs in Mathematics, Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao. BCAM SpringerBriefs.
- [17] Emmanuele DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993. MR1230384
- [18] Serena Dipierro and Enrico Valdinoci, *A simple mathematical model inspired by the Purkinje cells: from delayed travelling waves to fractional diffusion*, *Bull. Math. Biol.* **80** (2018), no. 7, 1849–1870, DOI [10.1007/s11538-018-0437-z](https://doi.org/10.1007/s11538-018-0437-z). MR3814763
- [19] Serena Dipierro, Enrico Valdinoci, and Vincenzo Vespi, *Decay estimates for evolutionary equations with fractional time-diffusion*, *J. Evol. Equ.* **19** (2019), no. 2, 435–462, DOI [10.1007/s00028-019-00482-z](https://doi.org/10.1007/s00028-019-00482-z). MR3950697
- [20] Alberto Farina and Enrico Valdinoci, *Regularity and rigidity theorems for a class of anisotropic nonlocal operators*, *Manuscripta Math.* **153** (2017), no. 1-2, 53–70, DOI [10.1007/s00229-016-0875-6](https://doi.org/10.1007/s00229-016-0875-6). MR3635973
- [21] Alessio Fiscella and Enrico Valdinoci, *A critical Kirchhoff type problem involving a nonlocal operator*, *Nonlinear Anal.* **94** (2014), 156–170, DOI [10.1016/j.na.2013.08.011](https://doi.org/10.1016/j.na.2013.08.011). MR3120682
- [22] Marina Ghisi and Massimo Gobbino, *Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: time-decay estimates*, *J. Differential Equations* **245** (2008), no. 10, 2979–3007, DOI [10.1016/j.jde.2008.04.017](https://doi.org/10.1016/j.jde.2008.04.017). MR2454809
- [23] Enrico Giusti and Graham Hale Williams, *Minimal surfaces and functions of bounded variation*, Vol. 2, Springer, 1984.
- [24] Daniel Hauer and José M. Mazón, *Kurdyka-Lojasiewicz-Simon inequality for gradient flows in metric spaces*, arXiv e-prints (2017), available at [1707.03129](https://arxiv.org/abs/1707.03129).
- [25] ———, *Regularizing effects of homogeneous evolution equations: the case of homogeneity order zero*, *J. Evol. Equ.*, posted on 2019, DOI [10.1007/s00028-019-00502-y](https://doi.org/10.1007/s00028-019-00502-y).
- [26] Teruo Ikebe and Tosio Kato, *Uniqueness of the self-adjoint extension of singular elliptic differential operators*, *Arch. Rational Mech. Anal.* **9** (1962), 77–92, DOI [10.1007/BF00253334](https://doi.org/10.1007/BF00253334). MR0142894
- [27] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado, and J. H. T. Bates, *The role of fractional calculus in modeling biological phenomena: a review*, *Commun. Nonlinear Sci. Numer. Simul.* **51** (2017), 141–159, DOI [10.1016/j.cnsns.2017.04.001](https://doi.org/10.1016/j.cnsns.2017.04.001). MR3645874
- [28] Jukka Kempainen, Juhana Siljander, Vicente Vergara, and Rico Zacher, *Decay estimates for time-fractional and other non-local in time subdiffusion equations in  $\mathbb{R}^d$* , *Math. Ann.* **366** (2016), no. 3-4, 941–979, DOI [10.1007/s00208-015-1356-z](https://doi.org/10.1007/s00208-015-1356-z). MR3563229
- [29] Toma M. Marinov, Nelson Ramirez, and Fidel Santamaria, *Fractional integration toolbox*, *Fract. Calc. Appl. Anal.* **16** (2013), no. 3, 670–681, DOI [10.2478/s13540-013-0042-7](https://doi.org/10.2478/s13540-013-0042-7). MR3071207
- [30] Ralf Metzler and Joseph Klafter, *The random walk’s guide to anomalous diffusion: a fractional dynamics approach*, *Phys. Rep.* **339** (2000), no. 1, 77, DOI [10.1016/S0370-1573\(00\)00070-3](https://doi.org/10.1016/S0370-1573(00)00070-3). MR1809268
- [31] T. M. Michelitsch, B. A. Collet, A. P. Riascos, A. F. Nowakowski, and F. C. G. A. Nicolleau, *Recurrence of random walks with long-range steps generated by fractional Laplacian matrices on regular networks and simple cubic lattices*, *J. Phys. A* **50** (2017), no. 50, 505004, 29, DOI [10.1088/1751-8121/aa9008](https://doi.org/10.1088/1751-8121/aa9008). MR3738798
- [32] Hoai-Minh Nguyen, Andrea Pinamonti, Marco Squassina, and Eugenio Vecchi, *New characterizations of magnetic Sobolev spaces*, *Adv. Nonlinear Anal.* **7** (2018), no. 2, 227–245, DOI [10.1515/anona-2017-0239](https://doi.org/10.1515/anona-2017-0239). MR3794886
- [33] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez, *A fractional porous medium equation*, *Adv. Math.* **226** (2011), no. 2, 1378–1409, DOI [10.1016/j.aim.2010.07.017](https://doi.org/10.1016/j.aim.2010.07.017). MR2737788



- [34] Stefania Patrizi and Enrico Valdinoci, *Relaxation times for atom dislocations in crystals*, Calc. Var. Partial Differential Equations **55** (2016), no. 3, Art. 71, 44, DOI 10.1007/s00526-016-1000-0. MR3511786
- [35] Maria Michaela Porzio, *On decay estimates*, J. Evol. Equ. **9** (2009), no. 3, 561–591, DOI 10.1007/s00028-009-0024-8. MR2529737
- [36] E. È. Saftenku, *Modeling of slow glutamate diffusion and AMPA receptor activation in the cerebellar glomerulus*, J. Theoret. Biol. **234** (2005), no. 3, 363–382, DOI 10.1016/j.jtbi.2004.11.036. MR2139665
- [37] Nikola Sandrić, *On transience of Lévy-type processes*, Stochastics **88** (2016), no. 7, 1012–1040, DOI 10.1080/17442508.2016.1178749. MR3529858
- [38] Enrico Valdinoci, *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA **49** (2009), 33–44. MR2584076
- [39] Juan Luis Vázquez, *The porous medium equation*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007. Mathematical theory. MR2286292
- [40] Rico Zacher, *Maximal regularity of type  $L_p$  for abstract parabolic Volterra equations*, J. Evol. Equ. **5** (2005), no. 1, 79–103, DOI 10.1007/s00028-004-0161-z. MR2125407