

The Greenberg Functor is Site Cocontinuous

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Abstract In this paper we show that it is possible to define a topology on the category of formal schemes over a ring of p -adic integers such that the left adjoint of the Greenberg Transform is a site cocontinuous functor when we equip the category of schemes over the residue field with the étale topology. We show furthermore that this topology allows us to give an isomorphism between the corresponding fundamental groups, and use this isomorphism to show that it is possible to geometrize the quasicharacters of a p -adic torus by a local system on a formal scheme over the ring of p -adic integers.

1 Introduction

The Greenberg Transform, and its left adjoint (often called the Greenberg Functor), are two functors that are ubiquitous in arithmetic algebraic geometry. The Greenberg Transform itself first appeared in a proto-form in Serge Lang's thesis [18] and was further explored and formally introduced by Marvin J. Greenberg in [12] and [13], where Greenberg showed that the transform that bears his name admits a left adjoint, as well as how to use the pair of functors to study torsors and cohomology. More recently, these functors have appeared in the book [6] by Bosch, Lütkebohmert, and Raynaud on Néron Models; in the work of Buim to study p -jet spaces (cf. [7]); in the work of Nicaise and Sebag to study motivic properties of (formal) schemes (cf. [20] and [21]); in the work of Cunningham and Roe in [11] to geometrize quasicharacters of p -adic tori; in the work of Yu in the study of smooth models as they are used in Bruhat-Tits theory (cf. [23]); in the work of Bertapelle and Tong to study the Picard group and the pro-algebraic structures of Serre (cf. [2]); and in the work of Bhatt and Scholze in defining and proving the

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representability of the positive p -adic loop group functor (cf. Proposition 9.2 of [3]).

While the Greenberg functor and the Greenberg Transform have given myriad tools with which to study schemes and their arithmetic properties, especially by relating the mixed-characteristic case of formal schemes over $\mathrm{Spec} R$ to schemes over the residue field $\mathrm{Spec} k$, the Greenberg Transform was written in a pre-Grothendieck language, which made applying the rich theory around it more difficult. It was this issue that lead Bertapelle and González-Avilés to revisit and recast the Greenberg Transform into scheme (and formal scheme)-theoretic language of modern algebraic geometry in [1]. This significantly helped in the study of the Greenberg Transform, as it established many site-theoretic and geometric properties of the functor itself, and studied the Greenberg Transform in great detail. However, in contrast to the in-depth study and development of the Greenberg Transform, the theory surrounding the Greenberg functor is less developed.

In this paper, we work with and study the Greenberg functor and consider an application to local systems. In particular, we show the Greenberg functor is site-cocontinuous for certain Grothendieck topologies on $\mathbf{Sch}/_{\mathrm{Spec} k}$ and $\mathbf{FSch}/_{\mathrm{Spec} R}$, where k is the residue field of a complete integral extension R/\mathbb{Z}_p and $\mathbf{FSch}/_{\mathrm{Spec} R}$ is the category of formal schemes over $\mathrm{Spec} R$. More precisely, we will define topologies on the category $\mathbf{FSch}/_{\mathrm{Spec} R}$ for which the Greenberg functor $\mathfrak{h} : \mathbf{Sch}/_{\mathrm{Spec} k} \rightarrow \mathbf{FSch}/_{\mathrm{Spec} R}$ becomes cover-reflecting for the étale and fppf topologies. This means that if $\{\varphi_i : X_i \rightarrow X \mid i \in I\}$ is a collection of morphisms over X in $\mathbf{Sch}/_{\mathrm{Spec} k}$, and if $\{F\varphi_i : FX_i \rightarrow FX \mid i \in I\}$ generates a covering sieve over FX , then the set $\{\varphi_i : X_i \rightarrow X \mid i \in I\}$ is a cover over X . This topology, which we will call K for the moment (cf. Definition 2 for an explicit description), allows us to show that there is a canonical isomorphism of fundamental groups

$$\pi_1^{\acute{E}t}(X, x) \cong \pi_1^K(\mathfrak{h} X, \mathfrak{h} x)$$

for any scheme X over $\mathrm{Spec} k$. In particular, from this group isomorphism, we show that there is an isomorphism of the category of étale local systems over a k -scheme X with the category of local systems in the K topology over $\mathfrak{h} X$.

Theorem 1.1 (cf. Theorem 5.1) *For any group scheme G over $\mathrm{Spec} k$ with geometric point g of G , if \mathfrak{h} is the Greenberg functor, then there is an isomorphism of categories*

$$\mathbf{Rep}(\pi_1^{\acute{E}t}(G, g)) \cong \mathbf{Rep}(\pi_1^K(\mathfrak{h} G, \mathfrak{h} g)).$$

In particular, this gives an isomorphism of categories

$$\mathbf{Loc}_{\acute{E}t}(G) \cong \mathbf{Loc}_K(\mathfrak{h} G).$$

Our study of the Greenberg functor is motivated in two directions:

1. The first motivation comes from wishing to better understand the Greenberg Transform and a desire to use it in arithmetic algebraic geometry, since understanding a right adjoint functor is to understand its left adjoint, and dually. Furthermore, a deeper understanding of this left adjoint and how it affects Grothendieck topologies on its domain will lead to a comparison between Grothendieck topologies on schemes over fields of prime characteristic and Grothendieck topologies on categories of formal schemes over p -adically complete rings.
2. The second motivation comes from [11] in which the authors begin with a torus T over a p -adic field F , take its Néron model N_T over its ring of integers \mathcal{O}_F , and then take the Greenberg Transform of N_T , to define a scheme X over $\text{Spec } k$ for which there is an identification of groups

$$T(F) \cong N_T(\mathcal{O}_F) \cong X(k).$$

By then taking the category of étale local systems over X and using the Trace of Frobenius, they arrive at the following: Any quasicharacter of $T(F)$ is geometrized as the Trace of Frobenius of some étale local system \mathcal{L} . This causes one to ask whether it is possible to construct a category of local systems in some topology over a formal scheme over $\text{Spec } \mathcal{O}_F$ for which this new local system geometrizes the quasicharacter $\chi : T(F) \rightarrow \mathbb{C}^*$ directly.

In an attempt to address both of the considerations above, we are lead to study a class of functors, which we call geometrically adhesive (cf. Definition 1); Corollary 2 shows that the Greenberg functor is geometrically adhesive. With this notion, we build a topology which allows the transfer of sites from categories of schemes to categories of formal schemes in the sense that the fundamental groups are preserved by the transfer. This is Theorem 4.1, which is the main theorem of the paper:

Theorem 1.2 (cf. Theorem 4.1) *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a geometrically adhesive functor and assume that there is a Grothendieck topology J on \mathcal{C} for which $\mathbf{Shv}(\mathcal{C}, J)_{\text{icf}}$ is a Galois category. Then there is a topology K on \mathcal{D} and a functor $F^* : \mathbf{Shv}(\mathcal{D}, K)_{\text{icf}} \rightarrow \mathbf{Shv}(\mathcal{C}, J)_{\text{icf}}$ such that if F^* is fully faithful, there is an isomorphism of fundamental groups*

$$\pi_1^J(X, x) \cong \pi_1^K(FX, Fx)$$

for all objects X of \mathcal{C} and geometric points x of X .

As an application, we show that the pullback functor

$$\mathfrak{h}^* : \mathbf{Shv}(\mathbf{FSch}_{/\text{Spec } R}, J) \rightarrow \mathbf{Shv}(\mathbf{Sch}_{/\text{Spec } k}, J)$$

is fully faithful (cf. Lemma 5). Furthermore, we derive:

Corollary 1. *If J is a topology on $\mathbf{Sch}/_{\mathrm{Spec} k}$ for which $\mathbf{Shv}(\mathbf{Sch}/_{\mathrm{Spec} k}, J)$ is a Galois category and if \mathfrak{h} is the Greenberg functor, then for any k -scheme X there is an isomorphism of fundamental groups*

$$\pi_1^J(X, x) \cong \pi_1^K(\mathfrak{h} X, \mathfrak{h} x).$$

2 An Introduction to Geometrically Adhesive Functors, with Motivation

We begin by recalling the full-level Greenberg Transform for a complete field extension K/\mathbb{Q}_p ; cf. [1] for a modern account and [12] and [13] for the introduction of the functors. Begin by assuming that R/\mathbb{Z}_p is an absolutely unramified extension ring of \mathbb{Z}_p with residue field k/\mathbb{F}_p and fraction field K/\mathbb{Q}_p . Furthermore, if $n \in \mathbb{N}$, define the scheme

$$S_n := \mathrm{Spec} \frac{R}{\mathfrak{p}^{n+1}}$$

so that

$$\mathrm{Spec} R \cong \mathrm{Spec} \left(\lim_{\leftarrow} \frac{R}{\mathfrak{p}^{n+1}} \right) \cong \lim_{\rightarrow} \mathrm{Spec} \frac{R}{\mathfrak{p}^{n+1}} = \lim_{\rightarrow} S_n,$$

where the limit is calculated in the category **Cring** of commutative rings with identity and the colimit is calculated in the category **AffSch** of affine schemes. Then the full-level Greenberg Transform is a functor $\mathrm{Gr} : \mathbf{FSch}/_{\mathrm{Spec} R} \rightarrow \mathbf{Sch}/_{\mathrm{Spec} k}$, where $\mathbf{FSch}/_{\mathrm{Spec} R}$ is the category of formal schemes over the scheme $\mathrm{Spec} R$, has the left adjoint $\mathfrak{h} : \mathbf{Sch}/_{\mathrm{Spec} k} \rightarrow \mathbf{FSch}/_{\mathrm{Spec} R}$, i.e., we have an adjunction diagram

$$\begin{array}{ccc} & \mathfrak{h} & \\ & \curvearrowright & \\ \mathbf{Sch}/_{\mathrm{Spec} k} & \perp & \mathbf{FSch}/_{\mathrm{Spec} R} \\ & \curvearrowleft & \\ & F^* & \end{array}$$

In order to give a particularly concrete description of \mathfrak{h} , we now assume that the ramification degree $e = [K : \mathbb{Q}_p]/[k : \mathbb{F}_p] = 1$, i.e., K is unramified over \mathbb{Q}_p . The left adjoint \mathfrak{h} then can be seen as the colimit of the functors

$$h_n : \mathbf{Sch}/_{\mathrm{Spec} k} \rightarrow \mathbf{Sch}/_{S_n} \hookrightarrow \mathbf{FSch}/_{\mathrm{Spec} R}$$

where each h_n is defined by, for each scheme $X = (|X|, \mathcal{O}_X)$ in $\mathbf{Sch}/_{\mathrm{Spec} k}$,

$$h_n|X| := |X|$$

and

$$h_n \mathcal{O}_X := W_{n+1} \mathcal{O}_X$$

where $W_n \mathcal{O}_X$ is the sheaf of length n Witt Vectors in \mathcal{O}_X ; note that the action of $W_{n+1} \mathcal{O}_X$ is given by, for each open set $U \subseteq |X|$,

$$\left(W_{n+1} \mathcal{O}_X \right) (U) = W_{n+1} \left(\mathcal{O}_X(U) \right),$$

where $W_{n+1}(\mathcal{O}_X(U))$ is the ring of length $n+1$ Witt Vectors with coefficients in $\mathcal{O}_X(U)$. It is worth observing that naïvely these functors do not admit a colimit; however, since there is a natural map

$$S_m \rightarrow S_n$$

whenever $m \leq n$, as it comes from the correspondence $\mathbf{Sch}(S_m, S_n) \cong \mathbf{Cring}(R/\mathfrak{p}^{n+1}, R/\mathfrak{p}^{m+1})$; in this way a scheme over S_m is a scheme over S_n by simply post-composing by map $S_m \rightarrow S_n$, which itself allows us to regard each functor $h_m : \mathbf{Sch}/_{\mathrm{Spec} k} \rightarrow \mathbf{Sch}/_{S_m}$ as a functor $h_m : \mathbf{Sch}/_{\mathrm{Spec} k} \rightarrow \mathbf{Sch}/_{S_n}$. Through this process, we derive a natural transformation $h_n \rightarrow h_{n+1}$ for all $n \in \mathbb{N}$. Taking the colimit of these functors h_n , together with all necessary embeddings of categories $\mathbf{Sch}/_{S_n} \rightarrow \mathbf{FSch}/_{\mathrm{Spec} R}$, gives the desired colimit

$$\mathfrak{h} : \mathbf{Sch}/_{\mathrm{Spec} k} \rightarrow \mathbf{FSch}/_{\mathrm{Spec} R}.$$

From these definitions a routine calculation implies that for each scheme $X = (|X|, \mathcal{O}_X)$ in $\mathbf{Sch}/_{\mathrm{Spec} k}$, we have that

$$\mathfrak{h}|X| = |X|$$

and

$$\mathfrak{h} \mathcal{O}_X = W \mathcal{O}_X,$$

where $W \mathcal{O}_X$ is the sheaf of Witt Vectors on \mathcal{O}_X . A standard gluing argument then allows one to show that if $\{U_i \mid i \in I\}$ is an open gluing of a scheme X in $\mathbf{Sch}/_{\mathrm{Spec} k}$, then

$$\mathfrak{h} \left(\bigcup_{i \in I} U_i \right) = \mathfrak{h} X = \bigcup_{i \in I} \mathfrak{h} U_i;$$

such a geometric condition is very powerful and is easier to work with than the mystical and difficult to understand Greenberg Transform itself. In particular, we call such a functor *geometrically adhesive* because it preserves all possible gluing!

Remark 1. The union written above is meant simply as a short hand for a specific colimit. Explicitly, for locally ringed spaces U and V , we write

$$U \cup V := U \coprod_X V$$

as the gluing pushout of along their intersection $X \rightarrow U$ and $X \rightarrow V$. Similarly, we write $U \cap V$ to denote the pullback of U and V along the morphisms $U \rightarrow U \cup V$ and $V \rightarrow U \cup V$. In particular, the diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow \lrcorner & & \downarrow \\ V & \longrightarrow & U \cup V \end{array}$$

in **LRS** is simultaneously a pushout and pullback diagram.

Remark 2. If the algebraic field extension F/\mathbb{Q}_p has nontrivial ramification, say as in the quadratic extension $\mathbb{Q}_p(\sqrt{p})$, then there is an analogous story to the one told above that may be used to construct the Greenberg Transform and its left adjoint \mathfrak{h} . It essentially the same construction, save for now we take products of the $W_n \mathcal{O}_X$ sheaves to build up the Eisenstein equation of ramification termwise, and then build up our Witt Vector structure with these ramifications in mind. Explicit details may be found in [1], but will not be particularly relevant for this paper, save for in providing intuition for the explicit case to which we wish to apply our results.

We will now codify some notation useful as we proceed in the section. In particular, let \mathcal{C} and \mathcal{D} be subcategories of the category **LRS** of locally ringed spaces or slice categories of subcategories of **LRS**.

Definition 1. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be *geometrically adhesive* if, whenever $U = (|U|, \mathcal{O}_U) \in \text{Ob } \mathcal{C}/_X$ has a collection of open subobjects $\{V_j \mid j \in J\}$ with

$$\bigcup_{j \in J} V_j = U,$$

then

$$F \left(\bigcup_{j \in J} V_j \right) = \bigcup_{j \in J} FV_j.$$

Remark 3. For the reader familiar with the work of Lack, Cockett, et al. (cf. [8], [9], [10], amongst others), one may notice the similarity of a join restriction functor between join restriction categories, or perhaps more readily with adhesive functors between adhesive categories (cf. [17], for instance), and our given definition of a geometrically adhesive functor above. There are similarities, especially in the fact that these are all getting towards some sort of manifold-type construction and intuition, but we have given the definition above for the following reasons:

- The definition of a geometrically adhesive functor is intimately tied to the sheaves that come equipped with a locally ringed space, and geometrically adhesive functors are built to bring this perspective to mind and to task.
- We are not giving any extra attention to restriction or join properties of the categories \mathcal{C} or \mathcal{D} , if nontrivial versions of said structures even exist, save for potentially in incidental circumstances.
- The definition of join restriction functors is tied to the abstract differential geometry of Cartesian differential categories. Because we are focusing only on the algebro-geometric aspects induced by these functors, we provide an algebraic dual to the more analytic perspective afforded by join restriction functors.
- These are different from the adhesive functors of Lack (save for perhaps morally) in the following way: A geometrically adhesive functor only asks to preserve gluings of open subobjects between arbitrary categories of locally ringed spaces, while adhesive functors in the sense of Lack are functors between adhesive categories that preserve pushouts against arbitrary monomorphisms. In particular, Proposition 3 shows that in general these are different notions.
- Proposition 3 shows, in addition to differentiating adhesive functors from geometrically adhesive functors, that geometrically adhesive functors are not necessarily cocontinuous.

Remark 4. Not all functors are geometrically adhesive. For instance, consider the functor $\Gamma : \mathbf{Sch} \rightarrow \mathbf{Sch}$ given by

$$\Gamma(X) \mapsto \text{Spec}(\mathcal{O}_X(X)).$$

This functor is not geometrically adhesive because it destroys nonaffine schemes and their affine gluings. In particular, recall that if X is affine and only if

$$\Gamma(X) \cong \Gamma(\text{Spec } A) = \text{Spec}(\mathcal{O}_A(|\text{Spec } A|)) = \text{Spec } A.$$

To see that this is not geometrically adhesive, consider the scheme $\mathbb{P}_{\mathbb{Z}}^1$ with the gluing

$$\begin{array}{ccc}
 & \mathbb{P}_{\mathbb{Z}}^1 & \\
 \nearrow & & \nwarrow \\
 \mathbb{A}_{\mathbb{Z}}^1 & & \mathbb{A}_{\mathbb{Z}}^1 \\
 \uparrow & & \uparrow \\
 \text{Spec } \mathbb{Z}[x, x^{-1}] & \xrightarrow{\cong} & \text{Spec } \mathbb{Z}[y, y^{-1}]
 \end{array}$$

where the bottom isomorphism is induced by the **Cring** morphism $\varphi : \mathbb{Z}[x, x^{-1}] \rightarrow \mathbb{Z}[y, y^{-1}]$ given by $x \mapsto y^{-1}$. Then we find that

$$\Gamma(\mathbb{P}_{\mathbb{Z}}^1) = \text{Spec}(\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^1}(\mathbb{P}_{\mathbb{Z}}^1)) = \text{Spec } \mathbb{Z}$$

so taking Γ of the whole diagram gives the commuting diagram

$$\begin{array}{ccc} & \text{Spec } \mathbb{Z} & \\ \exists! \nearrow & & \nwarrow \exists! \\ \mathbb{A}_{\mathbb{Z}}^1 & & \mathbb{A}_{\mathbb{Z}}^1 \\ \uparrow & & \uparrow \\ \text{Spec } \mathbb{Z}[x, x^{-1}] & \xrightarrow{\cong} & \text{Spec } \mathbb{Z}[y, y^{-1}] \end{array}$$

which is not a gluing diagram because, amongst other reasons, $\mathbb{A}_{\mathbb{Z}}^1$ is not a subscheme of $\text{Spec } \mathbb{Z}$.

Example 1. Let \mathcal{D} be a subcategory of **LRS** such that $\text{Spec } 0 \in \text{Ob } \mathcal{D}$. Then the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ defined by $FU := \text{Spec } 0$ is geometrically adhesive.

Proposition 1. *Adhesive functors need not reflect isomorphisms.*

Proof. Let F be the functor given in the above example and let $\mathcal{C} := \mathbf{Sch}$. Consider the schemes $U_i, U_j := \mathbb{A}_{\mathbb{Z}}^1$ with gluing

$$U_i \cup U_j = \tilde{U} := \mathbb{P}_{\mathbb{Z}}^1$$

and define $U := \text{Spec } \mathbb{Z}$. We then calculate that

$$F(U) = F(\text{Spec } \mathbb{Z}) = \text{Spec } 0 = F(\mathbb{A}_{\mathbb{Z}}^1 \cup \mathbb{A}_{\mathbb{Z}}^1) = F(\mathbb{P}_{\mathbb{Z}}^1) = F(\tilde{U})$$

but, of course, $U = \text{Spec } \mathbb{Z} \not\cong \mathbb{P}_{\mathbb{Z}}^1 = \tilde{U}$.

Proposition 2. *Adhesive functors need not be faithful.*

Proof. The ‘‘affinization’’ functor F above does the job for sufficiently chosen categories \mathcal{C} of locally ringed spaces. In particular, take $\mathcal{C} = \mathbf{Sch}$.

Proposition 3. *Geometrically adhesive functors need not be cocontinuous. In particular, geometrically adhesive functors need not preserve gluings along closed subobjects.*

Proof. As above, the proof is by example. Consider the ring

$$A := \{(a, f) \mid a \in \mathbb{Z}_p, f \in \mathbb{Z}_p[x], a \equiv f(0) \pmod{p}\} = \mathbb{Z}_p \times_{\mathbb{F}_p} \mathbb{Z}_p[x]$$

and define the set of objects

$$\mathcal{C}_0 := \{\text{Spec } A, \text{Spec } \mathbb{Z}_p, \text{Spec } \mathbb{F}_p, \mathbb{A}_{\mathbb{Z}_p}^1, \mathbb{P}_{\mathbb{Z}_p}^1, \text{Spec } \mathbb{Z}_p[x, x^{-1}]\}$$

with morphism set

$$\mathcal{C}_1 := \{f \mid \exists X, Y \in \mathcal{C}_0. f \in \mathbf{Sch}(X, Y)\}$$

and take \mathcal{C} to be the category $(\mathcal{C}_0, \mathcal{C}_1)$. Note that the only object with non-trivial open subobjects is $\mathbb{P}_{\mathbb{Z}_p}^1$, which is the gluing of two copies of $\mathbb{A}_{\mathbb{Z}_p}^1$ along the isomorphism $\mathrm{Spec} \mathbb{Z}_p[x, x^{-1}] \rightarrow \mathrm{Spec} \mathbb{Z}_p[x, x^{-1}]$ given by $x \mapsto x^{-1}$. Furthermore, observe that the object $\mathrm{Spec} A$ is a gluing along a closed subscheme by an argument of [22], i.e., $\mathrm{Spec} A$ is the pushout

$$\begin{array}{ccc} \mathrm{Spec} \mathbb{F}_p & \xrightarrow{s} & \mathrm{Spec} \mathbb{Z}_p \\ (p, x) \downarrow & & \downarrow \iota_1 \\ \mathbb{A}_{\mathbb{Z}_p}^1 & \xrightarrow{\iota_2} & \mathrm{Spec} A \end{array}$$

where the closed immersion $\mathrm{Spec} \mathbb{F}_p \rightarrow \mathrm{Spec} \mathbb{Z}_p$ picks out the special fibre and the closed immersion $\mathrm{Spec} \mathbb{F}_p \rightarrow \mathbb{A}_{\mathbb{Z}_p}^1$ picks out the closed point (p, x) .

Define a functor $F : \mathcal{C} \rightarrow \mathbf{Sch}$ as follows: If $X \in \mathcal{C}_0$, define

$$FX := \begin{cases} \mathrm{Spec} \mathbb{Z}_p & \text{if } X = \mathrm{Spec} \mathbb{F}_p; \\ X & \text{else} \end{cases}$$

and if $f \in \mathcal{C}_1$, define

$$F(f) := \begin{cases} f & \text{if } \mathrm{Dom} f \neq \mathrm{Spec} \mathbb{F}_p; \\ \mathrm{id}_{\mathrm{Spec} \mathbb{Z}_p} & \text{if } f = \mathrm{id}_{\mathrm{Spec} \mathbb{F}_p}; \\ \varphi & \text{if } \mathrm{Dom} f = \mathrm{Spec} \mathbb{F}_p, f = \varphi \circ s, \varphi : \mathrm{Spec} \mathbb{Z}_p \rightarrow X, X \neq \mathrm{Spec} \mathbb{F}_p; \end{cases}$$

this fully defines the functor because for any $X \in \mathcal{C}_0$, if $X \neq \mathrm{Spec} \mathbb{F}_p$ then any map $\mathrm{Spec} \mathbb{F}_p \rightarrow X$ factors through the special fibre of $\mathrm{Spec} \mathbb{Z}_p$, i.e., there exists a morphism $g : \mathrm{Spec} \mathbb{Z}_p \rightarrow X$ making the diagram

$$\begin{array}{ccc} \mathrm{Spec} \mathbb{F}_p & \xrightarrow{f} & X \\ & \searrow s & \nearrow \varphi \\ & \mathrm{Spec} \mathbb{Z}_p & \end{array}$$

commute in \mathbf{Sch} . In particular, with this the verification that F is a functor is trivial and omitted.

The functor F just preserves the gluing of $\mathbb{P}_{\mathbb{Z}_p}^1$ by construction, and since this is the only nontrivial open subobject gluing, this shows that F is geometrically adhesive. On the other hand, F sends the pushout diagram defining $\mathrm{Spec} A$ to the diagram, where $x : \mathrm{Spec} \mathbb{Z}_p \rightarrow \mathbb{A}_{\mathbb{Z}_p}^1$ is the spectrum of the map $\mathbb{Z}_p[x] \rightarrow \mathbb{Z}_p$ corresponding to evaluation at $x = 0$,

$$\begin{array}{ccc}
\mathrm{Spec} \mathbb{Z}_p & \xlongequal{\quad} & \mathrm{Spec} \mathbb{Z}_p \\
\downarrow x & & \downarrow \iota_1 \\
\mathbb{A}_{\mathbb{Z}_p}^1 & \xrightarrow{\quad \iota_2 \quad} & \mathrm{Spec} A
\end{array}$$

which is evidently not a pushout in **Sch**; explicitly, the pushout of $\mathrm{Spec} \mathbb{Z}_p \xleftarrow{\quad} \mathrm{Spec} \mathbb{Z}_p \xrightarrow{x} \mathbb{A}_{\mathbb{Z}_p}^1$ is $\mathrm{Spec} B$, where B is the ring

$$B = \{(a, f) \mid a \in \mathbb{Z}_p, f \in \mathbb{Z}_p[x], f(0) = a\} = \mathbb{Z}_p \times_{\mathbb{Z}_p} \mathbb{Z}_p[x] \cong \mathbb{Z}_p[x].$$

But then F does not preserve the closed gluing of $\mathrm{Spec} A$ and does not preserve the pushout diagram. This proves that geometrically adhesive functors need not be cocontinuous or preserve gluings along closed subobjects.

Proposition 4. *Adhesive functors need not be full.*

Proof. Consider the functor \mathfrak{h} extended to all schemes X via the same assignment, i.e., via $\mathfrak{h} X = (|\mathfrak{h} X|, \mathfrak{h} \mathcal{O}_X)$ where

$$|\mathfrak{h} X| = |X|$$

and the sheaf $\mathfrak{h} \mathcal{O}_X$ is defined by, for all $U \subseteq |X|$ open,

$$\mathfrak{h} \mathcal{O}_X(U) = W(\mathcal{O}_X(U))$$

where W is the p -typical Witt vector functor à la Borger; see [4] and [5] for details. A routine calculation (when we identify $\mathbf{FSch}/_{\mathrm{Spec} R}$ as the ind-closure of $\mathbf{Sch}/_{\mathrm{Spec} R}$ and then give all affine schemes the trivial filtration) shows that

$$\mathbf{FSch}(\mathrm{Spec} \mathbb{Z}_p, \mathrm{Spec} \mathbb{F}_p) \cong \mathbf{Cring}(\mathbb{F}_p, \mathbb{Z}_p) = \emptyset.$$

However, since there is a unique continuous map $\tau : \{\eta, s\} \rightarrow \{*\}$, it is the case that we could have a sheaf morphism $\tau^\sharp : \mathfrak{h} \mathcal{O}_{\mathbb{F}_p} \rightarrow \tau_* \mathfrak{h} \mathcal{O}_{\mathbb{Z}_p}$. Explicitly we note that

$$\mathfrak{h} \mathcal{O}_{\mathbb{Z}_p}(U) = W(\mathcal{O}_{\mathbb{Z}_p}(U)) = \begin{cases} \mathbb{Z}_p \oplus \mathfrak{a} & \text{if } U = |\mathrm{Spec} \mathbb{Z}_p|; \\ \mathbb{Q}_p^{\mathbb{N}} & \text{if } U = \{\eta\}; \\ 0 & \text{if } U = \emptyset \end{cases}$$

where $\mathfrak{a} = \bigoplus_{n \geq 1} p^n \mathbb{Z}_p$ and the multiplication in $\mathbb{Z}_p \oplus \mathfrak{a}$ is by the rule

$$(a, x)(b, y) = (ab, ay + bx + xy).$$

The morphism $\varphi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \oplus \mathfrak{a}$ given by $a \mapsto (a, 0)$ is a ring homomorphism. On the other hand we see that for all $U \subseteq |\mathrm{Spec} \mathbb{F}_p|$ open,

$$\mathfrak{h} \mathcal{O}_{\mathbb{F}_p}(U) = W(\mathcal{O}_{\mathbb{F}_p}(U)) = \begin{cases} \mathbb{Z}_p & \text{if } U = \{*\}; \\ 0 & \text{if } U = \emptyset. \end{cases}$$

Then we can define a sheaf morphism $\tau^\sharp : \mathfrak{h} \mathcal{O}_{\mathbb{F}_p} \rightarrow \tau_* \mathfrak{h} \mathcal{O}_{\mathbb{Z}_p}$ because such a morphism of locally ringed spaces only sees global sections and the diagram

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\varphi} & \mathbb{Z}_p \oplus \mathfrak{a} \\ \exists! \downarrow & & \downarrow \exists! \\ 0 & \xrightarrow{\quad} & 0 \end{array}$$

commutes; moreover, it is not hard to see that this is the only such sheaf morphism between $\mathfrak{h} \mathcal{O}_{\mathbb{F}_p}$ and $\tau_* \mathfrak{h} \mathcal{O}_{\mathbb{Z}_p}$. Thus we find that

$$\mathbf{FSch}(\mathfrak{h} \text{Spec } \mathbb{Z}_p, \mathfrak{h} \text{Spec } \mathbb{F}_p) = \{(\tau, \tau^\sharp)\} \neq \emptyset = \mathbf{Sch}(\text{Spec } \mathbb{Z}_p, \text{Spec } \mathbb{F}_p).$$

This shows that \mathfrak{h} is not full when extended to a functor $\mathfrak{h} : \mathbf{Sch} \rightarrow \mathbf{FSch}$.

We now move from these definitions and remarks to give some properties of geometrically adhesive functors. In particular we will show that if U is an open subobject of V in \mathcal{C} (that is there is an open immersion $i : U \rightarrow V$) and if $F : \mathcal{C} \rightarrow \mathcal{D}$ is geometrically adhesive, then FU is an open subobject of FV (more explicitly, the map $Fi : FU \rightarrow FV$ is an open immersion).

Proposition 5. *Let $i : U \rightarrow V$ be an open immersion in \mathcal{C} and assume that $F : \mathcal{C} \rightarrow \mathcal{D}$ is geometrically adhesive. Then $Fi : FU \rightarrow FV$ is an open immersion.*

Proof. Begin by recalling that the fact that F is geometrically adhesive implies that if $\{V_i \rightarrow V \mid i \in I\}$ is an open covering of V , then

$$FV = F \left(\bigcup_{i \in I} V_i \right) = \bigcup_{i \in I} FV_i.$$

Thus it follows that each FV_i remains a subobject of FV , and hence we conclude that $|FV'| \subseteq |FV|$ whenever V' is an open subobject of V . Therefore we have that FU is a subobject of FV .

Finally, to see that FU is an open subobject of FV , we note that since F is a functor between (slice) categories of locally ringed spaces, F takes sheaves to sheaves and hence is at least Zariski continuous. Now consider the map $i : U \rightarrow V$ and observe that we can deduce, from the functoriality of F and the pushforward/pullback adjunction $i^{-1} \dashv i_* : \mathbf{Shv}(V) \rightarrow \mathbf{Shv}(U)$, that the following deduction may be made: First, observe that since i is an open immersion, there is an isomorphism of U -sheaves $i^b : i^{-1} \mathcal{O}_V \cong \mathcal{O}_U$; pushing this through the adjunction gives rise to the sheaf morphism

$$i^\sharp : \mathcal{O}_V \rightarrow i_* \mathcal{O}_U .$$

Applying the functor F on sheaves then gives the sheaf morphism

$$F(i^\sharp) : F \mathcal{O}_V \rightarrow (Fi)_* F \mathcal{O}_U$$

which is equivalent by the functoriality of F to the sheaf morphism

$$F(i)^\sharp : \mathcal{O}_{FV} \rightarrow F(i)_* \mathcal{O}_{FU} .$$

Passing this through the adjunction $F(i)^{-1} \dashv F(i)_*$ sends the morphism $F(i)^\sharp$ to the map

$$F(i)^\flat : F(i)^{-1} \mathcal{O}_{FV} \rightarrow \mathcal{O}_{FU}$$

which is equivalent, again by the functoriality of F , to the map

$$F(i^\flat) : (Fi)^{-1} F \mathcal{O}_V \rightarrow F \mathcal{O}_U .$$

Now, because i^\flat is an isomorphism it follows that $F(i^\flat) = F(i)^\flat$ is as well. Therefore $(Fi)^{-1} \mathcal{O}_{FV} \rightarrow \mathcal{O}_{FU}$ is an isomorphism of FU -sheaves, and hence the map $Fi : FU \rightarrow FV$ is an open immersion. This completes the proof of the proposition.

We now proceed to show that geometrically adhesive functors preserve pullbacks defined by open gluings, i.e., that such functors preserve intersections. This will be necessary because it will allow us to show that geometrically adhesive functors preserve pullbacks along covers induced by pretopologies. As we proceed with this, we assume the following:

1. The category \mathcal{C} has the property that if U and V are objects of \mathcal{C} , then the union space $U \cup V$ and intersection space $U \cap V$ exists in \mathcal{C} .

In this way we can regard any two objects as open subobjects of some larger object, and we can view their intersection as an open subspace of U, V , and $U \cup V$.

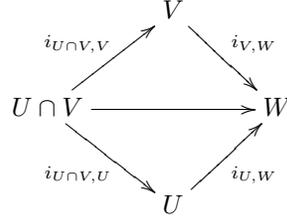
Theorem 2.1 *If U and V are objects in \mathcal{C} and $F : \mathcal{C} \rightarrow \mathcal{D}$ is geometrically adhesive, then*

$$F(U \cap V) = FU \cap FV .$$

Proof. Let U and V be objects of \mathcal{C} and assume that there are open immersions $U, V \rightarrow W$, for some $W \in \text{Ob } \mathcal{C}$. However, since U and V are glued along their intersection, it suffices to prove the proposition for the locally ringed space

$$W := U \coprod_{U \cap V} V = U \cup V ,$$

where the pushout $U \coprod_{U \cap V} V$ is the gluing of U and V along the subspace $U \cap V$; note also that in this definition, since U and V are open in W , $U \cap V$ is open and the immersion $U \cap V \rightarrow W$ factors as

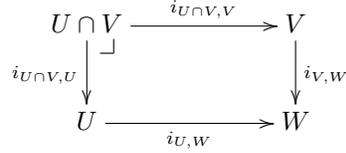


where each of the i morphisms are open immersions.

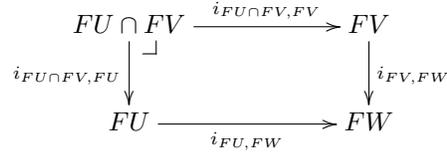
We will first show that the canonical map $F(U \cap V) \rightarrow FU \cap FV$ is an open immersion. Begin with the observations that

$$FW = F(U \cup V) = FU \cup FV = FU \coprod_{FU \cap FV} FV$$

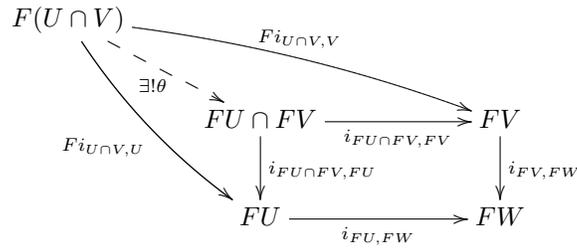
and that $U \cap V$ is the pullback



in \mathcal{C} . Thus we get that $FU \cap FV$ is the pullback



and so applying F gives rise to a unique morphism $\theta : F(U \cap V) \rightarrow FU \cap FV$ making the diagram



commute in \mathcal{D}_Y . Moreover, since F preserves open immersions by Proposition 5, the morphisms $F i_{U \cap V, V}$ and $F i_{U \cap V, U}$ are both open immersions in \mathcal{D}_Y . Thus, from the equations

$$i_{FU \cap FV, FU} \circ \theta = Fi_{U \cap V, U}$$

and

$$i_{FU \cap FV, FV} \circ \theta = Fi_{U \cap V, V}$$

it follows that θ is both monic and open, and hence an open immersion.

We will now show that

$$FU \coprod_{FU \cap FV} = FU \cup FV = FW = FU \coprod_{F(U \cap V)} FV,$$

as this will imply that $F(U \cap V) \cong FU \cap FV$; the fact that $F(U \cap V)$ is an open subobject of $FU \cap FV$ will then complete the proof. To do this assume that there is an object S of \mathcal{D}/Y and that there are morphisms $\varphi : FU \rightarrow S$ and $\psi : FV \rightarrow S$ such that the rectangle

$$\begin{array}{ccc} FU & \xrightarrow{\varphi} & S \\ i_{FU \cap FV, FU} \uparrow & & \uparrow \psi \\ FU \cap FV & \xrightarrow{i_{FU \cap FV, FV}} & FV \end{array}$$

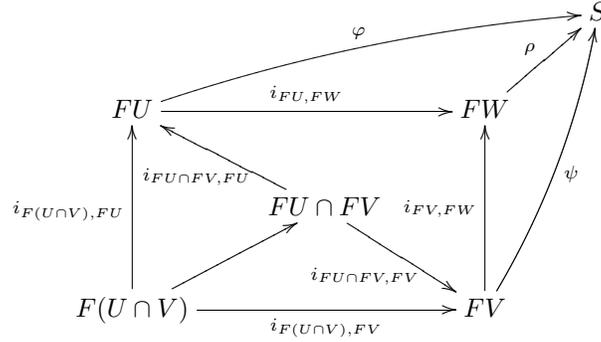
commutes. Then there exists a unique morphism $\rho : FW \rightarrow S$ making the diagram

$$\begin{array}{ccccc} & & & & S \\ & & & \varphi & \nearrow \\ & & & & \exists! \rho \\ FU & \xrightarrow{i_{FU, FW}} & FW & \xrightarrow{\psi} & \\ i_{FU \cap FV, FU} \uparrow & & i_{FV, FW} \uparrow & & \\ FU \cap FV & \xrightarrow{i_{FU \cap FV, FV}} & FV & \xrightarrow{\psi} & \end{array}$$

commutes. Now observe that since the open immersion

$$i_{F(U \cap V), FU \cap FV} : F(U \cap V) \rightarrow FU \cap FV$$

makes the diagram



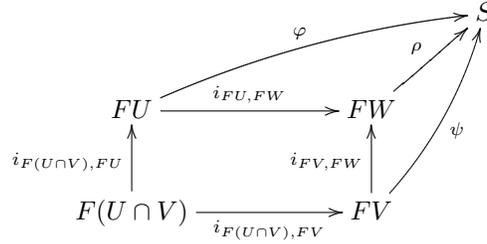
commute, where the arrow $F(U \cap V) \rightarrow FU \cap FV$ is the open immersion $i_{F(U \cap V), FU \cap FV}$; thus using that

$$i_{F(U \cap V), FU} = i_{FU \cap FV, FU} \circ i_{F(U \cap V), FU \cap FV}$$

and

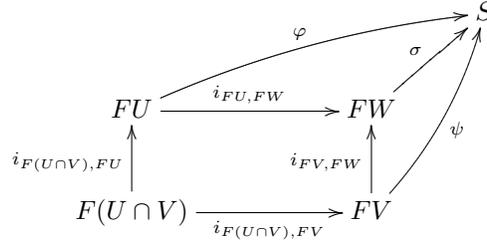
$$i_{F(U \cap V), FV} = i_{FU \cap FV, FV} \circ i_{F(U \cap V), FU \cap FV}$$

as open immersions, we have that the diagram

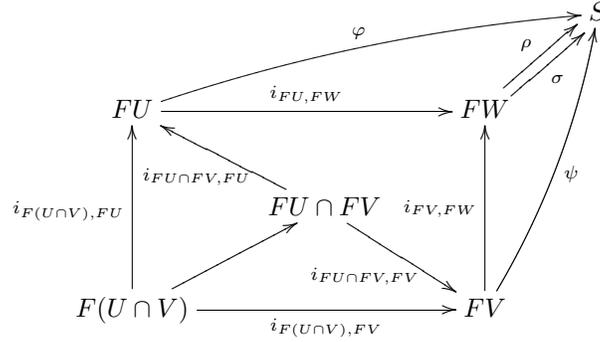


commutes as well. Thus, to show that $FW = FU \coprod_{F(U \cap V)} FV$ as well, we need only show that ρ is the unique such map making this diagram commute.

With this in mind, assume that there exists a morphism $\sigma : FW \rightarrow S$ making the diagram



commute as well. But then we derive that the diagram



commutes with the equations

$$\sigma \circ i_{FU,FW} = \varphi$$

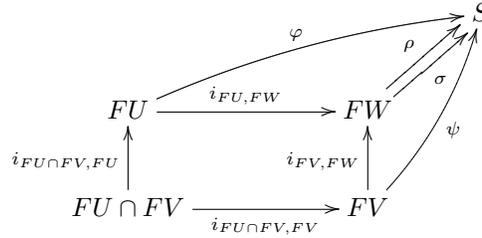
and

$$\sigma \circ i_{FV,FW} = \psi$$

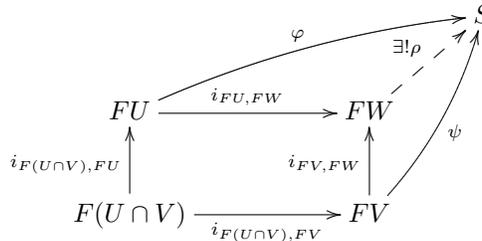
holding by assumption. But then since

$$\varphi \circ i_{FU \cap FV,FU} = \psi \circ i_{FU \cap FV,FV}$$

it follows that the diagram



commutes as well. Thus the universal property of FW as the pushout $FU \amalg_{FU \cap FV} FV$ gives that $\rho = \sigma$ and so there is a unique morphism $\rho : FW \rightarrow S$ making the diagram



commute. This shows that if P is the pushout

$$P := FU \coprod_{F(U \cap V)} FV$$

then FW is a subobject of P .

To prove that P is a subobject of FW , assume that there exist morphisms $\varphi' : FU \rightarrow S'$ and $\psi' : FV \rightarrow S'$ making the diagram

$$\begin{array}{ccc} FU & \xrightarrow{\varphi'} & S' \\ i_{F(U \cap V), FU} \uparrow & & \uparrow \psi' \\ F(U \cap V) & \xrightarrow{i_{F(U \cap V), FV}} & FV \end{array}$$

commute. Now let $\rho' : P \rightarrow S'$ be the unique map out of the pushout making

$$\begin{array}{ccc} & & S' \\ & \nearrow \varphi' & \\ FU & \xrightarrow{\iota_1} & P \\ i_{F(U \cap V), FU} \uparrow & & \uparrow \iota_2 \\ F(U \cap V) & \xrightarrow{i_{F(U \cap V), FV}} & FV \\ & \searrow \psi' & \end{array} \quad \begin{array}{c} \exists! \rho' \\ \nearrow \\ \searrow \end{array}$$

commute. Factorizing the morphisms $i_{F(U \cap V), FU}$ and $i_{F(U \cap V), FV}$ through $FU \cap FV$ then allows us to produce the commuting diagram:

$$\begin{array}{ccccc} & & & & S' \\ & & & & \nearrow \varphi' \\ & & & & \rho' \\ & & & & \uparrow \\ & & & & \psi' \\ FU & \xrightarrow{\iota_1} & P & \xrightarrow{\rho'} & \\ \uparrow i_{F(U \cap V), FU} & \nearrow i_{FU \cap FV, FU} & \uparrow \iota_2 & & \\ & FU \cap FV & & & \\ \downarrow i_{FU \cap FV, FV} & \searrow i_{FU \cap FV, FV} & \downarrow \iota_2 & & \\ F(U \cap V) & \xrightarrow{i_{F(U \cap V), FV}} & FV & & \end{array}$$

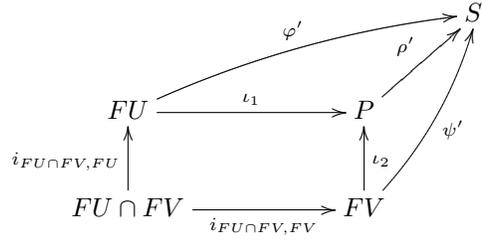
Observe that because P is the pushout of FU and FV over the open subobject $F(U \cap V)$, ι_1 and ι_2 are both open immersions as well. Furthermore, since $F(U \cap V)$ is an open subobject of $FU \cap FV$ and since the morphisms $i_{FU \cap FV, FU}$ and $i_{FU \cap FV, FV}$ are both open immersions as well, we have that

$$\iota_1 \circ i_{FU \cap FV, FU} = \iota_2 \circ i_{FU \cap FV, FV}.$$

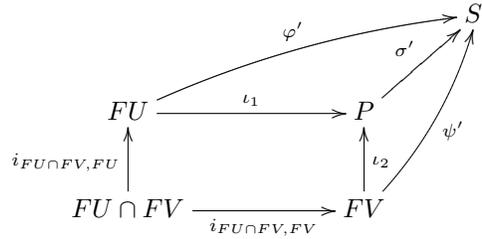
We then compute that

$$\varphi' \circ i_{FU \cap FV, FU} = \rho' \circ \iota_1 \circ i_{FU \cap FV, FU} = \rho' \circ \iota_2 \circ i_{FU \cap FV} = \psi' \circ i_{FU \cap FV, FV}$$

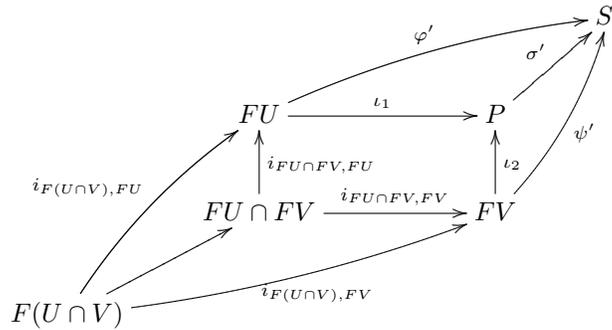
which shows that the diagram



commutes. To see that this does so universally through ρ' , assume that there exists a $\sigma' : P \rightarrow S'$ such that



commutes. However, consider now the diagram



and note that this composite implies that

$$\begin{array}{ccccc}
& & & & S' \\
& & & \nearrow \varphi' & \\
FU & \xrightarrow{\iota_1} & P & \nearrow \sigma' & \\
\uparrow i_{F(U \cap V), FU} & & \uparrow \iota_2 & \nearrow \psi' & \\
F(U \cap V) & \xrightarrow{i_{F(U \cap V), FV}} & FV & &
\end{array}$$

commutes. Using the universal property of the pushout P then gives us that $\sigma' = \rho'$ and hence shows that P is a subobject of FW by a factorization of the universal property.

Now that we have shown that FW is a subobject of $FU \amalg_{F(U \cap V)} FV$ and $FU \amalg_{F(U \cap V)} FV$ is a subobject of FW , it follows that

$$FU \amalg_{FU \cap FV} FV = FU \cup FV = FW = FU \amalg_{F(U \cap V)} FV$$

and so we derive from this that $FU \cap FV \cong F(U \cap V)$; however, since the map

$$i_{F(U \cap V), FU \cap FV} : F(U \cap V) \rightarrow FU \cap FV$$

is an open immersion, it follows that $FU \cap FV = F(U \cap V)$ and we are done.

Proposition 6. *Let R be a ring object in **LRS**. Then the functor $h_R : \mathcal{C} \rightarrow \mathbf{LRS}$ defined by, for a locally ringed space $X = (|X|, \mathcal{O}_X)$,*

$$|h_R X| = |X|$$

and, for opens $U \subseteq |X|$,

$$\mathcal{O}_{h_R X}(U) := \mathcal{C}(U, R)$$

is geometrically adhesive.

Proof. The verification that this satisfies the topological side of the definition is trivial, while the sheaf-theoretic side follows from a standard gluing argument.

Corollary 2. *If $\mathrm{Gr} : \mathbf{FSch}/\mathrm{Spec} R \rightarrow \mathbf{Sch}/\mathrm{Spec} k$ is the Greenberg transform, then the left adjoint $(\mathfrak{h} \dashv \mathrm{Gr}^R) : \mathbf{Sch}/\mathrm{Spec} k \rightarrow \mathbf{FSch}/\mathrm{Spec} R$ is geometrically adhesive.*

Proposition 7. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{A}$ are geometrically adhesive, then $G \circ F : \mathcal{C} \rightarrow \mathcal{A}$ is geometrically adhesive.*

Proof. Immediate from the calculation

$$\begin{aligned} (G \circ F) \left(\bigcup_{i \in I} U_i \right) &= G \left(F \left(\bigcup_{i \in I} U_i \right) \right) = G \left(\bigcup_{i \in I} F U_i \right) = \bigcup_{i \in I} G(F U_i) \\ &= \bigcup_{i \in I} (G \circ F)(U_i). \end{aligned}$$

3 The Adhesive Site

From the definition of geometrically adhesive functors and Proposition 5, we see that geometrically adhesive functors have good behaviour with respect to open gluings of schemes and open immersions of schemes. We would now like to define a topology on the codomain of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ so that the gluing condition

$$F \left(\bigcup_{i \in I} U_i \right) \cong \bigcup_{i \in I} F U_i$$

becomes instead an intrinsic property of the functor preserving coverages that are allowed to occur with respect to whatever topology on \mathcal{C} that we have at hand. In particular, we will show that under the construction of this topology, the Greenberg functor \mathfrak{h} is site-cocontinuous (cf. Corollary 3).

As we proceed, we recall briefly the notion of a sieve on a category. A sieve S on an object U of \mathcal{C} is a subfunctor of the representable functor on U , i.e., a monomorphism $S \rightarrow \mathcal{C}(-, U)$ in the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$. Results from classical Grothendieck topos theory (cf. [14]) show that a Grothendieck topology is completely determined by its covering sieves in a nonambiguous way. Thus we will work with sieves on the highest level possible, and specialize to working with coverages in the sense of Grothendieck pretopologies when we need to work with explicit covers.

Begin by letting $C := \{\varphi_i : U_i \rightarrow U \mid i \in I\}$ be a set of covering morphisms on an object U in \mathcal{C} . We then define the *sieve generated by C* to be the functor $(C) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ where first we define a set

$$(C) := \{\varphi_i \circ \psi \mid \varphi_i \in C, \text{Dom } \varphi_i = \text{Codom } \psi\}$$

and then defining the action of (C) on any object X of \mathcal{C} via

$$(C)(X) := \{\theta \in (C) \mid \text{Dom } \theta = X\},$$

while the action of (C) on morphisms is simply by precomposition. It is readily checked that (C) defines a sieve on U . In a similar vein, if \mathcal{C} is a collection of sieves we wish to make into covering sieves on a category, we write

$$\langle C \mid C \in \mathcal{C} \rangle$$

for the Grothendieck topology generated by the sieves C in \mathcal{C} , i.e., for the minimal Grothendieck topology J on \mathcal{C} which contains all the sieves C as covering sieves. We will use both of these notions to define a topology on \mathcal{D} in terms of a given topology on \mathcal{C} .

Definition 2. Let J be a fixed Grothendieck topology on a category of locally ringed spaces \mathcal{C} and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an geometrically adhesive functor where \mathcal{D} is also a category of locally ringed spaces. We then generate a topology \mathcal{A}_F^J on \mathcal{D} by taking

$$\mathcal{A}_F^J := \langle (FC) \mid C \text{ is a } J\text{-covering sieve} \rangle$$

where, if C is a sieve on $U \in \text{Ob } \mathcal{C}$,

$$FC := \{F\varphi : FV \rightarrow FU \mid \varphi \in C\}.$$

The topology \mathcal{A}_F^J is then called the *J - F -geometrically adhesive site on \mathcal{D}* .

Remark 5. If the functor F and the site J are clear from context, we will simply refer to the site \mathcal{A}_F^J as the *geometric adhesive site* on \mathcal{D} instead of the *F - J -geometric adhesive site* on \mathcal{D} .

Lemma 1. *If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a geometrically adhesive functor and if (\mathcal{C}, J) is a site, then the functor $F : (\mathcal{C}, J) \rightarrow (\mathcal{D}, \mathcal{A}_F^J)$ is cover reflecting.*

Proof. This is immediate from construction. Since the topology \mathcal{A}_F^J is minimally generated by FJ , a subfunctor $S \rightarrow \mathcal{D}(-, FU)$ is an \mathcal{A}_F^J -covering sieve on FU if and only if there exists an $R \in J(U)$ such that $FR \subseteq S$. However, this is what it means to be cover reflecting.

Corollary 3. *The Greenberg functor $\mathfrak{h} : \mathbf{Sch}/_{\text{Spec } k} \rightarrow \mathbf{FSch}/_{\text{Spec } R}$ is site-cocontinuous.*

Proof. From the discussion around the definition of what it means to be cover reflecting in [16], a functor of sites is cocontinuous if and only if it is cover reflecting.

The cover reflecting property of the \mathcal{A}_F^J topology is a very practical one, as it will allow us to prove an intuitive result: It allows us to show that if a pretopology τ generates the topology J on \mathcal{C} , then $F\tau$ generates \mathcal{A}_F^J . However, we cannot in general say that there is a pretopology generated by $F\tau$ on \mathcal{D} , as \mathcal{D} need not admit pullbacks.

Proposition 8. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be geometrically adhesive and assume that the site (\mathcal{C}, J) is generated by the pretopology τ and that \mathcal{D} admits pullbacks. Then the pretopology $\tau' = \langle F\tau \rangle$ on \mathcal{D} generates \mathcal{A}_F^J .*

Proof. Observe that if $\mathcal{D}(FU, V) = \emptyset = \mathcal{D}(V, FU)$, then there is nothing to say, as the only covering sieves of V are trivial. Moreover, by the Cover

Reflecting Property it suffices to show that if $R \in \mathcal{A}_F^J(FU)$ is a covering sieve, then there exists a cover $C \in J(U)$ for which $FC \subseteq R$, as we can refine along covers formed from FU_i 's to collections that come from functorial images of J -covers through the functor F .

To prove this let ρ be the maximal pretopology on \mathcal{D} which generates \mathcal{A}_F^J and let $\{\psi_i : V_i \rightarrow FU \mid i \in I\} \in \rho(FU)$ be given such that

$$\{V_i \rightarrow FU \mid i \in I\} \subseteq R;$$

such a cover exists because ρ generates \mathcal{A}_F^J . Now, since \mathcal{A}_F^J is cover reflecting, there exists a collection of morphisms $\{F\varphi_j : FU_j \rightarrow FU \mid j \in J\}$ which refines $\{\psi_i : V_i \rightarrow FU \mid i \in I\}$. However, we also have that

$$\{F\varphi_j : FU_j \rightarrow FU \mid j \in J\} = F\{\varphi_j : U_j \rightarrow U \mid j \in J\}$$

and $C := \{\varphi_j : U_j \rightarrow U \mid j \in J\} \in \tau(U)$. But then it follows that if S is the J -covering sieve generated by this cover, $(FS) \subseteq R$. Since the cover $\langle FC \rangle \in \tau'(FU)$, we get that $\langle FC \rangle \subseteq (FS) \subseteq R$ and so it follows that τ' generates \mathcal{A}_F^J .

Remark 6. Note that the above proof does not show that the pretopology $\sigma = \langle F\tau \rangle$ is the *maximal* pretopology generating \mathcal{A}_F^J . I do not think that this will happen, in general, but this remains open.

Many of the results we will now give are presented in this section for the purpose of having tools to compute the fundamental groups of the sheaf toposes $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ so that we can have comparisons of the nature, for locally ringed spaces $U \in \text{Ob } \mathcal{C}$ and a geometrically adhesive functor $F : \mathcal{C} \rightarrow \mathcal{D}$,

$$\pi_1^J(U) \stackrel{?}{\rightarrow} \pi_1^{\mathcal{A}_F^J}(FU), \quad \pi_1^{\mathcal{A}_F^J}(FU) \stackrel{?}{\rightarrow} \pi_1^J(U)$$

at least in the case in which both U and FU are connected. We will use these in particular to study the adjunction $\mathfrak{h} \dashv \text{Gr}^R : \mathbf{Sch}_{/\text{Spec } k} \rightarrow \mathbf{FSch}_{/\text{Spec } R}$ and to hopefully understand why, at least in the case in which J is the étale topology on $\mathbf{Sch}_{/\text{Spec } k}$, passing through the Greenberg Transform allows us to geometrize quasi-characters and see things over R that simply do not come from the étale site over $\text{Spec } R$; for details see [11] on the motivation for this idea.

In order to discuss how to move these results over, we need to prove two key lemmas which will allow us to reduce proving that a presheaf $P : \mathcal{D}^{\text{op}} \rightarrow \mathbf{Set}$ is an \mathcal{A}_F^J -sheaf to checking that it satisfies the sheaf axiom on images of J -covers. This will then allow us to prove Lemma 3 below, which itself is essential for the proof of Theorem 3.1.

As we proceed, we will assume the following: The categories \mathcal{C} and \mathcal{D} have pullbacks, and if J is a topology on \mathcal{C} generated by pretopology τ , then $\rho = \langle F\tau \rangle$ is the pretopology on \mathcal{D} generated by τ . Note that by Proposition 8 ρ generates \mathcal{A}_F^J , so it suffices to argue if presheaves P on \mathcal{D} are \mathcal{A}_F^J -sheaves by

checking on ρ -coverages by a standard result of site theory (cf. Proposition 1 of [19]). However, we need to prove the lemma below, save for with one observation at hand. Note that since \mathcal{A}_F^J is generated by FJ , any nontrivial cover of an object V of \mathcal{D} can be refined by some cover of the form $\{g_i : FU_i \rightarrow V \mid i \in I\}$. Thus, when one considers the sheaf condition for a presheaf P on \mathcal{D} , if V is an object of \mathcal{D} with $\mathcal{D}(FU, V) \neq \emptyset$ for some U in \mathcal{C} , it suffices to consider refinements of $\{V_j \rightarrow V \mid j \in J\}$ of the form $\{FU_i \rightarrow V \mid i \in I\}$.

Lemma 2. *Let V be an object of \mathcal{D} such that there exists a cover $D := \{h_j : V_j \rightarrow V \mid j \in J\} \in \rho(V)$ for which there is a refinement $\{g_i : FU_i \rightarrow V \mid i \in I\}$ that makes $\{FU_i \rightarrow FX \mid i \in I\}$ the functorial image of a cover $C := \{U_i \rightarrow X \mid i \in I\}$, where*

$$X := \bigcup_{i \in I} U_i.$$

Then a presheaf P satisfies the sheaf axiom with respect to the cover D if and only if P satisfies the sheaf axiom with respect to $F(C)$.

Proof. Let us begin by fixing some notation. The map

$$e_V : P(V) \rightarrow \prod_{j \in J} P(V_j)$$

is the pairing map $e_V = \langle P(h_j) \rangle_{j \in J}$ (and analogously $e_{FX} = \langle P(\iota_{FU_i}) \rangle_{i \in I}$), while since $\{g_i : FU_i \rightarrow V \mid i \in I\}$ refines D , for each $j \in J$ there exists an $i \in I$ such that there is a factorization

$$\begin{array}{ccc} FU_i & \xrightarrow{g_i} & V \\ & \searrow \varphi_{ij} & \nearrow h_j \\ & & V_j \end{array}$$

in \mathcal{D} . There then is a map, for any presheaf P on \mathcal{D} , $\alpha : \prod_{j \in J} P(V_j) \rightarrow \prod_{i \in I} P(FU_i)$ which is given as follows: For each $j \in J$, find all $i \in I$ for which there are factorizations as above, say indexed by the set I_j , and then construct the map

$$\begin{array}{ccc} \prod_{j \in J} P(V_j) & \xrightarrow{\alpha_j} & \prod_{i \in I_j} P(FU_i) \\ \pi_j \downarrow & \nearrow \langle \varphi_{ij} \rangle_{i \in I_j} & \\ P(V_j) & & \end{array}$$

in **Set**; the map α then takes the form $\alpha = \langle \alpha_j \rangle_{j \in J}$. In this same way we can get a map $\beta : \prod_{j, j' \in J} P(V_j \times_V V_{j'}) \rightarrow \prod_{i, i'} P(FU_i \times_{FU} FU_{i'})$ by taking $\beta = \langle P(\varphi_{ij} \times \varphi_{i'j'}) \rangle$.

\implies : This direction is clear by performing a refinement argument and lifting any map $f : Y \rightarrow \prod_{i \in I} P(FU_i)$ along the $P\varphi_{ij}$, iterating through all the $j \in J$, and then using that this lift factors through both the equalizer e_V and the unique map $P(g) : PV \rightarrow P(FX)$.

\impliedby : We now assume that P satisfies the sheaf axiom with respect to the cover D . Now consider the commuting diagram

$$\begin{array}{ccc} P(V) & \xrightarrow{e_V} & \prod_{j \in J} P(V_j) \rightrightarrows \prod_{j, j' \in J} P(V_j \times_V V_{j'}) \\ \downarrow P g & & \downarrow \alpha \\ P(FX) & \xrightarrow{e_{FX}} & \prod_{i \in I} P(FU_i) \rightrightarrows \prod_{i, i' \in I} P(FU_i \cap FU_{i'}) \\ & & \downarrow \beta \end{array}$$

where the maps α, β, e_V , and e_{FX} are defined as above, and e_{FX} is an equalizer by assumption of P satisfying the sheaf axiom on FC . Now suppose that there exists some set Z and a morphism $f : Z \rightarrow \prod_{j \in J} P(V_j)$ such that the diagram

$$Z \xrightarrow{f} \prod_{j \in J} P(V_j) \rightrightarrows \prod_{j, j' \in J} P(V_j \times_V V_{j'})$$

commutes in **Set**. It then follows by construction that the diagram

$$Z \xrightarrow{\alpha \circ f} \prod_{i \in I} P(FU_i) \rightrightarrows \prod_{i, i' \in I} P(FU_i \times_{FU} FU_{i'})$$

commutes in **Set**, so by the universal property of an equalizer there exists a unique function $k : Z \rightarrow P(FX)$ making the diagram

$$\begin{array}{ccc} FX & \xrightarrow{e_{FX}} & \prod_{i \in I} P(FU_i) \rightrightarrows \prod_{i, i' \in I} P(FU_i \times_{FU} FU_{i'}) \\ \uparrow & & \uparrow \\ \exists! k \downarrow & \nearrow \alpha \circ f & \\ V & & \end{array}$$

commute. This implies that in particular, the diagram

$$\begin{array}{ccc} P(FX) & \xrightarrow{e_{FX}} & \prod_{i \in I} P(FU_i) \rightrightarrows \prod_{i, i' \in I} P(FU_i \times_{FU} FU_{i'}) \\ \uparrow \exists! k & & \uparrow \alpha \\ V & \xrightarrow{f} & \prod_{j \in J} P(V_j) \rightrightarrows \prod_{j, j' \in J} P(V_j \times_V V_{j'}) \\ & & \uparrow \beta \end{array}$$

commutes. We now claim that k factors through $P(V)$ and $P(g)$. To see this note that since

$$\alpha \circ f = e_{FX} \circ k = \langle P(\iota_{FU_i}) \rangle_{i \in I} \circ k = \langle P \iota_{FU_i} \circ k \rangle_{i \in I}$$

it suffices to argue on the FU_i by virtue of the fact that $FX = \bigcup FU_i$ and a colimit is determined by the maps out of its colimiting objects. Now, since for each $j \in J$ we can find an $i \in I$ such that $g_i = h_j \circ \varphi_{ij}$, we need only make local gluing arguments. In particular, fix some $s \in P(FX)$ and note that $e_{FX}(s) = (s_i)_{i \in I}$ produces an element whose image in the product of pullback sections $\prod_{i, i' \in I} P(FU_i \times_{FX} FU_{i'})$ is the same under P applied to either pullback map; call this image

$$t = P(\pi_1^{i, i'})(s_i)_{i \in I} = P(\pi_2^{i, i'})(s_i)_{i \in I} = (s_{i, i'})_{i, i' \in I}.$$

However, from the refinement condition on covers and the commutativity of the diagram, we can find some section $(v_j)_{j \in J} \in \prod_{j \in J} P(V_j)$ for which

$$\alpha(v_j)_{j \in J} = (s_i)$$

and hence find some $t' = (v_{j, j'})_{j, j' \in J} \in \prod_{j, j' \in J} P(V_j \times_V V_{j'})$ which maps through β to t . Explicitly,

$$\beta(t') = \beta(s_{j, j'}) = (s_{i, i'})_{i, i' \in I} = t.$$

This in turn allows us to conclude that

$$P(\pi_1^{j, j'})(v_j) = P(\pi_2^{j, j'})(v_j)$$

by applying β and then using the commutativity of the diagram. However, writing $s \in P(FX)$ as $s = k(z)$ for some $z \in Z$ gives that

$$\alpha(v_j)_{j \in J} = (\alpha \circ f)(z) = (e_{FX} \circ k)(z)$$

and hence the section $(s_i)_{i \in I} = e_{FX}(s)$ comes from simultaneously a uniquely given section over FX and a section on each of the V_j . Thus we can find some $v \in P(V)$ for which

$$P(g)(v) = s$$

and

$$e_V(v) = (v_j)_{j \in J},$$

by the geometry of the V_j and FX , and the choice of these v determines a function $\gamma : Z \rightarrow P(V)$. That is, we have a factorization

$$\begin{array}{ccccc}
Z & & & & \\
& \searrow & & \searrow & \\
& & P(V) & \xrightarrow{P(g)} & P(FX) \\
& \searrow & \downarrow e_V & & \downarrow e_{FX} \\
& & \prod_{j \in J} P(V_j) & \xrightarrow{\alpha} & \prod_{i \in I} P(FU_i)
\end{array}$$

which makes the diagram

$$\begin{array}{ccc}
PV & \xrightarrow{e_V} & \prod_{j \in J} P(V_j) \rightrightarrows \prod_{j \in J} P(V_j \times_V V_{j'}) \\
\uparrow \exists \gamma & \nearrow f & \\
Z & &
\end{array}$$

commutes. However, this map γ can be easily seen to be unique as follows: If there exists a morphism $\delta : Z \rightarrow P(V)$ giving the same factorization, the fact that e_{FX} is an equalizer gives that

$$P(g) \circ \gamma = k = P(g) \circ \delta;$$

consequently, from the uniqueness of k and the fact that $P(g)$ is determined based on the gluing data of the FU_i , it follows that $\gamma = \delta$. However, this implies that the diagram

$$\begin{array}{ccc}
PV & \xrightarrow{e_V} & \prod_{j \in J} P(V_j) \rightrightarrows \prod_{j \in J} P(V_j \times_V V_{j'}) \\
\uparrow \exists! & \nearrow f & \\
Z & &
\end{array}$$

commutes and so e_V is an equalizer, as was to be shown.

Remark 7. The above lemma shows that if V is an object of \mathcal{D} with $V \cong FX$ for all objects X of \mathcal{C} , then we can characterize the \mathcal{A}_F^J -covering sieves of V as follows: Assume that the J -cover $\{V_j \rightarrow V \mid j \in J\}$ of V is refined by $\{FX_i \rightarrow V \mid i \in I\}$ and assume that X is the gluing of the X_i so that FX is the gluing of the FX_i . Let $\rho : FX \rightarrow V$ be the canonical map. Then, for any covering sieve S containing the cover $\{V_j \rightarrow V \mid j \in J\}$, we must be able to find a J -cover R of X which factors through covers on each of the X_i , and S must contain the set $\{\rho \circ F\varphi \mid \varphi \in R\}$.

Corollary 4. *Let $\mathcal{F} : \mathcal{D}^{\text{op}} \rightarrow \mathbf{Set}$ be a presheaf. Then \mathcal{F} is an \mathcal{A}_F^J -sheaf if and only if for all $U \in \text{Ob } \mathcal{C}$, the diagram*

$$\mathcal{F}(FU) \longrightarrow \prod_{i \in I} \mathcal{F}(FU_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(FU_i \cap FU_j)$$

is an equalizer for all J -covers $\{U_i \rightarrow U \mid i \in I\}$ of U .

Proof. For objects of \mathcal{D} that either receive or have morphisms from or into objects FX , for $X \in \mathcal{C}_0$, this follows from the above lemma. The remaining case follows from observing that if $V \in \mathcal{D}_0$ with the relations $\mathcal{D}(V, FX) = \emptyset = \mathcal{D}(FX, V)$ for all $X \in \mathcal{C}_0$, then the only \mathcal{A}_F^J covers on V are trivial. This in turn implies that any presheaf satisfies the sheaf axiom over V ; combining this with the prior lemma gives the corollary.

Lemma 3. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be geometrically adhesive and let J be a site on \mathcal{C} . Then the functor*

$$F^* : \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) \rightarrow [(\mathcal{C})^{\text{op}}, \mathbf{Set}]$$

given by $F^*(\mathcal{F}) := \mathcal{F} \circ F$ factors as:

$$\begin{array}{ccc} \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) & \xrightarrow{F^*} & [(\mathcal{C})^{\text{op}}, \mathbf{Set}] \\ & \searrow^{F^*} & \nearrow \\ & \mathbf{Shv}(\mathcal{C}, J) & \end{array}$$

Proof. Begin by observing that by Corollary 4 it suffices to prove that a presheaf on \mathcal{D} is in fact an \mathcal{A}_F^J -sheaf by using the functorial images of J -covers. Let \mathcal{F} be any \mathcal{A}_F^J -sheaf and let $\{\varphi_i : U_i \rightarrow U \mid i \in I\}$ be a J -covering in a pretopology τ generating J . Then consider the following diagram, where the equality between the second and third rows follows from:

$$\begin{array}{ccccc} (F^*\mathcal{F})(U) & \xrightarrow{\langle (F^*\mathcal{F})(\varphi_i) \rangle_{i \in I}} & \prod_{i \in I} (F^*\mathcal{F})(U_i) & \rightrightarrows & \prod_{i, j \in I} (F^*\mathcal{F})(U_i \cap U_j) \\ \parallel & & \parallel & & \parallel \\ \mathcal{F}(FU) & \xrightarrow{\langle \mathcal{F}(F\varphi_i) \rangle_{i \in I}} & \prod_{i \in I} \mathcal{F}(FU_i) & \rightrightarrows & \prod_{i, j \in I} \mathcal{F}(F(U_i \cap U_j)) \\ \parallel & & \parallel & & \parallel \\ \mathcal{F}(FU) & \xrightarrow{\langle \mathcal{F}(F\varphi_i) \rangle_{i \in I}} & \prod_{i \in I} \mathcal{F}(FU_i) & \rightrightarrows & \prod_{i, j \in I} \mathcal{F}(F(U_i) \cap F(U_j)) \end{array}$$

Since \mathcal{A}_F^J is generated by the pretopology $\langle F\tau \rangle$ defined by the J -covers (cf. Proposition 8 above), it follows that $\{F\varphi_i : FU_i \rightarrow FU \mid i \in I\}$ generates and refines an \mathcal{A}_F^J -cover; using that \mathcal{F} is an \mathcal{A}_F^J -sheaf implies that the bottom row in the diagram is an equalizer and hence that the top is as well. This proves the lemma.

Theorem 3.1 *The functor $F : \mathcal{C} \rightarrow \mathcal{D}$ induces an essential geometric morphism, perversely also named F ,*

$$F : \mathbf{Shv}(\mathcal{C}, J) \rightarrow \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J).$$

Proof. We will prove that the functor F^* preserves all small¹ colimits, as from here an appeal to Freyd's Adjoint Functor Theorem will show that F^* has a right adjoint. To do this, let $\{\mathcal{F}_i \mid i \in I\}$ be a family of \mathcal{A}_F^J -sheaves and let \mathcal{F} be the colimit

$$\mathcal{F} := \lim_{\rightarrow} \mathcal{F}_i$$

with colimit morphisms $\alpha_i : \mathcal{F}_i \rightarrow \mathcal{F}$. Now consider that to prove that F^* preserves small colimits, we must show that

$$F^* \left(\lim_{\rightarrow} \mathcal{F}_i \right) = \lim_{\rightarrow} (F^* \mathcal{F}_i);$$

unwrapping this definition shows that we must prove that for all $U \in \text{Ob } \mathcal{C}$,

$$\left(\lim_{\rightarrow} \mathcal{F}_i \right) (FU) \stackrel{?}{=} \lim_{\rightarrow} (\mathcal{F}_i(FU)).$$

To do this, we first observe that since $\mathbf{Shv}(\mathcal{C}, J)$ is a subtopos of the presehaf topos $[(\mathcal{C})^{\text{op}}, \mathbf{Set}]$, it suffices to compute whether or not the proposed equality holds by evaluating each natural transformation. However, consider that for each sheaf \mathcal{F}_i , there is an induced natural transformation $F^* \mathcal{F}_i \rightarrow F^* \mathcal{F}$ given from the horizontal composition in the pasting diagram below, where $r : \mathcal{D} \rightarrow (\mathcal{D})^{\text{op}}$ is the formal reflection of \mathcal{D} to its opposite category:

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{r} \mathcal{D}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{F}_i} \\ \Downarrow \alpha_i \\ \xrightarrow{\mathcal{F}} \end{array} \mathbf{Set}$$

Since the sheaves $F^* \mathcal{F}_i$ and $F^* \mathcal{F}$ act on \mathcal{C} through the above diagram, we find that in the comparison diagrams that only α_i varies. Thus, taking the colimit we find that

$$\lim_{\rightarrow} (\mathcal{F}_i \circ r \circ F) = \left(\lim_{\rightarrow} \mathcal{F}_i \right) \circ r \circ F = \mathcal{F} \circ r \circ F$$

so we have that

$$\lim_{\rightarrow} (\mathcal{F}_i(FU)) = \mathcal{F}(FU) = \left(\lim_{\rightarrow} \mathcal{F}_i \right) (FU)$$

¹ Here the word "small" means that we assume we're working in some Grothendieck Universe \mathcal{U} where all our "small" objects are of size $\alpha \leq \kappa$, where κ is some strongly inaccessible cardinal. This is the last comment we will make on this subject, and the reader who does not care may simply say that anything that is "small" is as large as some set that is not a proper class.

which proves that F^* preserves all small colimits. Thus, by Freyd's Adjoint Functor Theorem, F^* has a right adjoint F_* . Thus there is an adjunction

$$\begin{array}{ccc} & \xrightarrow{F^*} & \\ \mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J) & \perp & \mathbf{Shv}(\mathcal{C}/X, J) \\ & \xleftarrow{F_*} & \end{array}$$

which proves the first half of the essential geometric morphism.

We will now be done if we can exhibit the existence of a left adjoint to F^* , i.e., if we can prove that there is a functor $F_! : \mathbf{Shv}(\mathcal{C}, J) \rightarrow \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ such that there is an adjunction:

$$\begin{array}{ccc} & \xrightarrow{F_!} & \\ \mathbf{Shv}(\mathcal{C}, J) & \perp & \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) \\ & \xleftarrow{F^*} & \end{array}$$

However, we can show that if

$$\mathcal{F} = \varprojlim_{i \in I} \mathcal{F}_i$$

with projections $\rho_i : \mathcal{F} \rightarrow \mathcal{F}_i$, then we can calculate the limit in $\mathbf{Shv}(\mathcal{C}, J)$ as in the pasting diagram

$$\mathcal{C} \xrightarrow{F} \mathcal{D}/Y \xrightarrow{r} \mathcal{D}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \Downarrow \rho_i \\ \xrightarrow{\mathcal{F}_i} \end{array} \mathbf{Set}$$

just like the prior case. Dualizing the argument from here shows by Freyd's Adjoint Functor Theorem that F^* preserves all small limits and hence that there is an adjunction

$$\begin{array}{ccc} & \xrightarrow{F_!} & \\ \mathbf{Shv}(\mathcal{C}, J) & \perp & \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) \\ & \xleftarrow{F^*} & \end{array}$$

and hence we have the triple adjunction $F_! \dashv F^* \dashv F_*$ describing the essential geometric morphism $F : \mathbf{Shv}(\mathcal{C}, J) \rightarrow \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$.

Remark 8. Note that Theorem 3.1 cannot be deduced from [16] because, amongst other reasons, the functor F does not preserve terminal objects in general. For instance, if $F = \mathfrak{h} : \mathbf{Sch}_{\text{Spec } \mathbb{F}_p} \rightarrow \mathbf{FSch}_{/\text{Spec } \mathbb{Z}_p}$, then $F(\text{Spec } \mathbb{F}_p) = \text{Spf } \mathbb{Z}_p \neq \text{Spec } \mathbb{Z}_p$.

Lemma 4. *If p is a point of $\mathbf{Shv}(\mathcal{C}/X, J)$ then the composite geometric morphism $F \circ p$ is a point of $\mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J)$.*

Proof. Recall that the 2-category **Topos** of toposes has for morphisms geometric morphisms $\mathcal{E} \xrightarrow{f} \mathcal{F}$. Then since a point of $\mathbf{Shv}(\mathcal{C}/X, J)$ is a geometric morphism $p : \mathbf{Set} \rightarrow \mathbf{Shv}(\mathcal{C}/X, J)$ and $F : \mathbf{Shv}(\mathcal{C}/X, J) \rightarrow \mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J)$ is an essential geometric morphism by Theorem 3.1, we have that the diagram

$$\begin{array}{ccc} & \mathbf{Shv}(\mathcal{C}/X, J) & \\ p \nearrow & & \searrow F \\ \mathbf{Set} & \xrightarrow{F \circ p} & \mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J) \end{array}$$

commutes in **Topos**. Thus $F \circ p$ is a point of $\mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J)$.

Corollary 5. *The topos $\mathbf{Shv}(\mathbf{FSch}/_{\mathrm{Spf} \mathbb{Z}_p}, \mathcal{A}_{\mathfrak{h}}^{\acute{E}t})$ has a point.*

Proof. It is well-known that the étale topos $\mathbf{Shv}(\mathbf{Sch}/_{\mathrm{Spec} \mathbb{F}_p}, \acute{E}t)$ has points. Thus so does the sheaf topos $\mathbf{Shv}(\mathbf{FSch}/_{\mathrm{Spec} \mathbb{Z}_p}, \mathcal{A}_{\mathfrak{h}}^{\acute{E}t})$ by the above Lemma.

We now give some basic results about whether or not a geometrically adhesive functor induces a connected morphism of sheaf toposes. A key assumption that we will make here involves essential surjectivity of the geometrically adhesive functor F ; this assumption may be relaxed to a condition like that given in the statement of Lemma 2, but one does need to make some assumptions on the functor F and the categories \mathcal{C} and \mathcal{D} in order for F^* to be fully faithful. A counter-example is given below that discusses the failure of F^* to be faithful more precisely, but for the connectivity results below, it simply suffices to prove that F is essentially surjective.

Proposition 9. *Let $F : \mathcal{C}/X \rightarrow \mathcal{D}/Y$ be a full and essentially surjective geometrically adhesive functor. Then the functor F^* in the essential geometric morphism $F : \mathbf{Shv}(\mathcal{C}/X, J) \rightarrow \mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J)$ is full.*

Proof. Recall that since F is full, for all $U, U' \in \mathrm{Ob} \mathcal{C}/X$ we have that the map $\mathcal{C}/X(U, U') \rightarrow \mathcal{D}/Y(FU, FU')$ given by $\varphi \mapsto F\varphi$ is epic in **Set**. Now let $\mathcal{F}, \mathcal{G} \in \mathrm{Ob} \mathbf{Shv}(\mathcal{D}/Y, \mathcal{A}_F^J)$ and consider a natural transformation $\alpha : F^* \mathcal{F} \rightarrow F^* \mathcal{G}$. Then for all $\varphi : U \rightarrow U'$ in \mathcal{C}/X the diagram

$$\begin{array}{ccc} (F^* \mathcal{F})(U) & \xrightarrow{\alpha_U} & (F^* \mathcal{G})(U) \\ (F^* \mathcal{F})(\varphi) \uparrow & & \uparrow (F^* \mathcal{G})(\varphi) \\ (F^* \mathcal{F})(U') & \xrightarrow{\alpha_{U'}} & (F^* \mathcal{G})(U') \end{array}$$

which is equivalent to

$$\begin{array}{ccc}
\mathcal{F}(FU) & \xrightarrow{\alpha_U} & \mathcal{G}(FU) \\
\mathcal{F}(F\varphi) \uparrow & & \uparrow \mathcal{G}(F\varphi) \\
\mathcal{F}(FU') & \xrightarrow{\alpha_U} & \mathcal{G}(FU')
\end{array}$$

commutes in **Set**.

To prove that F^* is full, we simply must find a lift of α , i.e., a natural transformation $\beta : \mathcal{F} \rightarrow \mathcal{G}$ in such that $F^*\beta = \alpha$. To do this, we will construct β in two steps: First, define $\beta_{FU} : \mathcal{F}(FU) \rightarrow \mathcal{G}(FU)$ by setting

$$\beta_{FU} := \alpha_U.$$

Now assume that $V \in \text{Ob } \mathcal{D}/Y$ and find a $U \in \text{Ob } \mathcal{C}/X$ such that $V \cong FU$; this is possible by the essential surjectivity of F . Let $\psi : FU \rightarrow V$ be any fixed isomorphism between V and FU and define β_V by the rule

$$\beta_V := \mathcal{G}(\psi^{-1}) \circ \alpha_U \circ \mathcal{F}\psi.$$

Now let $\tilde{\varphi} : V \rightarrow V'$ be a morphism in \mathcal{D}/Y , where $V' \cong FU'$ for some $U' \in \text{Ob } \mathcal{C}/X$ through $\psi' : FU' \rightarrow V'$, and consider the commuting diagram

$$\begin{array}{ccc}
\mathcal{F}V & \xrightarrow{\mathcal{F}\psi} & \mathcal{F}(FU) \\
\mathcal{F}\tilde{\varphi} \uparrow & & \uparrow \exists \lambda \\
\mathcal{F}V' & \xrightarrow{\mathcal{F}\psi'} & \mathcal{F}(FU')
\end{array}$$

in **Set**. Note that λ exists because since $\mathcal{F}\psi'$ is an isomorphism, a direct calculation with the morphism

$$\lambda := \mathcal{F}\psi \circ \mathcal{F}\tilde{\varphi} \circ \mathcal{F}(\psi')^{-1}$$

shows that $\lambda \circ \mathcal{F}\psi' = \mathcal{F}\psi \circ \mathcal{F}\tilde{\varphi}$. However, by the fullness of F , there exists a morphism $\theta : U \rightarrow U'$ in \mathcal{C}/X such that

$$\lambda = \mathcal{F}(F\theta) = \mathcal{F}((\psi')^{-1} \circ \tilde{\varphi} \circ \psi).$$

This shows that the diagram

$$\begin{array}{ccc}
\mathcal{F}V & \xrightarrow{\mathcal{F}\psi} & \mathcal{F}(FU) \\
\mathcal{F}\tilde{\varphi} \uparrow & & \uparrow \exists \lambda \\
\mathcal{F}V' & \xrightarrow{\mathcal{F}\psi'} & \mathcal{F}(FU')
\end{array}$$

commutes in **Set**. Dually, we have that the diagram

$$\begin{array}{ccc} \mathcal{G}(FU) & \xrightarrow{\mathcal{G}(\psi^{-1})} & \mathcal{G}(V) \\ \mathcal{F}(F\theta) \uparrow & & \uparrow \mathcal{G}\tilde{\varphi} \\ \mathcal{G}(FU') & \xrightarrow{\mathcal{G}((\psi')^{-1})} & \mathcal{G}V' \end{array}$$

commutes in **Set**; thus, since the diagram

$$\begin{array}{ccc} \mathcal{F}(FU) & \xrightarrow{\alpha_U} & \mathcal{G}(FU) \\ \mathcal{F}(F\theta) \uparrow & & \uparrow \mathcal{G}(F\theta) \\ \mathcal{F}(FU') & \xrightarrow{\alpha_{U'}} & \mathcal{G}(FU') \end{array}$$

commutes by assumption, it follows that every cell in the diagram

$$\begin{array}{ccccccc} \mathcal{F}V & \xrightarrow{\mathcal{F}\psi} & \mathcal{F}(FU) & \xrightarrow{\alpha_U} & \mathcal{G}(FU) & \xrightarrow{\mathcal{G}(\psi^{-1})} & \mathcal{G}V \\ \mathcal{F}\tilde{\varphi} \uparrow & & \mathcal{F}(F\theta) \uparrow & & \uparrow \mathcal{G}(F\theta) & & \uparrow \mathcal{G}\tilde{\varphi} \\ \mathcal{F}V' & \xrightarrow{\mathcal{F}(\psi')} & \mathcal{F}(FU') & \xrightarrow{\alpha_{U'}} & \mathcal{G}(FU') & \xrightarrow{\mathcal{G}((\psi')^{-1})} & \mathcal{G}V' \end{array}$$

commutes and hence the outer rectangle commutes. But since we have by definition that $\beta_V = \mathcal{G}(\psi^{-1}) \circ \alpha_U \circ \mathcal{F}\psi$, it follows that the outer rectangle contracts to

$$\begin{array}{ccc} \mathcal{F}V & \xrightarrow{\beta_V} & \mathcal{G}V \\ \mathcal{F}\tilde{\varphi} \uparrow & & \uparrow \mathcal{G}\tilde{\varphi} \\ \mathcal{F}V' & \xrightarrow{\beta_{V'}} & \mathcal{G}V' \end{array}$$

which proves that $\beta : \mathcal{F} \rightarrow \mathcal{G}$ is a natural transformation. Then by construction we can show that

$$F^*\beta = \alpha,$$

which proves that F^* is full, as was desired.

Remark 9. With the Lemma 2 and Corollary 4 together suggest that in order to have the faithfulness of F^* that may be suggested by Lemma 3, it is necessary to have the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ satisfy the condition that for all $V \in \mathcal{D}_0 . \exists X \in \mathcal{C}_0 . (\mathcal{D}(V, FX) \neq \emptyset) \vee (\mathcal{D}(FX, V) \neq \emptyset)$. This may be made explicit by the following counter-example:

Let $\mathcal{C} = \mathbf{Sch}_{/\text{Spec } \mathbb{F}_p}$ and let $\mathcal{D} = \mathbf{FSch}_{/\text{Spec } \mathbb{Z}_p} \cup \{\text{Spec } \mathbb{Z}_\ell\}$, where $\text{gcd}(\ell, p) = 1$ and with the hom-set $\mathcal{D}(\text{Spec } \mathbb{Z}_\ell, \text{Spec } \mathbb{Z}_\ell) = \{\text{id}_{\text{Spec } \mathbb{Z}_\ell}\}$. Then

define a functor $h : \mathbf{Cat} \rightarrow \mathcal{D}$ by $h(X) = \mathfrak{h}X$ for all $X \in \mathcal{C}_0$ and observe that \mathfrak{h} is geometrically adhesive. However, F^* is not faithful as a functor of sheaf categories because you can choose any set on a presheaf over $\mathrm{Spec} \mathbb{Z}_\ell$ which is an $\mathcal{A}_\mathfrak{h}^J$ -sheaf over the $\mathbf{FSch}/_{\mathrm{Spec} \mathbb{Z}_p}$ component and get an $\mathcal{A}_\mathfrak{h}^J$ -sheaf. In particular, by taking the sheaves \mathcal{S} and \mathcal{T} to be defined by

$$\mathcal{T}(U) = \begin{cases} \{*\} & \text{if } U \in \mathbf{FSch}/_{\mathrm{Spec} \mathbb{Z}_p} \\ \{0, 1\} & \text{Else;} \end{cases}$$

and

$$\mathcal{S}(U) = \{*\}$$

for all $U \in \mathcal{D}$. Then $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_\mathfrak{h}^J)(\mathcal{S}, \mathcal{T}) = \{0, 1\}$, where the labels come from which element the natural transformation picks out on the map $\{*\} \rightarrow \{0, 1\}$ (so the sections over $\mathrm{Spec} \mathbb{Z}_\ell$). However, since the pullback h^* only sees sheaves that arise as taking $h(-)$ of \mathbb{F}_p -schemes, it is straightforward to calculate that

$$h^* \mathcal{S}(X) = \{*\} = h^* \mathcal{T}(X)$$

for all schemes X in $\mathbf{Sch}/_{\mathrm{Spec} \mathbb{F}_p}$ (together with the only possible morphisms). In particular, from this construction it follows that

$$\mathbf{Shv}(\mathcal{C}, J)(h^* \mathcal{S}, h^* \mathcal{T}) = \{\mathrm{id}_{h^* \mathcal{S}}\} \not\cong \{0, 1\} = \mathbf{Shv}(\mathcal{D}, \mathcal{A}_\mathfrak{h}^J)(\mathcal{S}, \mathcal{T}),$$

which shows that h^* is not faithful.

Proposition 10. *If $F : \mathcal{C}/_X \rightarrow \mathcal{D}/_Y$ is a geometrically adhesive, full, and essentially surjective functor, then F^* is fully faithful.*

Proof. We have already seen from Proposition 9 that F^* is full; we therefore only need to show that F^* is faithful. To do this, assume that there is a natural transformation $\gamma : \mathcal{F} \rightarrow \mathcal{G}$ such that $F^* \beta = F^* \gamma$ for some $\alpha : F^* \mathcal{F} \rightarrow F^* \mathcal{G}$. Since $F^* \beta = \alpha = F^* \gamma$, we have that $\gamma_{FU} = \alpha_U = \beta_{FU}$ for all $U \in \mathrm{Ob} \mathcal{C}/_X$. Now for each $V \in \mathrm{Ob} \mathcal{D}/_Y$, find a $U \in \mathrm{Ob} \mathcal{C}/_X$ and an isomorphism $\varphi : FU \rightarrow V$. Then since β and γ are natural transformations, we have that the diagrams

$$\begin{array}{ccc} \mathcal{F}V & \xrightarrow{\beta_V} & \mathcal{G}V \\ \mathcal{F}\psi \downarrow & & \downarrow \mathcal{G}\psi \\ \mathcal{F}(FU) & \xrightarrow{\beta_{FU}} & \mathcal{G}(FU) \end{array}$$

and

$$\begin{array}{ccc}
\mathcal{F}V & \xrightarrow{\gamma_V} & \mathcal{G}V \\
\mathcal{F}\psi \downarrow & & \downarrow \mathcal{G}\psi \\
\mathcal{F}(FU) & \xrightarrow{\gamma_{FU}} & \mathcal{G}(FU)
\end{array}$$

both commute. But then we calculate that

$$\begin{aligned}
\beta_V &= \text{id}_{\mathcal{G}V} \circ \beta_V = \mathcal{G}\psi^{-1} \circ \mathcal{G}\psi \circ \beta_V = \mathcal{G}\psi^{-1} \circ \beta_{FU} \circ \mathcal{F}\psi \\
&= \mathcal{G}\psi^{-1} \circ \gamma_{FU} \circ \mathcal{F}\psi = \mathcal{G}\psi^{-1} \circ \mathcal{G}\psi \circ \gamma_V = \text{id}_{\mathcal{G}V} \circ \gamma_V \\
&= \gamma_V
\end{aligned}$$

so it follows that $\beta = \gamma$ and hence F^* is fully faithful.

We would now like to show that there is an analogous version of the above proposition that holds even in the case that F is fully faithful but not essentially surjective. It will tell us that in the case in which the functor F is fully faithful, we can infer that both F^* reflects isomorphisms, as well as the fact that F^* is fully faithful. It uses the condition stated explicitly in Lemma 2: That for every object V of \mathcal{D} there exists an object X of \mathcal{C} for which $\mathcal{D}(V, FX) \neq \emptyset$ or $\mathcal{D}(FX, V) \neq \emptyset$.

Proposition 11. *Assume that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a fully faithful, geometrically adhesive functor and that for all objects V of \mathcal{D} there exists an object X of \mathcal{C} for which $\mathcal{D}(FX, V) \neq \emptyset$ or $\mathcal{D}(V, FX) \neq \emptyset$. Then F^* is isomorphism reflecting.*

Proof. Let \mathcal{F} and \mathcal{G} be \mathcal{A}_F^J -sheaves on \mathcal{D} such that $F^*\mathcal{F} \cong F^*\mathcal{G}$ as J -sheaves on \mathcal{C} . Then for all objects U of \mathcal{C} and for all morphisms φ of \mathcal{C} we get: From the isomorphism

$$F^*\mathcal{F}(U) \cong F^*\mathcal{G}(U),$$

we have that

$$\mathcal{F}(FU) \cong \mathcal{G}(FU);$$

and from the isomorphism

$$F^*\mathcal{F}(\varphi) \cong F^*\mathcal{G}(\varphi)$$

we also have

$$\mathcal{F}(F\varphi) \cong \mathcal{G}(F\varphi).$$

Furthermore, because both $F^*\mathcal{F}$ and $F^*\mathcal{G}$ are J -sheaves, it follows that for all J -sieves $S \in J(U)$, the diagram

$$(F^*\mathcal{F})(U) \longrightarrow \prod_{f \in S} (F^*\mathcal{F})(\text{Dom } f) \xrightarrow{\cong} \prod_{\substack{f \in S, g \in \mathcal{C}_1 \\ f \circ g \in \mathcal{C}_1}} (F^*\mathcal{F})(\text{Dom } g)$$

is an equalizer diagram, where the $f \in S$ are sieving morphisms; similarly, the same diagram with \mathcal{G} replaced with \mathcal{F} is an equalizer as well. Moreover, since the definition of the \mathcal{A}_F^J -topology shows that the \mathcal{A}_F^J covering sieves are generated by the sets $F(S)$, applying Corollary 4 and using the hypotheses in the proposition to avoid technical malfunctions that occur off the connected component² generated by the essential image of F allows us to check the isomorphism types of \mathcal{F} and \mathcal{G} along the FS 's. However, applying the definition of the functor F^* shows that the diagram above is equivalent to the diagram

$$\mathcal{F}(FU) \longrightarrow \prod_{f \in S} \mathcal{F}(F \text{Dom } f) \xlongequal{\quad} \prod_{\substack{f \in S, g \in \mathcal{C}_1 \\ f \circ g \in \mathcal{C}_1}} \mathcal{F}(F \text{Dom } g)$$

Using the fully faithfulness of F together with the isomorphisms $F \text{Dom } f \cong \text{Dom}(Ff)$ then shows that for every set FS , \mathcal{F} and \mathcal{G} are isomorphic on the desired covers. This implies that $\mathcal{F} \cong \mathcal{G}$ and completes the proof of the proposition.

We now need a lonely arithmetic geometric result for use later in this paper.

Lemma 5. *Let R/\mathbb{Z}_p be an integral extension with residue field k and let $\mathfrak{h} \dashv \text{Gr} : \mathbf{Sch}/_{\text{Spec } k} \rightarrow \mathbf{FSch}/_{\text{Spec } R}$ be the Greenberg adjunction. Then the pullback functor*

$$\mathfrak{h}^* : \mathbf{Shv}(\mathbf{FSch}/_{\text{Spec } R}, \mathcal{A}_{\mathfrak{h}}^J) \rightarrow \mathbf{Shv}(\mathbf{Sch}/_{\text{Spec } k}, J)$$

is fully faithful.

Proof. This follows from the fact that the topology $\mathcal{A}_{\mathfrak{h}}^J$ is lifted from J , from the fact that \mathfrak{h} is a left adjoint so there is always the canonical map $\varepsilon_{\mathfrak{X}} : \mathfrak{h}(\text{Gr}(\mathfrak{X})) \rightarrow \mathfrak{X}$ for any formal scheme \mathfrak{X} in $\mathbf{FSch}/_{\text{Spec } R}$, and from the structure of rings of Witt vectors and algebras of Witt vectors over rings of Witt vectors.

4 The Adhesive Fundamental Group

As a point of notation, if \mathcal{E} is a Grothendieck topos, then we will write \mathcal{E}_{lcf} for the full subtopos of locally constant, locally finite objects in \mathcal{E} . We will be making a study of these categories based on the restriction of the essential geometric morphism $F : \mathbf{Shv}(\mathcal{C}, J) \rightarrow \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ induced by a geometrically adhesive functor $F : \mathcal{C} \rightarrow \mathcal{D}$.

Throughout this section we make the following assumptions:

² By which we mean the connected components of the category as a graph.

- A1. $F : \mathcal{C} \rightarrow \mathcal{D}$ is geometrically adhesive and the pullback F^* is fully faithful;
- A2. $\mathbf{Shv}(\mathcal{C}, J)_{lcf}$ is a Galois category with fibre functor $\text{Fi} : \mathbf{Shv}(\mathcal{C}, J)_{lcf} \rightarrow \mathbf{FinSet}$;
- A3. For every $V \in \mathcal{D}_0$, there exists an $X \in \mathcal{C}_0$ for which $\mathcal{D}(V, FX) \neq \emptyset$ or $\mathcal{D}(FX, V) \neq \emptyset$.

We will use this set up to study the fundamental group on $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ with the provision that there is a fundamental group of $\mathbf{Shv}(\mathcal{C}, J)_{lcf}$ with which we can work. This will culminate with us showing that in some situations (and in particular in the case of the Greenberg Transform) that the fundamental groups coincide. However, before we do that we must show that we can still use the techniques afforded by the essential geometric morphism which are, fittingly, essential to us.

Proposition 12. *The functor $F^* : \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) \rightarrow \mathbf{Shv}(\mathcal{C}, J)$ restricts to a functor $F^* : \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)_{lcf} \rightarrow \mathbf{Shv}(\mathcal{C}, J)_{lcf}$.*

Proof. Immediate from the fact that F^* is exact and hence preserves all limits and colimits.

Proposition 13. *There is an essential geometric morphism $F : \mathbf{Shv}(\mathcal{C}, J)_{lcf} \rightarrow \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ with pullback F^* given as above.*

Proof. The proof of this proposition is formal, and follows in the same way as Theorem 3.1. That is, if $\{\mathcal{F}_i \mid i \in I\}$ is a finite family of functors with colimit \mathcal{F} and colimit maps $\alpha_i : \mathcal{F}_i \rightarrow \mathcal{F}$, consider the following family of 2-cells:

$$\mathcal{C}^{\text{op}} \begin{array}{c} \xrightarrow{F^* \mathcal{F}_i} \\ \Downarrow F^* \alpha_i \\ \xrightarrow{F^* \mathcal{F}} \end{array} \mathbf{Set}$$

Observing that the functor $F^* \mathcal{F}_i = \mathcal{F}(F(-)) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$, we can rewrite the 2-cell as

$$\mathcal{C}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{F}_i(F(-))} \\ \Downarrow \alpha_i \\ \xrightarrow{\mathcal{F}(F(-))} \end{array} \mathbf{Set}$$

which factors as

$$\mathcal{C}^{\text{op}} \begin{array}{c} \xrightarrow{F^{\text{op}}} \\ \Downarrow d_F \\ \xrightarrow{F^{\text{op}}} \end{array} \mathcal{D}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{F}_i} \\ \Downarrow \beta_i \\ \xrightarrow{\mathcal{F}} \end{array} \mathbf{Set}$$

and further simplifies to the diagram:

$$\mathcal{C}^{\text{op}} \xrightarrow{F^{\text{op}}} \mathcal{D}^{\text{op}} \begin{array}{c} \xrightarrow{\mathcal{F}_i} \\ \Downarrow \alpha_i \\ \xrightarrow{\mathcal{F}} \end{array} \mathbf{Set}$$

From this it follows that upon taking the colimit of the diagram that the colimiting aspect is calculated solely in the $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ 2-cell. Thus it follows that

$$\varinjlim_{i \in I} F^* \mathcal{F}_i = F^* \mathcal{F};$$

note that this is because $\mathbf{Shv}(\mathcal{C}, J)_{lcf}$ and $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)_{lcf}$ are the categories of locally constant, locally finite objects and morphisms between them. The fact that F^* preserves limits is shown mutatis mutandis, and the fact that this implies F^* admits both left and right adjoints comes from Freyd's Adjoint Functor Theorem and the fact that the codomain of F^* is a topos. Finally, the fact that $F^* \mathcal{F}$ remains locally constant and locally finite comes from the fact that F^* preserves all finite limits and finite colimits.

Lemma 6. *Let $\text{Fi} : \mathbf{Shv}(\mathcal{C}, J)_{lcf} \rightarrow \mathbf{FinSet}$ be a functor. Then:*

1. *If Fi is exact, so is $\text{Fi} \circ F^* : \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) \rightarrow \mathbf{FinSet}$.*
2. *If Fi reflects isomorphisms, so does $\text{Fi} \circ F^*$.*
3. *If Fi is pro-representable, so is $\text{Fi} \circ F^*$.*

Proof. Claim (1) follows immediately from the fact that F^* is exact because it has a left and right adjoint, and from the fact that the composite of exact functors is exact.

For part (2) we assume that there are sheaves $\mathcal{F}, \mathcal{G} \in \text{Ob } \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ such that

$$(\text{Fi} \circ F^*)(\mathcal{F}) \cong (\text{Fi} \circ F^*)(\mathcal{G}).$$

However, in this case because Fi reflects isomorphisms, we have that $F^* \mathcal{F} \cong F^* \mathcal{G}$, and so this reduces to the fact that F^* reflects isomorphisms. However, because we have assumed properties (A1) above, the result follows from that fact that F^* reflects isomorphisms and the fact that the composite of two isomorphism reflecting functors is again isomorphism reflecting.

To prove (3), assume that $F^* \mathcal{F}$ is representable and let $\{A_i \mid i \in I\}$ be the filtered inverse system of objects such that

$$\text{Fi}(\mathcal{F}) \cong \varinjlim_{i \in I} \mathbf{Shv}(\mathcal{C}, J)(A_i, \mathcal{F})$$

for all $\mathcal{F} \in \text{Ob } \mathbf{Shv}(\mathcal{C}, J)$. However, fix a $\mathcal{G} \in \text{Ob } \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ and consider that for each $i \in I$,

$$\mathbf{Shv}(\mathcal{C}, J)(A_i, F^* \mathcal{G}) \cong \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_! A_i, \mathcal{G})$$

so

$$(\text{Fi} \circ F^*)(\mathcal{G}) = \text{Fi}(F^* \mathcal{G}) \cong \varinjlim_{i \in I} \mathbf{Shv}(\mathcal{C}, J)(A_i, F^* \mathcal{G}) \cong \varinjlim_{i \in I} \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_! A_i, \mathcal{G}).$$

Because $\{A_i \mid i \in I\}$ is a filtered inverse system, the category I is filtered and so $\{F_! A_i \mid i \in I\}$ is a filtered inverse system by the covariance of $F_!$. This proves the lemma.

Note that the proof of the pro-representability of the functor $\text{Fi} \circ F^*$ used in an essential way the adjunction $F_! \dashv F^*$. Using this again in the same way we derive the lemma below:

Lemma 7. *If $\mathcal{F} \in \text{Ob } \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)_{lcf}$ and if A is a normal object for $\mathbf{Shv}(\mathcal{C}, J)_{lcf}$ such that*

$$\mathbf{Shv}(\mathcal{C}, J)(A, F^* \mathcal{F}) \cong \text{Fi}(F^* \mathcal{F})$$

then

$$\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_! A, \mathcal{F}) \cong \text{Fi} \circ F^*(\mathcal{F}).$$

Lemma 8. *Assume that $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)_{lcf}$ admits an exact isomorphism reflecting functor $I : \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J) \rightarrow \mathbf{FinSet}$. Then if an object A in $\mathbf{Shv}(\mathcal{C}, J)$ is non-initial and indecomposable, so is $F_! A$.*

Proof. Assume that $A \in \text{Ob } \mathbf{Shv}(\mathcal{C}, J)$ is indecomposable and find a non-trivial composition

$$A \cong \coprod_{i=1}^n A_i.$$

Then if we apply $F_!$ to the above coproduct we get that

$$F_! A = \coprod_{i=1}^n F_! A_i,$$

and this will be nontrivial if we can show at least one $F_! A_i \not\cong F_! A$ or $F_! A \not\cong \perp$.

We will first show that at least one $F_! A_i \not\cong \perp$. Consider that since the decomposition of A is nontrivial, there exists at least one object in the composite, call it A_k , such that $A_k \not\cong A$ and $A_k \not\cong \perp$. Because $A_k \not\cong \perp$, for each object U of \mathcal{C} ,

$$A_k(U) \neq \emptyset$$

as the initial sheaf \perp is uniquely defined by the equation

$$\perp(V) = \emptyset$$

for all objects X of \mathcal{C} . But then we observe that $\mathbf{Shv}(A_k, \perp) = \emptyset$, as no nonempty set can map to the empty set and if a sheaf is empty anywhere it is empty everywhere. Now assume that there is a map $\varphi : F_! A_k \rightarrow F_! \perp$; however, using that $F_!$ is a right adjoint shows that $F_! \perp = \perp$ and so there is a map $F_! A_k \rightarrow \perp$. Now observe that since F^* is left exact we have that

$$\begin{aligned}
\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!A_k, \perp) &= \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!A_k, F_!\perp) \\
&\cong \mathbf{Shv}(\mathcal{C}, J)(A_k, F^*F_!\perp) \\
&= \mathbf{Shv}(\mathcal{C}, J)(A_k, \perp) \\
&= \emptyset.
\end{aligned}$$

Therefore $F_!A_k \not\cong \perp$ as $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!A_k, F_!\perp) = \emptyset$.

We will now show that $F_!A_k \not\cong F_!A$. To do this aftpodac this were the case and note that from the assumption $F_!A_k \cong F_!A$ and from

$$F_!A \cong \prod_{i=1}^n F_!A_i$$

we have that

$$F_!A_k \cong \prod_{i=1}^n F_!A_i = F_!A_k \prod \left(\prod_{i=1, i \neq k}^n F_!A_i \right).$$

Now apply I to get that

$$I(F_!A_k) = I(F_!A_k) \prod_{i=1, i \neq k}^n I(F_!A_i) = I(F_!A_k) \sqcup \bigcup_{i=1, i \neq k}^n I(F_!A_i).$$

Now since the above equality of finite sets is a disjoint union it follows that $I(F_!A_i) = \emptyset$. However, because I is exact, $I(\perp) = \emptyset$ and since I reflects isomorphisms, this happens exactly when each $F_!A_i \cong \perp$.

On the other hand, if $A_k \not\cong A$ and A is indecomposable, by the above argument it follows that there is an A_j for which A_j is also nontrivial. Proceeding with the same line of thought and using that $F_!A_j \not\cong \perp$ and using that just above we have shown that $F_!A_j \not\cong \perp$ we have that

$$I(F_!A_j) \cong I(F_!A) \cong I(F_!A_k)$$

while simultaneously we have

$$I(F_!A_j) \sqcup I(F_!A_k) = I(F_!A).$$

This is evidently false, and so it must be the case that $F_!A_k \not\cong F_!A$. This proves that $F_!A$ is indecomposable.

Remark 10. Note that the above lemma can be false (or at least this proof fails) if we relax the assumption that the exact isomorphism reflecting functor I factors through **Set** and not **FinSet**, as infinite sets can be isomorphic to some of their proper subsets.

Remark 11. It is worth remarking that the proof we gave above used the Law of the Excluded Middle. I could not find a way that did not make use

of the Principle of Contraposition or Proof by Contradiction. Perhaps this should not be bothersome, however, because of the Boolean nature of Galois categories.

Corollary 6. *For every sheaf \mathcal{F} in $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$, there is an indecomposable sheaf \mathcal{G} in $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)$ such that*

$$\mathrm{Fi} \circ F^*(\mathcal{F}) \cong \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(\mathcal{G}, \mathcal{F}).$$

Proof. Begin by observing that since each functor $F^*\mathcal{F}$ is a J -sheaf on \mathcal{C} , there is a normal object N in $\mathbf{Shv}(\mathcal{C}, J)$ such that

$$\mathrm{Fi}(F^*\mathcal{F}) \cong \mathbf{Shv}(\mathcal{C}, J)(N, F^*\mathcal{F})$$

by Proposition 8.46 of [15]. Thus by Lemmas 7 and 8 we have that $F_!N$ is indecomposable and that

$$\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!N, \mathcal{F}) \cong \mathrm{Fi} \circ F^*(\mathcal{F}),$$

which concludes the proof.

Lemma 9. *If N is normal in $\mathbf{Shv}(\mathcal{C}, J)_{\mathrm{icf}}$ then $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!N, F_!N)$ is a group.*

Proof. Because N is normal, it is indecomposable and $N \not\cong \perp$. Thus we have that $F_!N$ by Lemma 8 that $F_!N$ is indecomposable. Now let $\varphi : F_!N \rightarrow F_!N$ be an endomorphism of $F_!N$ and consider the $(\mathcal{E}, \mathcal{M})$ -factorization of φ :

$$\begin{array}{ccc} F_!N & \xrightarrow{\varphi} & F_!N \\ & \searrow \varepsilon & \nearrow \mu \\ & X & \end{array}$$

where X is a subobject of $F_!N$, ε is a regular epic, and μ is monic. Then since φ cannot factor through \perp , $X \not\cong \perp$ and hence a nonzero subobject of $F_!N$; by the indecomposability of $F_!N$ this implies that $X = F_!N$. Therefore, by considering that

$$(\mathrm{Fi} \circ F^*)(\varepsilon) : (\mathrm{Fi} \circ F^*)(F_!N) \rightarrow (\mathrm{Fi} \circ F^*)(F_!N)$$

is an epimorphism from a finite set to itself, it follows that ε is monic and hence an automorphism of $F_!N$. Finally, since μ is monic, the map

$$(\mathrm{Fi} \circ F^*)(\mu) : (\mathrm{Fi} \circ F^*)(F_!N) \rightarrow (\mathrm{Fi} \circ F^*)(F_!N)$$

is a monomorphism from a finite set to itself. This implies that μ is then epic and hence an automorphism as well; from here it follows that $\varphi = \varepsilon \circ \mu$ is also an automorphism and we are done.

Corollary 7. *If N is a normal object in $\mathbf{Shv}(\mathcal{C}, J)_{lcf}$, $F_!N$ is normal in $\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)_{lcf}$.*

Proof. Begin by observing that

$$\mathbf{Shv}(\mathcal{C}, J)(N, N) \cong \mathrm{Fi}(N)$$

so we derive that

$$\mathbf{Shv}(\mathcal{D}, J)(F_!N, F_!N) \cong \mathbf{Shv}(N, F^*(F_!N)) \cong \mathrm{Fi}(F^*(F_!N)) \cong (\mathrm{Fi} \circ F^*)(F_!N)$$

because N is the normal object representing $\mathrm{Fi}(F^*F_!N)$. Finally, by Lemma 9 it follows that

$\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!N, F_!N)$ is a group and hence that $F_!N$ is normal.

We now proceed to show the last ingredient in our proof of the equivalence of fundamental groups: We must know that the automorphism group of normal objects N in $\mathbf{Shv}(\mathcal{C}, J)_{lcf}$ is isomorphic to the automorphism group of $F_!N$.

Lemma 10. *If N is a normal object in $\mathbf{Shv}(\mathcal{C}, J)$ then there is an isomorphism of groups*

$$\mathbf{Shv}(\mathcal{C}, J)(N, N) \cong \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!N, F_!N).$$

Proof. Begin by observing that Assumption (A1) gives that F^* is fully faithful; consequently the counit $\varepsilon : F_! \circ F^* \rightarrow \mathrm{id}_{\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)_{lcf}}$ is an isomorphism and $F_!$ is essentially surjective. Then by the property of adjoint functors, there is a natural isomorphism $\theta_{F_!N}$ of hom-sets

$$\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!N, F_!N) \xrightarrow{\theta_{F_!N}} \mathbf{Shv}(\mathcal{C}, J)(N, (F^* \circ F_!)N);$$

moreover, from the fact that F^* is fully faithful, this isomorphism factors as

$$\begin{array}{ccc} \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_!N, F_!N) & \xrightarrow{\theta_{F_!N}} & \mathbf{Shv}(\mathcal{C}, J)(N, (F^* \circ F_!)N) \\ \downarrow F^* & \nearrow \eta^* & \\ \mathbf{Shv}(\mathcal{C}, J)((F^* \circ F_!)N, (F^* \circ F_!)N) & & \end{array}$$

where the map η^* is precomposition by the unit of adjunction. Now, using that F^* is a conservative exact functor between toposes of locally constant, locally finite sheaves implies that there is a Beck-Chevalley condition for $(F_!, F^*)$ (cf. Page 179 of [19]). This induces a natural map $(F^* \circ F_!)N \rightarrow N$ as an algebra morphism for the monad induced by $F_! \dashv F^*$, and hence gives a further isomorphism

$$\mathbf{Shv}(\mathcal{C}, J)(N, (F^* \circ F_!)N) \cong \mathbf{Shv}(\mathcal{C}, J)(N, N).$$

Composing these all gives the desired isomorphism

$$\mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_1N, F_1N) \cong \mathbf{Shv}(\mathcal{C}, J)(N, N).$$

We can now state and prove the main theorem of the paper, and then show the geometrization result as a corollary. We reiterate the assumptions made throughout this section here for the sake of clarity.

Theorem 4.1 *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a geometrically adhesive functor, suppose that F^* is fully faithful, let X be an object in (\mathcal{C}, J) , and let x be a point of X making $(\mathbf{Shv}(\mathcal{C}_{/X}, J_{/X})_{lcf}, \mathbb{F}_x)$ into a Galois category. Moreover, suppose that for all $V \in \mathcal{D}$ there exists an object X of \mathcal{C} such that $\mathcal{D}(FX, V) \neq \emptyset$ or $\mathcal{D}(V, FX) \neq \emptyset$. Then there is an isomorphism of fundamental groups*

$$\pi_1^J(X, x) \cong \pi_1^{\mathcal{A}_F^J}(FX, Fx).$$

Proof. Applying the various results of this section show that there is a well-defined profinite fundamental group of $\mathbf{Shv}(\mathcal{D}_{/FX}, \mathcal{A}_{F/FX}^J)_{lcf}$ at FX and Fx whose normal objects are all of the form F_1N , where N is a normal object of $\mathbf{Shv}(\mathcal{C}_{/X}, J_{/X})_{lcf}$. This implies that the F_1N give the correct cofinal system to limit against. Taking then the isomorphism

$$\mathbf{Shv}(\mathcal{C}, J)(N, N) \cong \mathbf{Shv}(\mathcal{D}, \mathcal{A}_F^J)(F_1N, F_1N)$$

for all normal objects N , we get that

$$\begin{aligned} \pi_1^{\mathcal{A}_F^J}(FX, Fx) &\cong \varprojlim \mathbf{Shv}(\mathcal{D}_{/X}, \mathcal{A}_{F/X}^J)(F_1N_i, F_1N_i) \\ &\cong \varprojlim \mathbf{Shv}(\mathcal{C}_{/X}, J_{/X})(N_i, N_i) \\ &\cong \pi_1^J(X, x). \end{aligned}$$

Corollary 8. *If G is a group scheme over $\mathbf{Sch}_{/\mathrm{Spec} k}$ and if J is any topology on $\mathbf{Sch}_{/\mathrm{Spec} k}$ for which $\mathbf{Shv}(\mathbf{Sch}_{/\mathrm{Spec} k}, J)_{lcf}$ is a Galois category, then there is an isomorphism of fundamental groups*

$$\pi_1^{\acute{E}t}(G, x) \cong \pi_1^{\mathcal{A}_F^J}(\mathfrak{h}G, \mathfrak{h}x).$$

5 Applications to Local Systems

Here we would like to present an important corollary of Theorem 4.1 above. In particular, it says that we can geometrize quasicharacters of p -adic group schemes by using p -adic formal schemes and the \mathcal{A}_F^J topology. To see this, we let F be a p -adic field and let G be a group scheme over F with geometric point x . Then, as is well-known to representation theorists, étale local sys-

tems of G arise as ℓ -adic representations of the fundamental group, i.e., as representations

$$\pi_1(G, x) \rightarrow \mathrm{GL}(V)$$

where V is a vector space over $\overline{\mathbb{Q}}_\ell$, for some integer prime ℓ coprime to p . Thus, if one wishes to geometrize representations of a connected, reductive group G over a p -adic field F with residue field k , it suffices to consider \mathcal{A}_F^J -local systems on schemes of characteristic zero. This is a new result, as it allows us to (in theory, although at this point not in practice) translate some of the results in geometric representation theory to a characteristic zero analogue in formal schemes over the trait $\mathrm{Spec} \mathcal{O}_F$ of integers of F .

Theorem 5.1 *If G is a group scheme over $\mathrm{Spec} k$ with geometric point x , then there is an isomorphism of categories*

$$\mathbf{Rep} \left(\pi_1^{\acute{E}t}(G, x) \right) \cong \mathbf{Rep} \left(\pi_1^{\mathcal{A}_\mathfrak{h}^{\acute{E}t}}(\mathfrak{h} G, \mathfrak{h} x) \right).$$

In particular, upon restricting to the categories of admissible irreducible representations, we obtain an isomorphism of categories

$$\mathbf{AdRep} \left(\pi_1^{\acute{E}t}(G, x) \right) \cong \mathbf{AdRep} \left(\pi_1^{\mathcal{A}_\mathfrak{h}^{\acute{E}t}}(\mathfrak{h} G, \mathfrak{h} x) \right).$$

This theorem is simply a restatement of the fact that if $G \cong H$ as p -adic groups, then their categories of representations (and their categories of admissible representations) are isomorphic as well. To see how to apply this to quasicharacters of p -adic tori, we follow [11]. Let F be a p -adic field and let \mathcal{O}_F be the ring of integers of F . Then if T is a torus over F , it admits a Néron model N_T which is locally of finite type as a smooth, commutative group scheme over \mathcal{O}_F ; cf. [6] for details. Moreover, $N_T(\mathcal{O}_F) = T(F)$ and $\mathrm{Gr}(N_T)(k) = N_T(\mathcal{O}_F) = T(F)$.

In [11], the authors showed that the Trace of Frobenius gives a natural transformation between the category of quasicharacter sheaves on N_T to continuous representations $N_T(\mathcal{O}_F) \rightarrow \overline{\mathbb{Q}}_\ell^*$ which is surjective, i.e., the group homomorphism

$$\mathrm{Trace}(\mathrm{Frob}) : \mathbf{QCS}_{/\mathrm{iso}}(N_T) \rightarrow \mathbf{Top}(\mathbf{Grp})(N_T(\mathcal{O}_F), \overline{\mathbb{Q}}_\ell^*)$$

is surjective. Now, since quasicharacter sheaves on N_T arise as certain local systems on $\mathrm{Gr}(N_T)$, which in turn are representations of $\pi_1^{\acute{E}t}(\mathrm{Gr}(N_t), x)$, we can use the functoriality of \mathfrak{h} and the adhesive site $\mathcal{A}_\mathfrak{h}^{\acute{E}t}$ to lift these local systems to corresponding representations of the \mathfrak{h} - $\acute{E}t$ fundamental group together with the corresponding functorial lift of Frobenius. Mimicking Sections 4.5 and 4.7 of [11] then allows one to prove that the adhesive site is rich enough to geometrize quasicharacters of the torus T .

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