Fundamental Gap Estimate on Convex Domains of Sphere

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Eigenvalues of Laplacian

Given a bounded smooth domain $\Omega \subset M^n$, the eigenvalues equation of the Laplacian on Ω is $\Delta\phi = -\lambda\,\phi$

$$\text{Dirichlet: } (\phi|_{\partial\Omega} = 0) \;\; 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty$$

$$\text{Neumann: } (\phi_\nu|_{\partial\Omega} = 0) \; 0 = \mu_0 < \mu_1 \leq \mu_2 \cdots \to \infty$$

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Min-Max Principle

$$\lambda_k = \inf_V \sup_{\phi \neq 0, \phi \in V} \frac{\int_M |\nabla \phi|^2}{\int_M \phi^2},$$

where V ranges all k-dim subspaces of $H_0^1(\Omega)$ ($H^1(\Omega)$ resp.)

The Fundamental Gap

Estimating the gap between the first two eigenvalues, the fundamental (or mass) gap,

$$\Gamma(\Omega) = \begin{cases} \lambda_2 - \lambda_1 > 0 & \text{Dirichlet boundary} \\ \mu_1 - 0 > 0 & \text{Neumann boundary} \end{cases}$$

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Get optimal geometric upper and lower bounds for the gap.

Example: $\Omega = [-D/2, D/2]$ the interval with diameter D. Then,

$$\Gamma(\Omega) = \frac{3\pi^2}{D^2}, \text{ Dirichlet}; \quad \Gamma(\Omega) = \frac{\pi^2}{D^2}, \text{ Neumann}$$

Neumann —Sharp Lower Bound

Theorem

Let $\Omega \subset M^n$ *be a bounded convex domain with diameter* $\leq D$, *and* $\mathrm{Ric}_M \geq (n-1)K$. *Then*

$$\mu_1(\Omega) \geq \bar{\mu}_1(n, K, D),$$

where $\bar{\mu}_1(n, K, D)$ *is the first non-zero eigenvalue of*

$$L_{n,K,D}(\varphi) = \varphi''(s) - (n-1)\varphi' \operatorname{tn}_K(s)$$

on the interval $[-\frac{D}{2}, \frac{D}{2}]$ *with* $\varphi'(\pm\frac{D}{2}) = 0$.

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Here

$$\mathrm{tn}_K(s) = \begin{cases} \sqrt{K}\tan(\sqrt{K}s), & K > 0 \\ 0, & K = 0 \\ -\sqrt{-K}\tanh(\sqrt{-K}s) & K < 0. \end{cases}$$

Remarks

- Special Well Known Classical Cases:
  When $K = 0$, $\bar{\mu}_1(n, 0, D) = \left(\frac{\pi}{D}\right)^2$, this gives Zhong-Yang (1984) estimate.
  When $K > 0$, $\bar{\mu}_1(n, K, \frac{\pi}{\sqrt{K}}) = nK$. Combining with Myers' theorem this gives the Lichnerowicz (1958) eigenvalue comparison: $\mu_1(M) \geq \mu_1(\mathbb{S}^n_K) = nK$.

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- History:
  P. Kröger (1992, gradient estimate);
  Mufa Chen and Fengyu Wang (1994, probabilistic 'coupling method');
  Andrews and Clutterbuck (2013, modulus of continuity for heat equation);
  Yuntao Zhang-Kui Wang (2017) (elliptic proof)

Remarks II

- Generalizations:
  For weighted Laplacian: Bakry-Qian (2000, gradient estimate); Andrews-Ni (2012, modulus of continuity);
  For $RCD(K, N)$ spaces: Y. Jiang - H.C.Zhang (2016, gradient estimate); Cavalletti-Mondino (2017, optimal transport);
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- Sharpness: Equality is achieved iff when $n = 1, K = 0$ (Hang-Wang 2007) or $K > 0, D = \frac{\pi}{\sqrt{K}}$.
  For all n, K, D, there are convex domains (even closed manifolds) whose μ_1 is arbitrary close to $\bar{\mu}_1$. So the estimate is sharp.

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$$\Gamma(\Omega, V) \geq \frac{3\pi^2}{D^2}, \qquad D = \operatorname{diam}\Omega.$$

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The lower bound is approached when $V = 0$, and domain is a thin rectangular box.

History on the Fundamental Gap Conjecture

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Question

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Question

What about convex domains in $\mathbb{S}^n$?

Lee-Wang (1987): $\Gamma(\Omega) \geq \frac{\pi^2}{D^2}$
Ling (1993): $\Gamma(\Omega, V) > \frac{\pi^2}{D^2}$

Main Results

Theorem (Seto-Wang-Wei2016, He-Wei2017, Dai-Seto-Wei2017)

Let $\Omega \subset S^n$ be a strictly convex domain with diameter D, $\lambda_i (i = 1, 2)$ be the first two eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. Then

$$\Gamma(\Omega) = \lambda_2 - \lambda_1 \geq 3\frac{\pi^2}{D^2}.$$

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Seto-Wang-Wei2016, for $n \geq 3, D \leq \pi/2$.
He-Wei2017, for $n \geq 3, D < \pi$.
Dai-Seto-Wei2017, all $n \geq 2, D < \pi$.

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Step 3: The gap estimate: $\lambda_2 - \lambda_1 \geq 3\frac{\pi^2}{D^2}$ by two-point maximum principle

Reduce to Neumann Gap

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$$\Delta w = -(\lambda_2 - \lambda_1) w - 2\langle \nabla \log \phi_1, \nabla w \rangle,$$

where w satisfies the Neumann boundary condition.

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where w satisfies the Neumann boundary condition.

This goes back to Singer-Wong-Yau-Yau1985 (weighted measure $\phi_1^2 dvol$)

Super log-concavity

Theorem (Dai-Seto-Wei2017)

Given $\Omega \subset \mathbb{M}^n_K$ a bounded strict convex domain with diameter D, let $\phi_1 > 0$ be a first eigenfunction of the Laplacian on Ω. Assume $K \geq 0$. Then for $\forall x, y \in \Omega$, with $x \neq y$,

$$\begin{aligned} \langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2})\rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2})\rangle \\ \leq -2\tfrac{\pi}{D} \tan\left(\tfrac{\pi d}{2D}\right) + (n-1)\,\mathrm{tn}_K(\tfrac{d}{2}), \end{aligned}$$

which gives

$$Hess\,(\log \phi_1) \leq -\left(\frac{\pi^2}{D^2} - \frac{n-1}{2}K\right) id.$$

This log-concavity is worse than the sphere model when $n \geq 3$ but better than sphere model when $n = 2$.

Gap Estimate

For all n, when $K \geq 0$,

$$\lambda_2 - \lambda_1 \geq 3\frac{\pi^2}{D^2}.$$

Sketch of the Proof

Let $w(x) = \dfrac{\phi_2(x)}{\phi_1(x)}$ and $\bar{w}(s) = \dfrac{\bar{\phi}_2(s)}{\bar{\phi}_1(s)}$, where

$$\Delta \phi_i = -\lambda_i \phi_i, \quad \phi_i|_{\partial\Omega} = 0$$

and

$$\begin{cases} \bar{\phi}_i'' + \bar{\lambda}_i \bar{\phi}_i = 0 \\ \bar{\phi}_i(\pm D/2) = 0 \end{cases}$$

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Compute

$$\Delta w = -(\lambda_2 - \lambda_1) w - 2\langle \nabla \log \phi_1, \nabla w \rangle,$$
$$\bar{w}'' = -(\bar{\lambda}_2 - \bar{\lambda}_1)\bar{w} - 2(\log \bar{\phi}_1)' \bar{w}'$$

satisfying Neumann boundary condition.

Consider the quotient of the oscillations

$$Q(x,y) = \frac{w(x) - w(y)}{\bar{w}\left(\frac{d(x,y)}{2}\right)}$$

on $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$, where $\Delta = \{(x,x) | x \in \overline{\Omega}\}$ is the diagonal.

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Apply maximal principle to $Q(x,y)$.

Assume the maximum of Q is attained at (x_0, y_0) and $x_0 \neq y_0$.
Denote $d_0 = \frac{d(x_0,y_0)}{2}$, $m = Q(x_0, y_0)$.
At the maximum point (x_0, y_0), $\forall E \in T_{x_0,y_0}\Omega \times \Omega$, we have

$$\nabla_E Q = 0 \;\Rightarrow\; \nabla w(y_0) = \nabla w(x_0) = -\frac{m}{2}\bar{w}'(d_0/2)e_n.$$

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$$\nabla^2_{E,E} Q = \frac{1}{\bar{w}(d_0)} \left(\nabla^2_{E,E}(w(y_0) - w(x_0)) - m\nabla^2_{E,E}\bar{w}(d_0)\right) \leq 0$$

Adding up $\nabla^2_{E_i,E_i} Q \leq 0$, $i = 1, \cdots n$ $(E_n = e_n \oplus (-e_n))$, we have

$$0 \geq \frac{\Delta w(x_0) - \Delta w(y_0)}{\bar{w}} - \frac{m}{\bar{w}} \sum_{i=1}^{n} \nabla^2_{E_i,E_i} \bar{w}.$$

Partial Laplace Comparison for Two-Point Distance Function

The Laplace comparison is for one point distance function — in $d(x, y)$, we fix x.
If we let both vary, one has

Theorem (Andrews and Clutterbuck 2013)

M^n with $\mathrm{Ric}_M \geq (n-1)K$, then (in weak sense)

$$\sum_{i=1}^{n-1} \nabla^2_{E_i, E_i} d(x, y) \leq -2(n-1)\,\mathrm{tn}_K\left(\frac{d(x, y)}{2}\right)$$

where $e_i \perp \nabla d$ are orthonormal and parallel and $E_i = e_i \oplus e_i$.

Partial Laplace Comparison for Two-Point Radial Function

Let $v(x,y) = \varphi(d(x,y))$. If $\mathrm{Ric}_{M^n} \geq (n-1)K$, then

$$\sum_{i=1}^{n-1} \nabla^2_{E_i,E_i} v(x,y) \leq -(n-1)\varphi' \,\mathrm{tn}_K\left(\frac{r(x,y)}{2}\right) \text{ if } \varphi' \geq 0$$

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This is very useful in estimating the modulus of continuity or oscillations!

$\bar{w}' \geq 0$, by two-points distance partial Laplace comparison, $\sum_{i=1}^{n-1} \nabla^2_{E_i,E_i} \bar{w} \leq -(n-1)\,\mathrm{tn}_K\, \bar{w}'$.

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$$
\begin{aligned}
0 \geq\ & -(\lambda_2-\lambda_1)m + \frac{m\bar{w}'}{\bar{w}}(n-1)\,\mathrm{tn}_K - \frac{m}{\bar{w}}\bar{w}'' \\
& -\frac{m\bar{w}'}{\bar{w}}(\langle \nabla \log \phi_1(y_0), e_n\rangle - \langle \nabla \log \phi_1(x_0), e_n\rangle) \\
=\ & -(\lambda_2-\lambda_1)m + (\bar{\lambda}_2 - \bar{\lambda}_1)m \\
& + \frac{m\bar{w}'}{\bar{w}}(-2\tfrac{\pi}{D}\tan\left(\tfrac{\pi d}{2D}\right) + (n-1)\,\mathrm{tn}_K(\tfrac{d}{2})) \\
& -\frac{m\bar{w}'}{\bar{w}}(\langle \nabla \log \phi_1(y_0), e_n\rangle - \langle \nabla \log \phi_1(x_0), e_n\rangle) \\
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\geq\ & -(\lambda_2-\lambda_1)m + (\bar{\lambda}_2-\bar{\lambda}_1)m
\end{aligned}
$$

by the super log-concavity, as $m, \bar{w}$ are positive.

When the maximum of Q is attained at (x_0, y_0) and $x_0 = y_0$, it can be proved similarly by a limiting process.
Note that as $y \to x$,

$$Q(x, \gamma'(0)) = \frac{2\langle \nabla w(x), \gamma'(0) \rangle}{\bar{w}'(0)}$$

.

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Back to super log-concavity

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Step 1: Preservation of modulus of concavity

Step 2: Existence of solution

Step 3: Convergent to model solution as $t \to \infty$

Modulus of concavity

Definition

Given a semi-convex function u on a domain Ω, a function $\psi : [0, +\infty) \to \mathbb{R}$ is called a modulus of concavity for u if for every $x \neq y$ in Ω

$$\langle \nabla u(y), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla u(x), \gamma'(-\tfrac{d}{2}) \rangle \leq 2\psi\left(\frac{d(x,y)}{2}\right).$$

Preservation of Log-concavity

Theorem

Let $\Omega \subset \mathbb{M}^n_K$ be a strictly convex domain with diameter D. Assume $K \geq 0$. Let $u(x,t) = e^{-\lambda_1 t}\phi_1(x)$, where $\Delta\phi_1 = -\lambda_1\phi_1$. Suppose $\psi_0 : [0, \frac{D}{2}] \to \mathbb{R}$ is a Lipschitz continuous function such that $\psi_0 + \frac{n-1}{2}\,\mathrm{tn}_K$ is a modulus of concavity for $\log\phi_1$. If ψ is a solution of

$$
\begin{aligned}
\frac{\partial\psi(s,t)}{\partial t} &\geq \psi''(s,t) + 2\psi(s,t)\psi'(s,t) \\
&\quad - \mathrm{tn}_K(s)\left[\psi'(s,t) + \psi^2(s,t) + \lambda_1\right], \\
\psi(\cdot,0) &= \psi_0(\cdot); \\
\psi(0,t) &= 0,
\end{aligned}
$$

where $\psi' = \frac{\partial}{\partial s}\psi$ and $\psi'' = \frac{\partial^2}{\partial s^2}\psi$, then $\psi(\cdot,t) + \frac{n-1}{2}\,\mathrm{tn}_K$ is a modulus of concavity for $\log u(\cdot,t)$ for each $t \geq 0$.

Model Equation

Note $f = (\log \bar{\phi}_1)' = -\frac{\pi}{D} \tan\left(\frac{\pi s}{D}\right)$ satisfies

$$f' + f^2 + \left(\frac{\pi}{D}\right)^2 = 0.$$

Model Equation

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On the other hand, the stationary solutions of ψ satisfy

$$0 = (\psi'(s) + \psi^2(s) + \lambda_1)' - 2\,\mathrm{tn}_K(s)(\psi' + \psi^2(s) + \lambda_1).$$

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Solving the ODE $y' - 2\,\mathrm{tn}_K(s)y = 0$, we have $y = y(0)K\,\mathrm{cs}_K^{-2}(s)$. Hence an initial condition $y(0) = 0$ would imply the trivial solution in y, which is equivalent to $\psi' + \psi^2 + \lambda_1 = 0$. The condition $y(0) = 0$ can be obtained by adding the condition $\psi'(0) = -\lambda_1$.

Thank you!

Figure: makes positive curvature!