Extra twisted connected sums and their ν-invariants

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Australian-German Workshop
on
Differential Geometry in the Large
8. 2. 2019

Outline

- G_2-geometry
  Intro, properties, examples, questions
- The ν-invariant
  Differential topology, definition of ν, properties, first examples
- Extra twisted connected sums
  Construction, properties, problems
- Computation of the ν-invariant
  Computations with η-invariants, examples, questions

G_2-geometry—Holonomy

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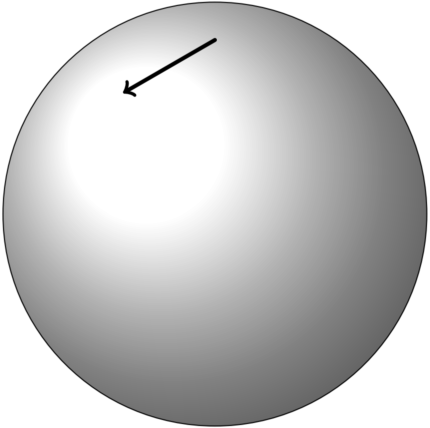

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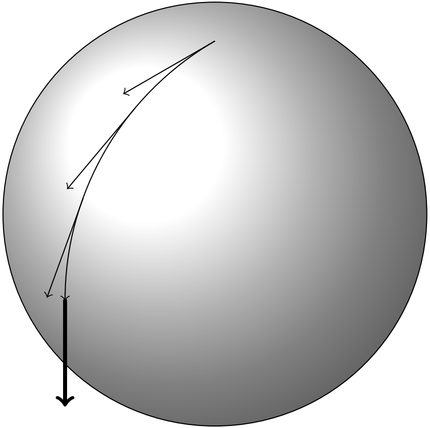

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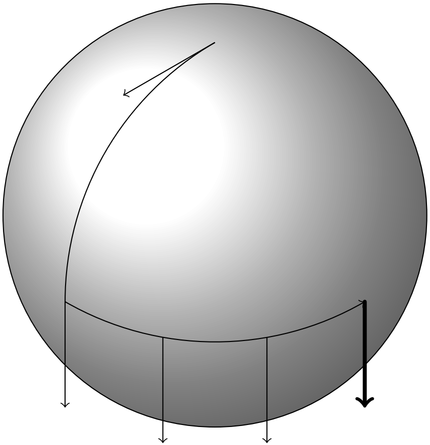

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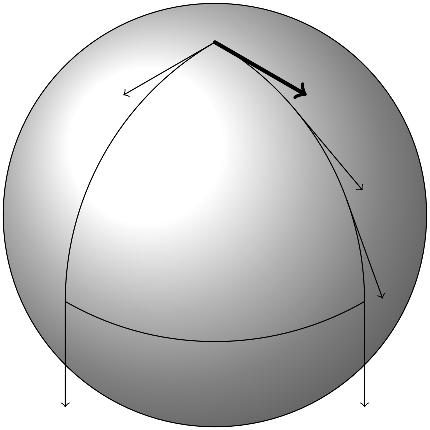

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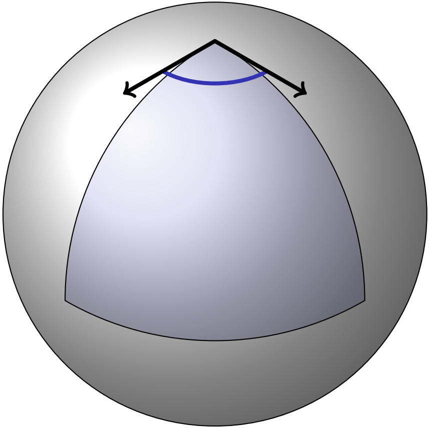

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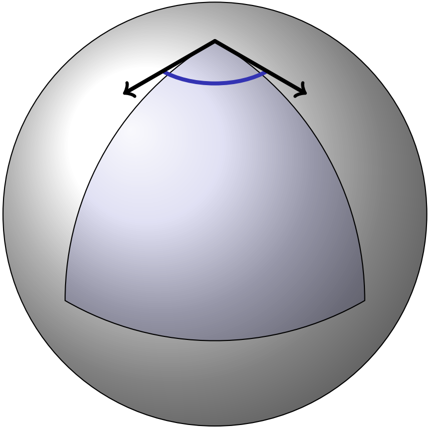
$$\mathrm{Hol}(S^2, g^{\mathrm{rnd}}) \cong SO(2)$$



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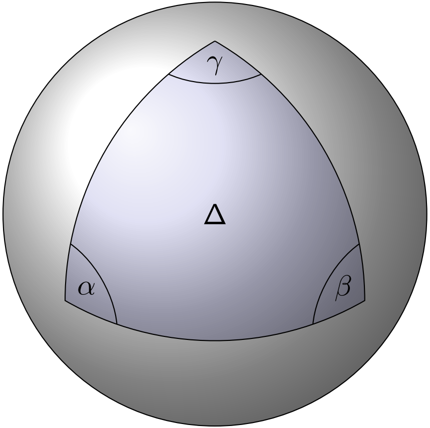
Related: the spherical area formula

$$A(\Delta) = \alpha + \beta + \gamma - \pi\ .$$



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$U(k)$	$2k$		J		*Kähler*
$Sp(\ell) \cdot Sp(1)$	4ℓ	const	$\langle I, J, K \rangle$		*Quat. Kähler*
$SU(k)$	$2k$	0	J, Ω	2	*Calabi-Yau*
$Sp(\ell)$	4ℓ	0	I, J, K, Ω	$\ell + 1$	*hyper Kähler*
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Physical motivation

- In string theory, spacetime takes the form $\mathbb{R}^{3,1} \times V$, where V is Calabi-Yau
- In M-theory, spacetime takes the form $\mathbb{R}^{3,1} \times M$, where M is a G_2-manifold
- Possible relations to other physical theories

Hence, many fruitful interactions possible

Characterisations of the Lie group G_2 give characterisations of G_2-manifolds

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- G_2 is compact, simply connected, so G_2-manifolds are Riemannian and spin
  The stabiliser of a nonzero spinor in Spin(7) is G_2
  A Riemannian 7-manifold (M, g) has $\mathrm{Hol}(M, g) \subset G_2$
  if and only if it is spin and there exists a nonzero parallel spinor

Let $\varphi \in \Omega^3(M)$ be positive and closed
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These are all known obstructions against holonomy G_2

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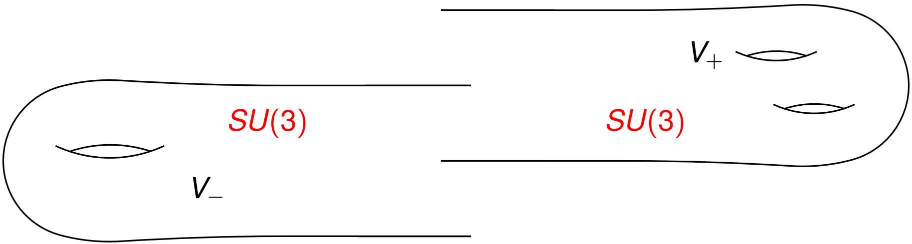

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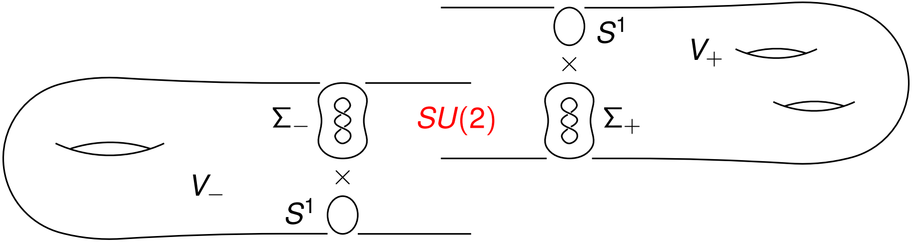

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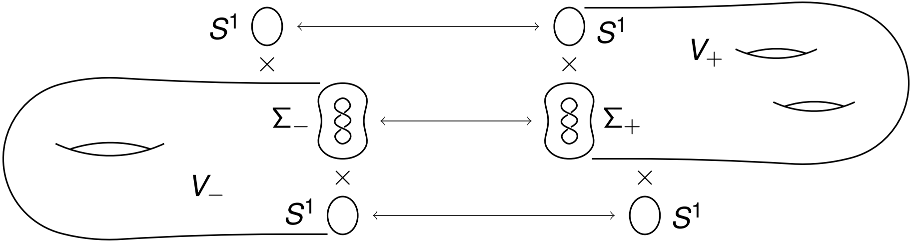

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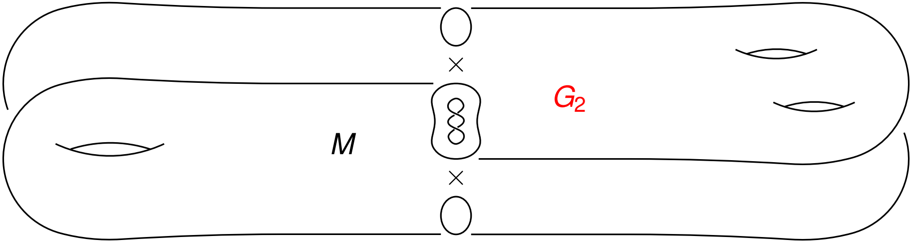

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  Singularities allow massless particles to appear in M-theory

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- Riemannian metric, spin structure, and a unit spinor (up to sign)

The ν-invariant—G_2-structures

We want to describe G_2-manifolds using differential topology

Definition

A G_2-structure on a seven-manifold M is a reduction of the $GL(7, \mathbb{R})$-frame bundle to a bundle with structure group G_2

Equivalent descriptions

- Positive three form φ on M
- Riemannian metric, spin structure, and a unit spinor (up to sign)

Idea. Use nowhere vanishing spinors to describe and distinguish G_2-structures

The ν-invariant—First questions

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Diarmuid Crowley and Johannes Nordström also asked

- Are there pairs of G_2-manifolds that are homeomorphic but not diffeomorphic?

Assume that M is a simply connected oriented spin 7-manifold with $H^4(M)$ torsion-free

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Let $\tilde{d} = \mathrm{lcm}(4, d/2)$

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Important relations

$$\xi(M,s) \equiv 7\nu(M,s) \mod 12$$

$$\frac{\xi(M,s) - 7\nu(M,s)}{12} \equiv \mu(M) \mod \gcd(28, \tilde{d}/4)$$

Let σ_0, σ_1 be two nowhere vanishing spinors. Extend to $\bar{\sigma} \in \Gamma(\mathcal{S}^+(M \times [0,1]))$
A generic $\bar{\sigma}$ will have nondegenerate isolated zeros because

$$\mathrm{rk}\,\mathcal{S}^+(M \times [0,1]) \quad = \quad 8 \quad = \quad \dim(M \times [0,1])$$

The ν-invariant—Comparing G_2-structures

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Orient $\mathcal{S}^+(M \times [0,1])$ and count with signs

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Can we write $\Delta\nu(M; \sigma_0, \sigma_1) = \nu(M, \sigma_0) - \nu(M, \sigma_1) \in \mathbb{Z}/48$?

Idea. If M is spin, then M is the spin boundary of some compact 8-manifold W
Extend σ to $\bar{\sigma} \in \Gamma(\mathcal{S}^+ W)$, then $\#\bar{\sigma}^{-1}(0)$ depends on W—not well-defined yet!

The ν-invariant—Cobordism definition

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- $\chi(W)$—Euler characteristic of W
- $\mathrm{sign}(W)$—signature of W

Definition (Crowley-Nordström)

Assume that $M = \partial W$ with W spin, compact. Define

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Problem. Given M, how to determine W with $M = \partial W$?

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- $\psi(\nabla^{\mathcal{S}M}, g^{\mathcal{S}M})$—Mathai-Quillen form in $\Omega^{\bullet}(\mathcal{S}M)$
- D_M—spin Dirac operator on $\Gamma(\mathcal{S}M)$
- B_M—odd signature operator $*d \pm d*$ on $\Omega^{\mathrm{ev}}(M)$
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Theorem (Crowley-G-Nordström)

$$\nu(M,\sigma) = 2\int_M \sigma^*\psi(\nabla^{\mathcal{S}M}, g^{\mathcal{S}M}) - 24(\eta + h)(D_M) + 3\eta(B_M) \in \mathbb{Z}/48$$

Proof.

Use $$2e(\nabla^{\mathcal{S}^+W}) = e(\nabla) + 48\hat{A}(\nabla)^{[8]} - 3L(\nabla)^{[8]} \in \Omega^8(W)$$ □

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The ν-invariant—The extended ν-invariant

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Let (M, g) be a compact manifold with $\mathrm{Hol}(M, g) \subset G_2$. Put

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- $\bar{\nu}(M, g) = 0$ if M admits an orientation reversing isometry

What about the known examples by Joyce and Kovalev?

- $\bar{\nu}(M, g) = 0$ for all twisted connected sums
  (but $\xi(M, s) \neq 0$ is possible—Wallis, arXiv:1808.09443)

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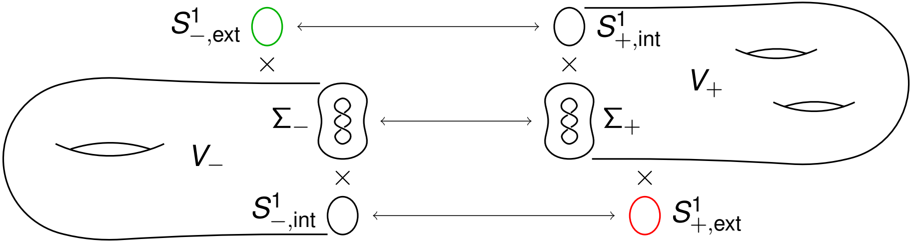
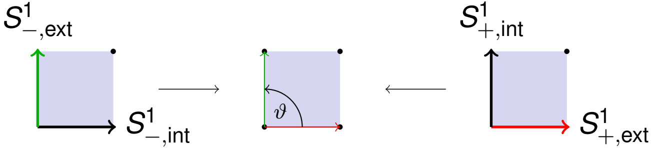
Answer. We will construct examples with $\bar{\nu}(M, g) \neq 0$
Using $\bar{\nu}(M, g)$, we will show that for some particular M, the G_2-moduli space $\mathcal{M}$ has several connected components

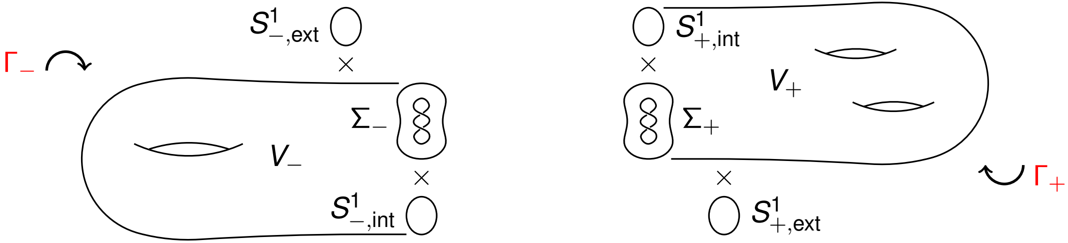
Recall twisted connected sums



Gluing of tori at angle $\vartheta = \frac{\pi}{2}$ between exterior circles



Extra twisted connected sums



Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts both on $V_\pm$ and on $S^1_{\pm,\mathrm{ext}}$

The induced action on $\partial V_\pm$ has to fix $\Sigma_\pm$ pointwise

The actions on $S^1_{\pm,\mathrm{int}}$ and $S^1_{\pm,\mathrm{ext}}$ have to be free

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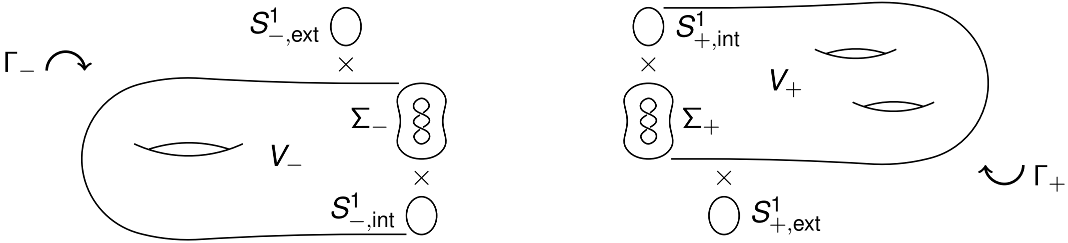

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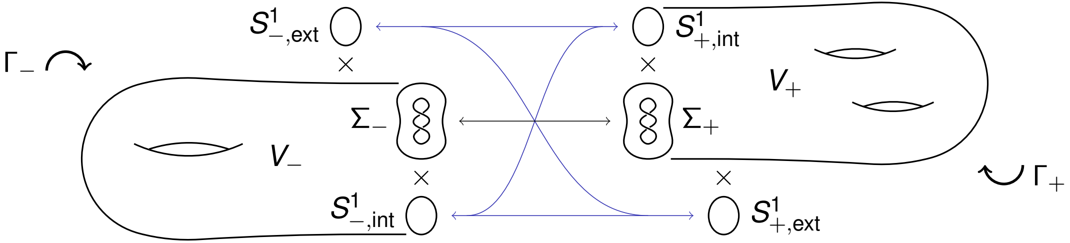

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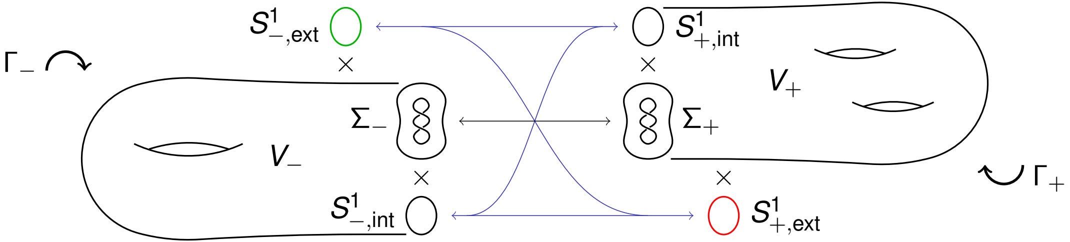
Then $(S^1_{\pm,\text{int}} \times S^1_{\pm,\text{ext}})/\Gamma_\pm$ is again a flat 2-torus
If both the tori and the K3 surfaces are isometric,
we can glue $M_\pm = (V_\pm \times S^1_{\pm,\text{ext}})/\Gamma_\pm$ at various angles ϑ

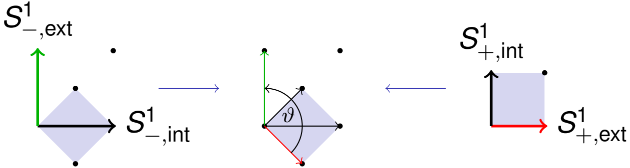
Extra twisted connected sums—Construction

Extra twisted connected sums



Modified gluing of tori at angle $\vartheta = \frac{3}{4}\pi$

$\Gamma_- \cong \mathbb{Z}/2$



$\Gamma_+ \cong \{0\}$

Extra twisted connected sums—Construction

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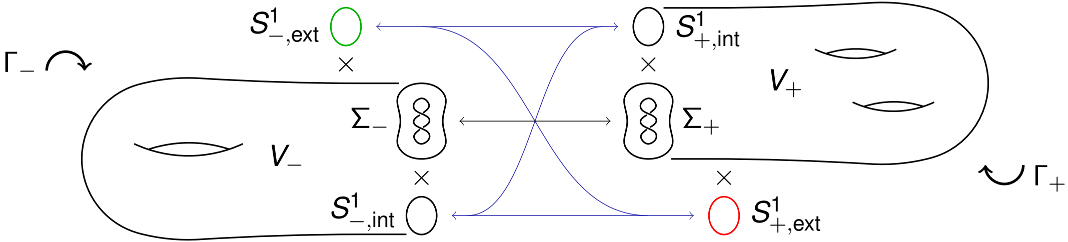

Modified gluing of tori at angle $\vartheta = \frac{2}{3}\pi$

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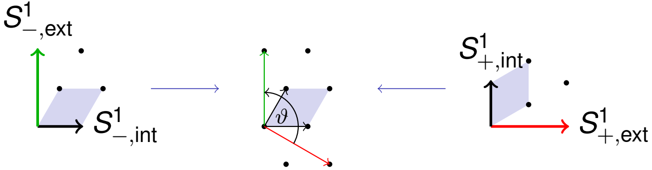

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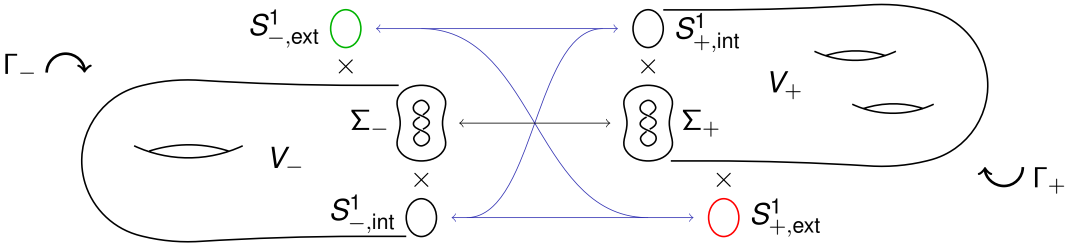
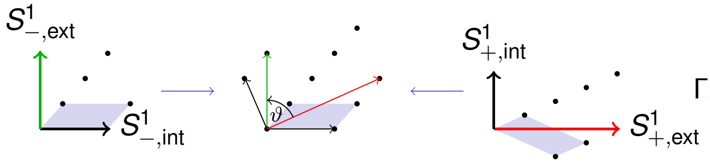
Extra twisted connected sums



Modified gluing of tori at angle $\vartheta = \arccos(\frac{1}{\sqrt{6}})$

$\Gamma_- \cong \mathbb{Z}/3$



$\Gamma_+ \cong \mathbb{Z}/4$

The CY structures on $V_\pm$ induce G_2-structures on $M_\pm$ and on $\Sigma_\pm \times T^2_\pm \times \mathbb{R}$

Then we need an isomorphism of G_2-manifolds $\Sigma_+ \times T^2_+ \times \mathbb{R} \xrightarrow{\cong} \Sigma_- \times T^2_- \times \mathbb{R}$

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Let $u_\pm$, $v_\pm$ be coordinates on the interior and exterior circles, respectively

$$u_- = -u_+ \cos\vartheta + v_+ \sin\vartheta \qquad \text{and} \qquad v_- = u_+ \sin\vartheta + v_+ \cos\vartheta$$

Let $t = t_- = -t_+$ be the coordinate in the cylinder direction

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Let $\omega_1^\pm, \omega_2^\pm, \omega_3^\pm \in \Omega^{2,+}(\Sigma)$ be hyperkähler triples, $\omega_1^\pm$ is the Kähler form from $V_\pm$

The CY structures on $V_\pm$ induce G_2-structures on $M_\pm$ and on $\Sigma_\pm \times T^2_\pm \times \mathbb{R}$

Then we need an isomorphism of G_2-manifolds $\Sigma_+ \times T^2_+ \times \mathbb{R} \xrightarrow{\cong} \Sigma_- \times T^2_- \times \mathbb{R}$

Let $u_\pm$, $v_\pm$ be coordinates on the interior and exterior circles, respectively

$$u_- = -u_+ \cos\vartheta + v_+ \sin\vartheta \qquad \text{and} \qquad v_- = u_+ \sin\vartheta + v_+ \cos\vartheta$$

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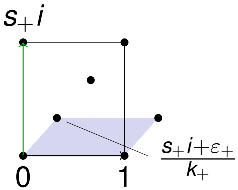
$$\varphi = dv_\pm \wedge \omega_1^\pm + du_\pm \wedge \omega_2^\pm + dt_\pm \wedge \omega_3^\pm + dt_\pm \wedge du_\pm \wedge dv_\pm$$
$$\omega_1^- = \cos\vartheta\, \omega_1^+ + \sin\vartheta\, \omega_2^+ \qquad \omega_2^- = \sin\vartheta\, \omega_1^+ - \cos\vartheta\, \omega_2^+ \qquad \omega_3^- = -\omega_3^+$$

Both the torus matching and the K3 matching depend on the gluing angle ϑ

Assume that $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ acts on $V_\pm \times S^1_{\pm,\mathrm{ext}}$
A torus matching is described by

- A number $\varepsilon_+ \in (\mathbb{Z}/k_+)^\times$ if $k_+ > 1$ describing the Γ_+-action



Extra twisted connected sums—Matching the tori

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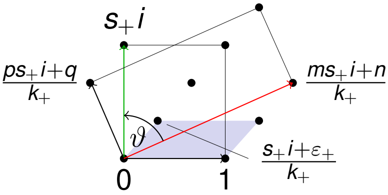
- A gluing matrix $G = \left(\begin{smallmatrix} m & p \\ n & q \end{smallmatrix}\right)$ with $\det G = -k_+k_-$ and $mq \leq 0$, $np \geq 0$

satisfying some extra conditions (only finitely many choices for ε_+, G possible)



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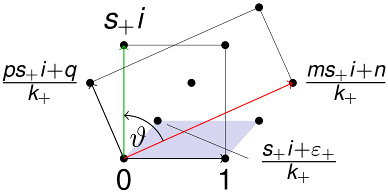

satisfying some extra conditions (only finitely many choices for ε_+, G possible)

From G, recover

- The aspect ratios $s_+ = \frac{\ell(S^1_{+,\mathrm{ext}})}{\ell(S^1_{+,\mathrm{int}})} = \sqrt{-\frac{nq}{mp}}$ and $s_- = \frac{\ell(S^1_{-,\mathrm{ext}})}{\ell(S^1_{-,\mathrm{int}})} = \sqrt{-\frac{mn}{pq}}$
- The gluing angle $\vartheta = \arg(ms_+ + in) \in (-\pi, \pi]$
- The fundamental group $\pi_1(M) \cong \mathbb{Z}/p$

If M is an extra twisted connect sum, identify $\Sigma = \Sigma_+ \cong \Sigma_-$, let

$$N_\pm = \mathrm{im}\big(H^2(V_\pm) \to H^2(\Sigma)\big) \quad \subset \quad L = H^2(\Sigma) \cong {E_8}^2 \oplus U^3$$

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Matching Problem. To construct M, find $N_+, N_- \subset L$ and positive classes $[\omega_1^\pm] \in L_{2\vartheta} \cap N_{\pm,\mathbb{R}}$ and $[\omega_2^\pm] \in L_{2\vartheta} \cap N_{\pm,\mathbb{R}}^\perp$ of length 1 satisfying (*)

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All this works for “easy” $V_\pm$ if $\mathrm{rk}\, N_+ = \mathrm{rk}\, N_-$ and $N_{\pm,\mathbb{R}} \subset L_{2\vartheta}$

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In the following, we will only need the configuration (N_+, N_-), the gluing matrix G, and the number ε_+

With respect to modified Atiyah-Patodi-Singer boundary conditions, define

$$\bar{\nu}(M_\pm, g) = 3\eta_{\mathrm{APS}}(B_{M_\pm}; L_{B,\pm}) - 24\eta_{\mathrm{APS}}(D_{M_\pm}; L_{D,\pm}) \quad \in \quad \mathbb{R}$$

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Assume $\vartheta \in (0, \pi)$, put $\rho = \pi - 2\vartheta \in (-\pi, \pi)$, and define

$$m_\rho(L; N_+, N_-) = \operatorname{sign}\rho \left(\#\{\, j \mid \alpha_j^- \in \{\pi - |\rho|, \pi\} \,\} - 1 + 2\,\#\{\, j \mid \alpha_j^- \in (\pi - |\rho|, \pi) \,\} \right)$$

Put $\operatorname{sign} 0 = 0$, then $m_\rho(L; N_+, N_-) = 0$ for ordinary twisted connected sums

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From the gluing formulas for η-invariants by Bunke, Kirk-Lesch and others, we get

Theorem (Crowley-G-Nordström)

$$\bar{\nu}(M, g) = \bar{\nu}(M_+, g) + \bar{\nu}(M_-, g) - 72\frac{\rho}{\pi} + 3m_\rho(L; N_+, N_-)$$

If $\Gamma_\pm \cong \{0\}$ or $\mathbb{Z}/2$ then $\bar{\nu}(M_\pm, g) = 0$

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There exists a spin 7-manifold M with

$$H^1(M) \cong H^2(M) \cong 0\;, \qquad H^4(M) \cong \mathbb{Z}^{97}\;, \quad \operatorname{div} p_1(TM) = 4$$

admitting three different G_2-holonomy metrics g_1, g_2, g_3 with

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There exists a spin 7-manifold M with

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The metric g_1 comes from a rectangular twisted connected sum
For g_2, g_3, take $\Gamma_+ \cong \Gamma_- \cong \mathbb{Z}/2$ and $\vartheta = \frac{\pi}{6}$

Let (M, g) be an extra twisted connected sum with $\Gamma_\pm \cong \mathbb{Z}/k_\pm$ where k_+, $k_- \in \{1, 2\}$ and $\vartheta \in \left\{\pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}, \frac{\pi}{2}\right\}$
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The ν-invariant mod 3 is an isomorphism $\nu\colon \Omega^7_{G_2} \to \mathbb{Z}/3$
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Question. Is the G_2-bordism class of (M, σ) an obstruction against holonomy G_2?

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Question. Is the G_2-bordism class of (M, σ) an obstruction against holonomy G_2?

Answer. No, there are examples with $3 \nmid \nu(M, \sigma)$.

This is our motivation to consider more complicated extra twisted connected sums

Let (M, g) be an extra twisted connected sum with $\Gamma_\pm \cong \mathbb{Z}/k_\pm$
where k_+, $k_- \in \{1, 2\}$ and $\vartheta \in \left\{\pm\frac{\pi}{6}, \pm\frac{\pi}{4}, \pm\frac{\pi}{3}, \frac{\pi}{2}\right\}$
In this case, 3 divides $\bar{\nu}(M, g) = -72\frac{\rho}{\pi} + 3m_\rho(L; N_+, N_-)$

Note that the G_2-bordism group in dimension 7 is $\Omega^7_{G_2} \cong \mathbb{Z}/3$
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Note. Recall that with $k_+ \geq 3$ or $k_- \geq 3$, we can get gluing angles $\vartheta \notin \mathbb{Q}\pi$
Because $\bar{\nu}(M, g) \in \mathbb{Z}$, expect $\bar{\nu}(M_\pm, g) \neq 0$ if $k_\pm > 2$

Computation of the ν-invariant—Adiabatic limits

Let $M_\pm = V_\pm \times S^1_{\pm,\text{ext}}$, rescale $S^1_{\pm,\text{ext}}$ by $a > 0$ to get $M_{\pm,a}$
The limit $a \to 0$ is called adiabatic limit

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From the adiabatic limit theorems of Bismut-Cheeger, Dai, G, we deduce

Theorem (G-Nordström)

Let $\gamma \in \Gamma_\pm$ be the generator that acts by $\frac{2\pi}{k_\pm}$ on $S^1_{\pm,\mathrm{ext}}$
Let $V_{\pm,j}$ be the set of isolated fixpoints of γ^j on $V_\pm$
Let $e^{i\alpha_{j,1}(p)}$, $e^{i\alpha_{j,2}(p)}$, $e^{i\alpha_{j,3}(p)}$ be the eigenvalues of γ^j on $T_pV_\pm$
with $\alpha_{j,1}(p) + \alpha_{j,2}(p) + \alpha_{j,3}(p) = 0$. Then

$$\lim_{a\to 0} \bar{\nu}(M_{\pm,a}) = \frac{3}{k_\pm} \sum_{j=1}^{k_\pm - 1} \cot \frac{\pi j}{k_\pm} \sum_{p \in V_{\pm,j}} \frac{\cos\frac{\alpha_{j,1}(p)}{2} \cos\frac{\alpha_{j,2}(p)}{2} \cos\frac{\alpha_{j,3}(p)}{2} - 1}{\sin\frac{\alpha_{j,1}(p)}{2} \sin\frac{\alpha_{j,2}(p)}{2} \sin\frac{\alpha_{j,3}(p)}{2}} \in \mathbb{Q}$$

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We will focus on examples without isolated fixpoints

Computation of the ν-invariant—Variational formula

By theorems of Bismut-Cheeger, Dai-Freed, the variational formula for the η-invariant of a Dirac type operator a manifold with boundary consists of

- the integral of a Chern-Simons form over the interior
- the degree-1-component of an η-form on the boundary

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Theorem (G-Nordström)

$$\bar{\nu}(M_\pm) - \lim_{a \to 0} \bar{\nu}(M_{\pm,a}) = F_{k_\pm,\varepsilon_\pm}(s_\pm) = 288 \int_0^{s_\pm} \tilde{\eta}(\mathbb{A})$$

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Bismut-Cheeger also give a formula for $\tilde{\eta}(\mathbb{A})$ as a sum over $\mathbb{Z}^2$ depending on a

Computation of the ν-invariant—Hyperbolic geometry

Represent the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ by $\tau \in \mathcal{H} = \{\, z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \,\}$
Then $\tilde{\eta} \in \Omega^1(\mathcal{H})$ is $SL(2,\mathbb{Z})$-invariant and satisfies $d\tilde{\eta} = -\frac{1}{4\pi}\, dA_{\mathrm{hyp}}$ by Bismut's family index theorem

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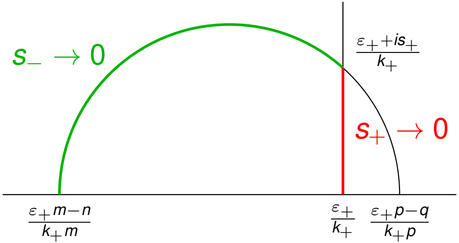
Adiabatic limits—geodesic rays



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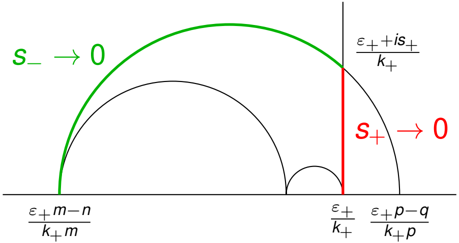
Adiabatic limits—geodesic rays $\tilde{\eta}(\mathbb{A}) = 0$ along families of rectangular ($k = 1$) or rhombic ($k = 2$) tori



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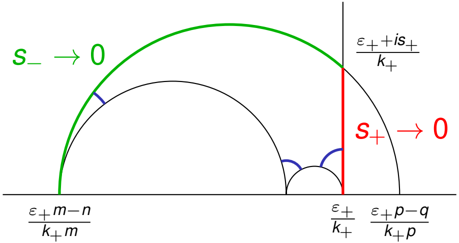
Cusps—families of adiabatic limits
Their contribution can be computed using a formula by Bunke and Ma



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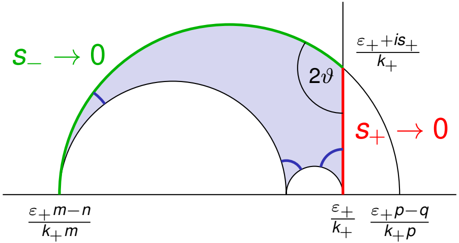

Compute $F_{k_-,\varepsilon_-}(s_-) + F_{k_+,\varepsilon_+}(s_+)$ using Stokes' theorem from the cusp contributions and the hyperbolic area formula
The angle 2ϑ at the finite corner cancels $-72\frac{\rho}{\pi}$ in the gluing formula

The logarithm of the Dedekind η-function is given by

$$L(\tau) = \frac{\pi i \tau}{12} - \sum_{n=1}^{\infty} \sum_{d|n} d^{-1}\, e^{2\pi i n \tau}$$

Theorem (G-Nordström-Zagier)

There exists a constant $c_{k_\pm,\varepsilon_\pm} \in \mathbb{Q}$ such that

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Compute the variational term using the functional equations

$$L(\tau + 1) = \frac{\pi i}{12} + L(\tau) \qquad \text{and} \qquad L\left(-\frac{1}{\tau}\right) = \frac{1}{2} \log \frac{\tau}{i} + L(\tau)$$

Example. The example with $\cos\vartheta = \frac{1}{\sqrt{6}}$ has $\bar{\nu}(M, g) = -65$
In particular, $3 \nmid \bar{\nu}(M, g)$, so it is not G_2-nullbordant

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Construct more examples

- Find more asymptotically cylindrical Calabi-Yau manifolds
- Understand their moduli space, make the K3 surfaces match
- Consider other constructions

Thanks for your attention!