Higher Willmore energies, Q-curvatures, and related global geometry problems.

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background:
G-. and Andrew Waldron: **Conformal hypersurface ... Loewner-Nirenberg-Yamabe problem. in press CAG, arXiv:1506.02723**, and **Renormalized Volume, CMP, 2017**; **G-.**, Waldon, **Commun. Contemp. Math., in press**, arXiv:1611.0834;

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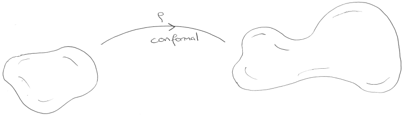
A conformal energy

It was observed by Blaschke 1929, Willmore 1965 and others that

$$\int_{\Sigma=surf} H^2 \, dA =: \textbf{Willmore energy} \quad \text{(or bending energy 1920)}$$

is invariant under the **conformal motions** of $\mathbb{E}^3$ (basically $SO(4,1)$ action). These are the local maps $\rho : \mathbb{E}^3 \to \mathbb{E}^3$ that preserve angles.



This property is linked to applications e.g. **solid mechanics** including **cell membranes**, **relativity** and **string theory**, and **geometric analysis** – Willmore conjecture concerns absolute minimisers.

The Willmore equation

So the Euler-Lagrange equation (wrt variation of embedding) for the Willmore energy is important:

$$\underbrace{\Delta_\Sigma H + 2H(H^2 - K)}_{\boxed{\mathcal{B} = \textbf{W}\text{illmore } \textbf{I}\text{nvariant}}} = 0; \qquad K = \text{Gauss curvature.}$$

The WI is an **extremely interesting invariant** for embeddings into Euclidean space $\mathbb{E}^3$,

$$\iota : \Sigma^2 \longrightarrow \mathbb{E}^3,$$

It is **conformally invariant** and has **linear leading term** $\Delta_\Sigma H$. A rare quality! One can show that it is a **fundamental conformal curvature quantity**.

Question: What is the meaning of the Willmore invariant? Are there analogues in higher dimensions? This turns out to be linked to the problem of finding **all** conformal hypersurface **invariants**!

Yamabe and its singular variant

(M,g) a $(n+1)$-dimensional Riemannian manifold (M,g).

Problem (Y): $\exists$ a smooth real-valued positive function u on M satisfying $\bar{g} := u^{-2}g$ has scalar curvature $\mathrm{Sc}^{\bar{g}} = -n(n+1)\eta$?

Set $u = \rho^{-2/(n-1)}$ then problem governed by the **Yamabe equation:**

$$\Big[-4\,\frac{n}{n-1}\,\Delta + \mathrm{Sc}\Big]\rho + n\,(n+1)\,\eta\rho^{\frac{n+3}{n-1}} = 0 \qquad \eta = 1, 0, -1. \quad (1)$$

– solved by Yamabe, Trudinger, Aubin, Schoen (1960–1984).

Problem (sY): $\exists$ a smooth real-valued ~~positive~~ function u on M satisfying $\bar{g} := u^{-2}g$ has scalar curvature $\mathrm{Sc}^{\bar{g}} = -n(n+1)\eta$?

[Set $u = \rho^{-2/(n-1)}$ then] problem governed by

$$S(u) := |du|^2 - \frac{2}{n+1}u\left(\Delta + \frac{\mathrm{Sc}}{2n}\right)u = \eta. \qquad (2)$$

Q: Solutions where u changes sign? If so nature of $\mathcal{Z}(u)$?

NB: $\exists$ example solutions $\boxed{\eta = 1}$ – e.g. on sphere .

Conformal-to-(almost)-Einstein equation

The metric $\bar{g} := u^{-2}g$ is **Einstein** i.e. $\mathrm{Ric}^{\bar{g}} = \lambda\bar{g}$ iff there is $u > 0$ s.t.

$$\text{trace-free}(\nabla_a\nabla_b u + P_{ab}u) = 0. \qquad \text{(AE) – almost Einstein eqn}$$

where $\mathrm{Ric} = (n-1)P + gJ$, $J := trace^g(P)$. We drop $u > 0$ and study. Best to replace u by conformal density of weight 1 $\sigma \in \Gamma(\mathcal{E}[1])$, where $\mathcal{E}[2n+2] \cong (\Lambda^{n+1}(TM))^2$. Then (AE) is conformally invariant. AE is overdetermined so study by prolongation. It is equivalent to the closed first order system:

$$\begin{aligned} \nabla_a\sigma - \mu_a &= 0 \\ \nabla_a\mu_b + P_{ab}\sigma + \rho\boldsymbol{g}_{ab} &= 0 \\ \nabla_a\rho - P_{ab}\mu^b &= 0 \end{aligned}$$

$\boldsymbol{g}$ is the conformal metric, $\otimes^{n+1}\boldsymbol{g} : (\Lambda^{n+1}(TM))^2 \xrightarrow{\simeq} \mathcal{E}[2n+2]$.

The tractor connection and D-operator

The above system can be collected into a linear connection – in fact the conformally invariant **tractor bundle** $\mathcal{T}$ and **connection** $\nabla^{\mathcal{T}}$. Given $\overline{g} \in \mathbf{c}$ this is given by

$$\mathcal{T} \stackrel{\overline{g}}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := (\Lambda^{n+1}TM)^{\frac{2}{2(n+1)}}$$

$$\nabla_a^{\mathcal{T}}(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a,\ \nabla \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a \rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a conformally invariant **tractor metric** h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b \mu^b = h(V, V).$$

There is also a 2^{nd} order conformally invariant **Thomas operator**:

$$\Gamma(\mathcal{E}[w]) \in f \mapsto D_A f \stackrel{\overline{g}}{=} \begin{pmatrix} (n+2w-1)wf \\ (n+2w-1)\nabla_a f \\ -(\Delta f + wJf) \end{pmatrix}$$

where J is $\mathrm{trace}^{\overline{g}}(P_{ab})$, so a number times $\mathrm{Sc}(\overline{g})$.

The scale tractor

If $I_A \stackrel{g}{=} (\sigma, \mu_a, \rho)$ is a parallel tractor then $\mu_a = \nabla_a \sigma$, and $\rho = -\frac{1}{n+1}(\Delta\sigma + \mathrm{J}\sigma)$. This gives the first statement of:

Proposition

I parallel implies $I_A = \frac{1}{n+1} D_A \sigma$. So $I \neq 0 \Rightarrow \sigma$ is nonvanishing on an open dense set $M_{\sigma\neq 0}$. On $M_{\sigma\neq 0}$, $\bar{g} = \sigma^{-2}\boldsymbol{g}$ is Einstein. Conversely if $\bar{g} = \sigma^{-2}\boldsymbol{g}$ is Einstein then $I := \frac{1}{n+1} D\sigma$ is parallel.

Drop parallel and study the **scale tractor** $I := \bar{D}\sigma := \frac{1}{n+1} D\sigma$:

$$I^2 := I^A I_A \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{n+1}\sigma(\mathrm{J} + \Delta)\sigma \qquad (3)$$

(cf. (2)) where g is any metric from $\mathbf{c}$ and ∇ its Levi-Civita connection. This is well-defined everywhere [on $(M, \mathbf{c}, \sigma)$], while where σ is non-zero, it computes

$$I^2 = -\frac{2}{n+1}\mathrm{J}^{\bar{g}} = -\frac{\mathrm{Sc}^{\bar{g}}}{n(n+1)} \quad \text{where} \quad \bar{g} = \sigma^{-2}\boldsymbol{g}.$$

$\therefore$ I^2 gives a **generalisation of the scalar curvature** $(\times \frac{-1}{n(n+1)})$.

Non-zero generalised scalar curvature I^2

The **singular Yamabe equation** is the equation $\boxed{I^2 = 1}$.

Proposition (G. JGP 2010)

Let $(M, \mathbf{c}, I)$ be Riemannian of signature with I^2 **positive**. *Then $\mathcal{Z}(\sigma)$, if not empty, is a smooth embedded separating hypersurface and the manifold M stratifies into $M = M_- \cup M_0 \cup M_+$ according to the strict sign of σ. The components $M \setminus M_\mp$ are conformal compactifications of $M_\pm$.*

Proof.

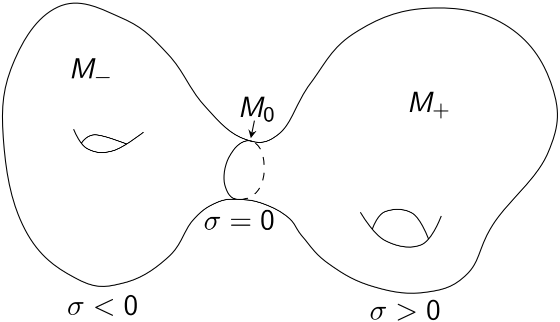
From $I^2 \stackrel{g}{=} \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) - \frac{2}{n+1}\sigma(\mathrm{J} + \Delta)\sigma$: Along $\mathcal{Z}(\sigma)$ we have

$$I^2 = \boldsymbol{g}^{ab}(\nabla_a\sigma)(\nabla_b\sigma) \quad (> 0 \quad \text{by assumption}).$$

in particular $\nabla\sigma$ is nowhere zero on $\mathcal{Z}(\sigma)$, and so σ is a **defining density**. Thus $\mathcal{Z}(\sigma)$ (if $\neq \emptyset$) is a smoothly embedded hypersurface by the implicit function theorem. □

The picture

Thus if $I^2 > 0$ we have:



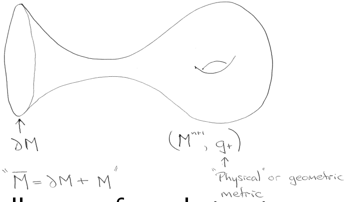
With $g_\pm = \sigma^{-2}\boldsymbol{g}$ on $M_\pm$. Since σ is a defining density for M_0 this exactly means that $M \setminus M_\mp$ **conformally compactifies** $(M_\pm, g_\pm)$. Then M_0 has a conformal structure $\mathbf{c}_0$ from $\boldsymbol{g}|_{\otimes^2 TM_0}$.

In particular this applies to a **singular Yamabe solution** $I^2 = 1$. But then we'll see there are **additional conditions** on the $(M_0, \mathbf{c}_o)$ **embedding** (and $(M_\pm, g_\pm)$ are asymptotically hyperbolic.)

Conformal compactification and Loewner-Nirenberg

A **conformal compactification** of a complete Riemannian manifold (M^{n+1}, g_+) is a manifold $\overline{M}$ with boundary ∂M s.t.:

- $\exists$ g on $\overline{M}$, with $g_+ = u^{-2} g$, where
- u a **defining f'n** for ∂M: $\partial M = \mathcal{Z}(u)$ & $du_p \neq 0$ $\forall p \in \partial M$.



$\Rightarrow$ canonically a conformal structure on boundary: $(\partial M, [g|_{\partial M}])$.

Our question/variant: Given g (or really $\mathbf{c} = [g]$) can we find a defining function $u \in C^\infty(\overline{M})$ for $\Sigma = \partial M$ s.t.

$$\mathrm{Sc}(u^{-2} g) = -n(n+1)?$$ NB: This satisfied for $\mathbb{H}^{n+1}$ Ball

Answer - Yes: Loewner-Nirenberg, Aviles and McOwen, ACF. Smoothly?

The obstruction density of ACF

Can we solve $\mathrm{Sc}(u^{-2}g) = -n(n+1)$? formally (i.e. power series) along the boundary? **Answer: No** - in general can get:

Theorem (Andersson, Chruściel, & Friedrich)

$$\exists u \in C^\infty(M) \quad s.t. \quad \mathrm{Sc}(u^{-2}g) = -n(n+1) + u^{n+1}\mathcal{B}_n.$$

and $\mathcal{B}_n|_{\partial M}$ is the obstrn to smooth soln of L-N. problem. Further

$$\mathcal{B}_2|_{\partial M} = \delta \cdot \delta \cdot \mathring{L} + \text{lower order}$$

is a conformal invariant of $\Sigma^2 = \partial M$.

In fact $\mathcal{B}_2 =$ **Willmore Invariant**!

Theorem. [G. + Waldron 2013] For $n \geq 2$ s.t.:

- $\mathcal{B}_n$ is a conformal invariant of $\Sigma = \partial M$.
- $\mathcal{B}_n$ generalises $\mathcal{B}_2 = \mathcal{B}$.

Idea. Use:

Lemma

$$\mathrm{Sc}(g_+) = -n(n+1) \Leftrightarrow I^2 := h(I, I) = 1$$

The conformal Eikonal equation

Problem: For a conformal manifold $(M, \mathbf{c})$ and an embedding $\iota : \Sigma \to M$ solve

$$I_A I^A = (\bar{D}_A \sigma)(\bar{D}^A \sigma) = 1 + O(\sigma^\ell), \qquad \bar{D} := \frac{1}{n+1} D$$

for ℓ as high as possible, and σ a Σ defining density.

Key point:

Proposition (G.+Waldron 2014)

$x := \sigma$, $y := -\frac{1}{I^2} I^A D_A$ generate an $\mathfrak{sl}(2)$! More generally:

$$[I \cdot D, \sigma] = I^2 (n + 1 + 2\boldsymbol{w}) \quad \boldsymbol{w} = \textit{weight operator}$$

From standard $\mathfrak{sl}(2)$ identities we have

$$[I \cdot D, \sigma^{k+1}] = I^2 \sigma^k (k+1)(n + k + 1 + 2\boldsymbol{w}),$$

and this allows an inductive solution (using also other tractor identities).

Lemma

Suppose that $\underline{\sigma \in \Gamma(\mathcal{E}[1])\ \textit{defines}\ \Sigma\ \textit{in}\ (M, \mathbf{c})}$ *and*

$$I_\sigma^2 = 1 + \sigma^k A_k \quad \textit{where} \quad A_k \in \Gamma(\mathcal{E}[-k])$$

is smooth on M, *and* $k \geq 1$, *then*

- *if* $k \neq (n+1)$ *then* $\exists\ f_k \in \Gamma(\mathcal{E}[-k])$ *s.t.* $\sigma' := \sigma + \sigma^{k+1} f_k$ *satisfies* $I_{\sigma'}^2 = 1 + \sigma^{k+1} A_{k+1}$, *where* A_{k+1} *smooth*;
- *if* $k = (n+1)$ *then:* $I_{\sigma'}^2 = I_\sigma^2 + O(\sigma^{n+2})$.

Proof.

Squaring with the tractor metric, using the $\mathfrak{sl}(2)$, etc

$$\begin{aligned}(\bar{D}\sigma')^2 &= (\bar{D}\sigma + \bar{D}(\sigma^{k+1} f_k))^2 \\ &= I_\sigma^2 + \frac{2}{n+1} I_\sigma \cdot D(\sigma^{k+1} f_k) + (\bar{D}(\sigma^{k+1} f_k))^2 \\ &= 1 + \sigma^k A_k + \frac{2\sigma^k}{n+1}(k+1)(n+1-k) f_k + O(\sigma^{k+1}).\end{aligned}$$

The distinguished defining density and obstruction density

Theorem (G.-, Waldron arXiv:1506.02723)

For Σ^n embedded in $(M^{n+1}, \mathbf{c})$ there is a distinguished defining density $\bar{\sigma}$, **unique** *modulo $+O(\sigma^{n+2})$, s.t.*

$$I_{\bar{\sigma}}^2 = 1 + \bar{\sigma}^{n+1} \mathcal{B}_{\bar{\sigma}}.$$

Moreover:

$$\mathcal{B} := \mathcal{B}_{\bar{\sigma}}|_\Sigma \in \Gamma(\mathcal{E}_\Sigma[-n-1])$$

is determined by $(M, \mathbf{c}, \Sigma)$ and is a **natural conformal invariant**.

For n even $\mathcal{B} = 0$ **generalises the Willmore equation** *in that:*

$$\mathcal{B} = \bar{\Delta}^{\frac{n}{2}} H + \textit{lower order terms};$$

while for n odd $\mathcal{B}$ has no linear leading term.

Corollary (ACF + above implies)

On a closed (M, g) if there is a sign changing smooth solution of sing. Yamabe: $|du|^2 - \frac{2}{n+1} u \left(\Delta + \frac{\mathrm{Sc}}{2n}\right) u = 1$ then $\Sigma := \mathcal{Z}(u)$ is a higher Willmore hypersurface – i.e. it satisfies $\mathcal{B} = 0$.

$\mathcal{B}$ is variational

For suitable regularisations $\overline{M}_\epsilon$ of conformally compact manifolds $\overline{M}$:

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}_\epsilon} \sqrt{g_+} = \frac{v_n}{\epsilon^n} + \cdots + \frac{v_1}{\epsilon} + \mathcal{A} \log \epsilon + V_{ren} + O(\epsilon).$$

Theorem (Graham 2016: PAMS 2017, arXiv:1606.00069)

If $g_+ = \bar{\sigma}^{-2}\mathbf{g}$, where $\bar{\sigma}$ an approximate solution of the sing. Yamabe problem then $\mathcal{A}$ a conformal invariant of $\Sigma \hookrightarrow M$ and

$$\frac{\delta \mathcal{A}}{\delta \Sigma} = \frac{(n+1)(n-1)}{2}\mathcal{B}$$

So the anomaly term in the renormalised volume expansion provides an **energy** with **functional gradient the obstruction density**, in other words an energy generalising the Willmore energy.

Extrinsic Q-curvature and the anomaly

In fact – also in analogy with the treatment of Poincaré-Einstein manifolds – there is nice local quantity giving the anomaly:

Theorem (G.- Waldron, CMP 2017, arXiv:1603.07367)

$$\mathcal{A} = \frac{1}{n!(n-1)!} \int_\Sigma Q$$

where, with $\tau \in \Gamma\mathcal{E}_+[1]$ *a scale giving the boundary metric,* $Q := (-I \cdot D)^n \log \tau$.

- Q here is an **extrinsically coupled** Q**-curvature** meaning e.g.

$$Q^{\widehat{g}_\Sigma} = e^{-nf}(Q^{g_\Sigma} + P_n f) \quad \text{where} \quad \widehat{g}_\Sigma = e^{2f} g_\Sigma$$

and for n even

$$P_n = \Delta_\Sigma^{\frac{n}{2}} + \text{lower order terms; } P_n \text{ FSA, and } P_n 1 = 0,$$

is an **extrinsically coupled** GJMS type operator. Q and P_n are from G.-, Waldron arXiv:1104.2991 = Indiana U.M.J. 2014.

Idea of proof

Use a Heaviside function θ to "cut off" an integral over all $\overline{M}$

$$\mathrm{Vol}_\epsilon = \int_{\overline{M}} \frac{dV^{g_\tau}}{\sigma^{n+1}} \theta(\frac{\sigma}{\tau} - \epsilon).$$

Then the divergent terms and anomaly are given by

$$v_k \sim \frac{d^{n-k}}{d\epsilon^{n-k}} \left(\epsilon^{n+1} \frac{d}{d\epsilon} \mathrm{Vol}_\epsilon \right) \Big|_{\epsilon=0},$$

So

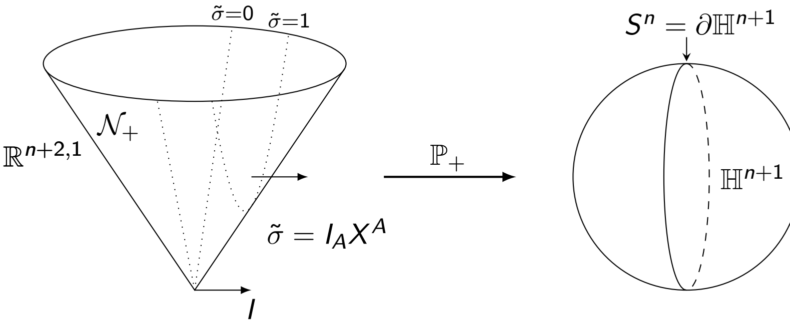
$$v_k \sim \int_{\overline{M}} \frac{\delta^{n-k}(\sigma)}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_{\overline{M}} \delta^{n-1}(\sigma) I \cdot D \log \tau$$

Then via identities, and the $sl(2)$ again

$$v_k \sim \int_\Sigma (I \cdot D)^{n-k} \frac{1}{\tau^k} \quad \text{and} \quad \mathcal{A} \sim \int_\Sigma (I \cdot D)^n \log \tau$$

Solutions to (sY) on the sphere

Conformal compactification of $\mathbb{H}^{n+1}$ by **symmetry breaking**:



• Affine parallel transport on $\mathbb{R}^{n+3}$ gives the **conformal tractor connection** on $(S^{n+1}, \mathbf{c})$. Thus spacelike I in $\mathbb{R}^{n+2,1}$ (with $I^2 = 1$) gives parallel tractor I and hence a solution of singular Yamabe $I^2 = 1$ s.t. $\mathcal{Z}(\sigma) \neq \emptyset$. In this case $\Sigma := \mathcal{Z}(\sigma)$ is **totally umbilic** (and higher Willmore). There are solutions on other closed mflds: G+Leitner: Commun. Contemp. Math. **12** (2010)

Above suggests: An "Obata type" question/conjecture

The Obata Theorem states (more than):

Theorem (Obata)

If $\bar{g}$ is a metric on the round sphere S^n that is conformal to the standard metric g and has constant scalar curvature, then $\bar{g}$ is Einstein.

NB: The metric here takes the form $\bar{g} = \sigma^{-2} g$ where σ is a smooth **nowhere vanishing function**, and the condition that it has constant scalar curvature is that the scale tractor satisfies $I_\sigma^2 = constant$ (where recall $I = \bar{D}\sigma$). Thus related to the Obata Theorem there is a very nice more general question:

Question

Let (S^n, g) be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, possibly with $\mathcal{Z}(\sigma)$ non-empty, such that $I_\sigma^2 = constant$. Then is $\bar{g} = \sigma^{-2} g$ necessarily Einstein on $S^n \setminus \mathcal{Z}(\sigma)$?

The Willmore link

Above question is especially interesting if $I^2 = const. > 0$. Then as a special case of the earlier general result (Corollary):

Corollary

Let (S^n, g) be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, with $\mathcal{Z}(\sigma)$ non-empty, such that $I^2_\sigma = const. > 0$. Then $\mathcal{Z}(\sigma)$ is a (higher) Willmore hypersurface.

Above we saw that $\exists$ solutions to $I^2_\sigma = const. \neq 0$ with $\mathcal{Z}(\sigma)$ totally umbilic. If $\bar{g} = \sigma^{-2} g$ is Einstein then $\mathcal{Z}(\sigma)$ umbilic is forced. Hence weaker question:

Question

Let (S^n, g) be the usual round sphere. Let $\sigma \in C^\infty(S^n)$, possibly with $\mathcal{Z}(\sigma)$ non-empty, such that $I^2_\sigma = constant > 0$. Then is $\mathcal{Z}(\sigma)$ necessarily totally umbilic?

THANK YOU FOR LISTENING

Abstract

Higher Willmore energies, Q-curvatures, and related global geometry problems.
The Willmore energy and its functional gradient (under variations of embedding) have recently been the subject of recent interest in both geometric analysis and physics, in part because of their link to conformal geometry. Considering a singular Yamabe problem on manifolds with boundary shows that these these surface invariants are the lowest dimensional examples in a family of conformal invariants for hypersurfaces in any dimension. The same construction and variational considerations shows that (on even dimensional hypersurfaces) the higher Willmore energy and its functional gradient are analogues of the integral of the celebrated Q-curvature conformal invariant and its function gradient (now with respect to metric variations) which is known as the Fefferman-Graham obstruction tensor (or the Bach tensor in dimension 4). In fact the link is deeper than this in that the Willmore energy we consider is an integral of an invariant that