Moment maps and non-reductive geometric invariant theory

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Recall **symplectic reduction** (Marsden–Weinstein 1974) and **symplectic implosion** (Guillemin–Jeffrey–Sjamaar 2002).

(X, ω) compact symplectic manifold
K compact Lie group with Lie algebra $\mathfrak{k}$ acting on (X, ω)
T maximal torus of K, Lie algebra $\mathfrak{t} \subseteq \mathfrak{k}$
$\mathfrak{t}^*_+$ = positive Weyl chamber
$K_\zeta = \{k \in K | (Ad^* k)\zeta = \zeta\}$, when $\zeta \in \mathfrak{t}^*_+$, with commutator subgroup $[K_\zeta, K_\zeta]$.

$\mu : X \to \mathfrak{k}^*$ **moment map** satisfies

$$d\mu_x(\xi).a = \omega_x(\xi, a_x) \qquad \forall x \in X, \xi \in T_x X, a \in \mathfrak{k}$$

and μ is K-equivariant (for the coadjoint action on $\mathfrak{k}^*$).

Special case: (X, ω) is **Kähler** and K acts holomorphically; the action extends to $G = K_{\mathbb{C}}$ = complexification of K.

Let $\zeta \in \mathfrak{k}^*$ be a regular value of $\mu : X \to \mathfrak{k}^*$, with stabiliser K_ζ for the coadjoint action. Then the

Marsden-Weinstein reduction at ζ

$$\mu^{-1}(\zeta)/K_\zeta$$

is a symplectic orbifold. Often we take $\zeta = 0$ to get the **'symplectic quotient'** $X/\!/K = \mu^{-1}(0)/K$.

$\mu^{-1}(0)/K$ has a stratified symplectic structure with more serious singularities when 0 is not a regular value of μ.

The **symplectic implosion** is also stratified symplectic, given by

$$X_{impl} = \mu^{-1}(\mathfrak{t}^*_+)/\sim$$

where $x \sim y \Leftrightarrow x = ky$ for some $k \in [K_\zeta, K_\zeta]$ with

$$\zeta = \mu(x) = \mu(y) \in \mathfrak{t}^*_+.$$

Example: $K = SU(2)$ so $\mathfrak{t}^*_+ = [0,\infty) = \{0\} \sqcup (0,\infty)$, and

$$X_{impl} = \frac{\mu^{-1}(0)}{SU(2)} \sqcup \mu^{-1}((0,\infty))$$

X_{impl} inherits a **symplectic structure and T-action** with moment map $X_{impl} \to \mathfrak{t}^*_+ \subseteq \mathfrak{t}^*$ induced by the restriction of μ.

Kähler case (Heinzner–Huckleberry–Loose, Sjamaar 1990s): $X/\!/K$, and hence X_{impl}, inherits a stratified Kähler structure, via $\mathrm{grad}\mu(x).a = i\, a_x\ (\forall a \in \mathfrak{k}) \rightsquigarrow \mu^{-1}(0)_{\mathrm{reg}}/K \cong$ (open subset of X)$/G$.

$(T^*K)_{impl}$ '**universal imploded cross-section**' is an affine algebraic variety over $\mathbb{C}$, embedded as the closure of a $K_{\mathbb{C}}$-orbit in a representation of K. In general

$$X_{impl} \cong (X \times (T^*K)_{impl})/\!/K$$

which is an algebraic variety if X is algebraic.

In the Kähler case $x \in X$ is semistable for $G = K_{\mathbb{C}}$ iff $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$, and $x \in X$ is stable iff $Gx \cap \mu^{-1}(0)_{\mathrm{reg}} \neq \emptyset$, defining open subsets $X^s \subseteq X^{ss}$ of X. If $x, y \in X^{ss}$ then $x \sim y$ iff $\overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$. The Kähler quotient is $X /\!\!/\!\!/ G = X^{ss}/\sim$.

The inclusion $\mu^{-1}(0) \to X^{ss}$ composed with the quotient map $X^{ss} \to X /\!\!/\!\!/ G$ is K-invariant and induces a bijection

$$X/\!/K = \mu^{-1}(0)/K \to X /\!\!/\!\!/ G$$

to the Kähler quotient, restricting on open subsets to

$$\mu^{-1}(0)_{\mathrm{reg}}/K \cong X^s/G \hookrightarrow X /\!\!/\!\!/ G \cong \mu^{-1}(0)/K.$$

E.g. Let $G = K_{\mathbb{C}}$ act linearly on a smooth projective variety $X \subseteq \mathbb{P}^n$ via $\rho : G \to GL(n+1)$. Assume $\rho(K) \subseteq U(n+1)$ so K preserves the Fubini-Study Kähler form on X. Then a moment map $\mu : X \to \mathfrak{k}^*$ is given by $\mu([x]).a = \dfrac{\bar{x}^T \rho_*(a) x}{2\pi i \|x\|^2} \in \mathbb{R}$ for $a \in \mathfrak{k}$.

Example: $X = (\mathbb{P}^1)^4$ where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = S^2 \subseteq \mathbb{R}^3$.
$K = SU(2)$ acting on X via rotations of S^2
$G = K_{\mathbb{C}} = SL(2;\mathbb{C})$ Möbius transformations

stability: $(y_1, y_2, y_3, y_4) \in X^s$ iff y_1, y_2, y_3, y_4 are distinct points in $\mathbb{P}^1$, with $X^s/G \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ via the cross ratio.

semistability: $(y_1, y_2, y_3, y_4) \in X^{ss}$ iff at most two of y_1, y_2, y_3, y_4 coincide in $\mathbb{P}^1$, with $X /\!/ G \cong \mathbb{P}^1$.

moment map $\mu : X \to \mathfrak{k}^* \cong \mathbb{R}^3$ is given by

$$\mu(y_1, y_2, y_3, y_4) = y_1 + y_2 + y_3 + y_4.$$

In this example $X /\!/ K = \mu^{-1}(0)/K$ is represented by *balanced configurations* of points on S^2, and the symplectic implosion $X_{impl} = \mu^{-1}(0)/K \sqcup \mu^{-1}((0,\infty))$ is its union with the configurations whose *centre of gravity lies on the positive x-axis*.

Link with alg geom/GIT (geometric invariant theory):
(Mumford, 1960s)
G cx **reductive** group, so $G = K_{\mathbb{C}}$ for maximal compact $K \leqslant G$;
X complex projective variety acted on by G.

We require a **linearisation** of the action (i.e. an ample line bundle L on X and a lift of the action to L; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho : G \to GL(n+1)$).

$$\begin{array}{ccccl} X & \rightsquigarrow & A(X) & = & \mathbb{C}[x_0,\ldots,x_n]/\mathcal{I}_X \\ | & & & & = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ | & & \cup| & & \\ \downarrow & & & & \\ X/\!/G & \Leftarrow & A(X)^G & & \text{algebra of invariants} \end{array}$$

G reductive implies that $A(X)^G$ is a *finitely generated* graded complex algebra so that $X/\!/G = \text{Proj}(A(X)^G)$ is a projective variety.

The rational map $X - - \to X /\!\!/ G$ fits into a diagram

$$\begin{array}{rccccl} & X & --\to & X/\!\!/G & \text{cx proj variety} \\ & \cup & & \| & \\ \text{semistable} & X^{ss} & \overset{\text{onto}}{\longrightarrow} & X/\!\!/G & \\ & \cup & & \cup & \text{open} \\ \text{stable} & X^{s} & \longrightarrow & X^{s}/G & \end{array}$$

where the morphism $X^{ss} \to X /\!\!/ G$ is G-invariant and surjective.

Topologically $\boxed{X /\!\!/ G = X^{ss}/\sim}$ where $x \sim y \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$, $x \in X^{ss}$ iff $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$, $x \in X^{s}$ iff $Gx \cap \mu^{-1}(0)_{\text{reg}} \neq \emptyset$ and

$$X /\!\!/ G = \mu^{-1}(0)/K$$

for a suitable moment map μ for the action of K.

Symplectic implosion links with (a very special case of) **non-reductive GIT**:

$B = T_{\mathbb{C}} U_{\text{max}}$ Borel subgroup (maximal soluble subgp) of $G = K_{\mathbb{C}}$ such that $G = KB$ and $K \cap B = T$.

U_{max} maximal unipotent subgroup of G normalised by T.

Fact: $K_{\mathbb{C}}/U_{\text{max}}$ is a quasi-affine variety whose algebra of regular functions $\mathcal{O}(K_{\mathbb{C}}/U_{\text{max}}) = \mathcal{O}(K_{\mathbb{C}})^{U_{\text{max}}}$ is finitely generated, so that $K_{\mathbb{C}}/U_{\text{max}}$ has a canonical affine completion

$$K_{\mathbb{C}} /\!\!/ U_{\text{max}} = Spec(\mathcal{O}(K_{\mathbb{C}})^{U_{\text{max}}}).$$

Thm (GJS): $K_{\mathbb{C}} /\!\!/ U_{\text{max}}$ has a K-invariant Kähler structure which is symplectically iso to the universal implosion $(T^*K)_{impl}$.

Cor: X affine or projective variety acted on linearly by $K_{\mathbb{C}} \Rightarrow$

$$X_{impl} \cong (X \times (K_{\mathbb{C}} /\!\!/ U_{\text{max}})) /\!\!/ K_{\mathbb{C}} \cong X /\!\!/ U_{\text{max}}.$$

There is a generalisation X_{implP} replacing U_{max} with the unipotent radical U_P of any parabolic subgroup P of $G = K_{\mathbb{C}}$.

What happens more generally with GIT for a non-reductive linear algebraic group H over $\mathbb{C}$?
Problem: We can't define a projective variety

$$X /\!\!/ H = \mathrm{Proj}(A(X)^H)$$

because $A(X)^H$ is not necessarily finitely generated.

Question: Can we define a sensible 'quotient' variety $X/\!\!/H$ when H is not reductive? If so, can we understand it geometrically? Using moment maps?

Partial answer: We can define open subsets X^s ('stable points') and X^{ss} ('semistable points') with a geometric quotient $X^s \to X^s/H$ and an 'enveloping quotient' $X^{ss} \to X /\!\!/ H$. BUT $X /\!\!/ H$ is **not necessarily projective** and $X^{ss} \to X /\!\!/ H$ is **not necessarily onto**. Also the Hilbert–Mumford criteria for (semi)stability do not generalise, at least not in an obvious way.

X projective variety with linear action of linear alg group H;
H has unipotent radical $U \trianglelefteq H$ with $R = H/U$ reductive.

We can try to study $X /\!\!/ H$ using a 'reductive envelope':
we look for a reductive G and $\phi : H \to G$ whose restriction to U is injective. Then $\exists$ an induced homomorphism $H \to G \times R$ and $(G \times R)$-action on the quasi-projective variety $G \times_U X = (G \times X)/U$.
We try to find a projective completion

$$\overline{G \times_U X}$$

with a $G \times R$-linearisation restricting to the given linearisation on X, such that $X /\!\!/ H$ can be identified with an open subset of the reductive GIT quotient

$$\overline{G \times_U X} /\!\!/ (G \times R).$$

Simple example: $U = \mathbb{C}^+$ and $\hat{U} = \mathbb{C}^+ \rtimes \mathbb{C}^*$ acting on $\mathbb{P}^n$.
$\exists$ coordinates s.t. $\mathbb{C}^*$ acts diagonally and the generator of $Lie(\mathbb{C}^+)$ has Jordan normal form with blocks of size $k_1 + 1, \ldots, k_q + 1$. So the linear $\mathbb{C}^+$ action extends to $G = SL(2)$, where

$$\mathbb{C}^+ = \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C} \} \leqslant G,$$

via $\mathbb{C}^{n+1} \cong \bigoplus_{i=1}^q Sym^{k_i}(\mathbb{C}^2)$, and the action of (a cover of) the $\mathbb{C}^+ \rtimes \mathbb{C}^*$ action extends to $GL(2)$. In this case the U-invariants are finitely generated (Weitzenböck's theorem) so we can define

$$\mathbb{P}^n /\!/ \mathbb{C}^+ = \text{Proj}((\mathbb{C}[x_0, \ldots, x_n])^{\mathbb{C}^+}).$$

Note: $G \times_{\mathbb{C}^+} \mathbb{P}^n \cong (G/\mathbb{C}^+) \times \mathbb{P}^n \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}^n \subseteq \mathbb{C}^2 \times \mathbb{P}^n \subseteq \mathbb{P}^2 \times \mathbb{P}^n$ via $(g, x) \mapsto (g\mathbb{C}^+, gx)$; the $\mathbb{C}^+$-invariants on $\mathbb{P}^n$ extend, and

$$\boxed{\mathbb{P}^n /\!/ \mathbb{C}^+ \cong (\mathbb{P}^2 \times \mathbb{P}^n) /\!/ SL(2) \cong (\mathbb{P}^n)_{impl}.}$$

Example when $(\mathbb{P}^n)^{ss} \to \mathbb{P}^n /\!\!/ \mathbb{C}^+$ is *not* onto:

$$\mathbb{P}^3 = \mathbb{P}(Sym^3(\mathbb{C}^2)) = \{ \text{ 3 unordered points on } \mathbb{P}^1\}.$$

Then $(\mathbb{P}^3)^{ss} = (\mathbb{P}^3)^s = \{$ 3 points on $\mathbb{P}^1$ with at most one at $\infty\}$ and its image in $\mathbb{P}^3 /\!\!/ \mathbb{C}^+ = (\mathbb{P}^3)^s/\mathbb{C}^+ \sqcup \mathbb{P}^3 /\!\!/ SL(2)$ is the open subset $(\mathbb{P}^3)^s/\mathbb{C}^+$ which does not include the 'boundary' point coming from $0 \in \mathbb{C}^2$.

If we quotient *not* just by $U = \mathbb{C}^+$ but by $\hat{U} = \mathbb{C}^+ \rtimes \mathbb{C}^*$, where $\mathbb{C}^*$ acts non-trivially on U, then we can modify the linearisation by multiplying by a rational character of $\hat{U}$. For some such choices of linearisation the 'boundary' point in the quotient by $\mathbb{C}^+$ coming from $0 \in \mathbb{C}^2$ becomes unstable for the induced action on $\mathbb{C}^*$, so we do get a surjective morphism

$$(\mathbb{P}^3)^{ss,\hat{U}} \xrightarrow{\text{onto}} \mathbb{P}^3 /\!\!/ \hat{U}.$$

Defn: Call a unipotent linear alg group U **graded unipotent** if $\exists\ \lambda : \mathbb{C}^* \to Aut(U)$ with all weights of the $\mathbb{C}^*$ action on $Lie(U)$ **strictly positive**. Then let $\hat{U} = U \rtimes \mathbb{C}^*$ be the induced semi-direct product.

Suppose that $\hat{U}$ acts linearly (with respect to an ample line bundle L) on a projective variety X. We can multiply the $\hat{U}$-linearisation by any character (or any rational character, after replacing L with $L^{\otimes m}$ for sufficiently divisible positive m), without changing the action. If we are willing to twist by an appropriate rational character, then GIT for the $\hat{U}$ action is nearly as well behaved as in the classical case for reductive groups.

Any linear algebraic group H over $\mathbb{C}$ is $U \rtimes R$ where $U \trianglelefteq H$ is its unipotent radical and $R \cong H/U$ is reductive. We say H **has internally graded unipotent radical** if R has a central one-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$ which grades U.

Thm: (Berczi, Doran, Hawes, K) Let U be graded unipotent acting linearly on a projective variety X, and suppose that the action extends to $\hat{U} = U \rtimes \mathbb{C}^*$. Suppose also that

$$(*) \quad x \in Z_{\text{min}} \Rightarrow \dim \text{Stab}_U(x) = 0$$

where Z_{min} is the union of connected components of $X^{\mathbb{C}^*}$ where $\mathbb{C}^*$ acts on the fibres of L with minimum weight. We can twist the action of $\hat{U}$ by a (rational) character so that 0 lies just above the minimum weight for the $\mathbb{C}^*$ action on X, and
(i) the ring $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is **finitely generated**, so that $X/\!/\hat{U} = \text{Proj}(A(X)^{\hat{U}})$ is **projective**;
(ii) $X/\!/\hat{U}$ is a **geometric quotient** of $X^{ss,\hat{U}} = X^{s,\hat{U}}$ by $\hat{U}$ and $X^{ss,\hat{U}}$ has a **Hilbert–Mumford** description.
Moreover, even without condition (*) there is a projective completion of $X^{s,\hat{U}}/\hat{U}$ which is a geometric quotient by $\hat{U}$ of an open subset $\tilde{X}^{ss}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X.

Examples of non-reductive groups H with internally graded unipotent radicals:

i) $H = \text{Aut}(Y)$ where Y is a complete toric variety;

ii) H a parabolic subgroup of a reductive group G;

iii) $H = \{k\text{-jets of germs of biholomorphisms of } (\mathbb{C},0)\}$

$$\cong \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ 0 & (a_1)^2 & \dots & p_{2k}(\underline{a}) \\ & & \dots & \\ 0 & 0 & \dots & (a_1)^k \end{pmatrix} : a_1 \in \mathbb{C}^*, a_2, \dots a_k \in \mathbb{C} \right\}$$

(and similarly when we replace $(\mathbb{C},0)$ with $(\mathbb{C}^m,0)$).

If H acts linearly on a projective variety X, and the linearisation is twisted by a suitable rational character of H and (*) holds, then this theorem applies to $X /\!/ H = (X /\!/ \hat{U}) /\!/ (R/\mathbb{C}^*)$,
which is $\text{Proj}(A(X)^H) = X^{ss}/\sim$ where the algebra of invariants $A(X)^H = (A(X)^{\hat{U}})^{R/\mathbb{C}^*}$ is finitely generated and $x \sim y$ as before.

When G reductive acts linearly on a projective variety X, $\exists$ a stratification ($=$ Morse stratification for $\|$moment map$\|^2$)

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

indexed by a finite subset $\mathcal{B}$ of a $+$ve Weyl chamber, with
(i) $S_0 = X^{ss}$, and for each $\beta \in \mathcal{B}$
(ii) $S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$ where P_β is a parabolic subgroup of G and Y_β^{ss} is an open subset of a projective subvariety $\overline{Y}_\beta$ of X.

$P_\beta = U_\beta \rtimes L_\beta$, where its unipotent radical U_β is graded by a central 1-parameter subgroup of its Levi subgroup L_β. To construct a quotient of (an open subset of) S_β by G we can study the linear action on $\overline{Y}_\beta$ of the parabolic subgroup P_β, appropriately twisted, and quotient first by $\hat{U}_\beta$ and then by the residual action of the reductive group $P_\beta/\hat{U}_\beta = L_\beta/\mathbb{C}^*$. We can use this to stratify moduli stacks and construct moduli spaces of unstable objects.

$H = U \rtimes R$, internally graded unipotent radical U, $R = K_{\mathbb{C}}$
$H \circlearrowright X \subseteq \mathbb{P}^n$ via $\rho : H \to GL(n+1)$ with $\rho(K) \subseteq U(n+1)$

Define $\mu_H : X \to \mathfrak{h}^*$ by $\mu_H([x]).a = \bar{x}^T \rho_*(a) x / \|x\|^2 \in \mathbb{C}$ for $a \in \mathfrak{h}$.

$X /\!\!/\!\!/ H$ = GIT quotient for appropriately twisted linearisation (after blowing up if need be).

When $H = R$ is reductive $X /\!\!/\!\!/ H \cong X/\!/K = \mu_H^{-1}(0)/K$.
Applications: Betti numbers, intersection pairings on $X /\!\!/\!\!/ H$...

When $H = P$ is a parabolic in a reductive G **and** the action of H extends to G, then $X /\!\!/\!\!/ H \cong (X /\!\!/\!\!/ U) /\!\!/\!\!/ R \cong X_{implP}/\!/K$. After twisting the linearisation by a suitable character of H (or equivalently adding a suitable central constant to μ_H), if $(*)$ holds we have

$$X /\!\!/\!\!/ H \cong \mu_H^{-1}(0)/K.$$

Back to simple example:

$\mathbb{P}^n /\!\!/ \mathbb{C}^+ \cong \mu_{SU(2)}^{-1}(\mathfrak{t}_+^*)$ / collapsing on the boundary $\mu_{SU(2)}^{-1}(0)$;

$\mathbb{P}^n /\!\!/ (\mathbb{C}^+ \rtimes \mathbb{C}^*) \cong \mu_{SU(2)}^{-1}(\mathfrak{t}_+^*) \cap \mu_{S^1}^{-1}(\xi)$ / (S^1 and collapsing).

Suppose $\mu_{SU(2)}^{-1}(\mathfrak{t}) \cap \mu_{S^1}^{-1}(\xi) \subseteq \mu_{SU(2)}^{-1}((\mathfrak{t}_+^*)^o) \cap \mu_{S^1}(\xi)$. Then

$$\mathbb{P}^n /\!\!/ (\mathbb{C}^+ \rtimes \mathbb{C}^*) \cong \mu_{SU(2)}^{-1}(\mathfrak{t}) \cap \mu_{S^1}^{-1}(\xi)/S^1 = \mu_{\mathfrak{t}^\perp}^{-1}(0) \cap \mu_{S^1}^{-1}(\xi)/S^1$$

where $\mu_{\mathfrak{t}^\perp} : \mathbb{P}^n \to \mathfrak{t}^\perp \cong \mathrm{Lie}\mathbb{C}^+$ is projection of $\mu_{SU(2)}$ onto $\mathfrak{t}^\perp$. So

$$\mathbb{P}^n /\!\!/ (\mathbb{C}^+ \rtimes \mathbb{C}^*) \cong \mu^{-1}(0)/S^1$$

where $\mu = (\mu_{\mathfrak{t}^\perp}, \mu_{S^1} - \xi) : \mathbb{P}^n \to \mathfrak{t}^\perp \times (\mathrm{Lie}S^1)^*$.

Is there a similar description of $X /\!\!/ H$ more generally?

Hope: Given (*) AND after twisting by a suitable character (add a suitable central constant to μ_H), then for a suitable projective embedding of X:
$\mu_{\widehat{U}}^{-1}(0)$ is a slice for the action of $U \rtimes \mathbb{R}^*$ on the open subset

$$X^{s,\widehat{U}} = \widehat{U}\mu_{\widehat{U}}^{-1}(0) \cong \widehat{U} \times_{S^1} \mu_{\widehat{U}}^{-1}(0)$$

of X, so that $\mu_{\widehat{U}}^{-1}(0)/S^1 \cong X^{s,\widehat{U}}/\widehat{U} = X /\!\!/ \widehat{U}$ and

$$\mu_H^{-1}(0)/K \cong (X /\!\!/ \widehat{U}) /\!\!/ (R/\mathbb{C}^*) \cong X /\!\!/ H.$$

Applications: calculating Betti numbers, generators for the cohomology ring and intersection pairings on

$$X /\!\!/ H = \mu_H^{-1}(0)/K,$$

via $\|\mu_H\|^2$ as an equivariantly perfect Morse function and Shaun Martin's approach to intersection pairings by reducing to torus quotients.

Kähler picture following Greb–Miebach (2018):
unipotent $U \leqslant G = K_{\mathbb{C}}$ simply-connected semisimple;
U acting holomorphically on (X, ω) compact Kähler.
Questions: (a) analogue of 'linear action'?
(b) analogue of 'reductive envelope' $\overline{G \times_U X}$?
(c) use of moment maps for K-action to construct and study quotients for U-action?
(d) constraints on ω to allow it to be extended to a K-invariant Kähler form on $\overline{G \times_U X}$?
(e) link with non-reductive GIT?

Thm (Greb–Miebach) TFAE: (1) $G \times_U X$ is Kähler;
(2) the U-action on X is 'meromorphic' (i.e. extends to meromorphic $\overline{U} \times X \dashrightarrow X$ for a suitable compactification $\overline{U}$);
(3) $\exists$ 'Hamiltonian G-extension': Z compact Kähler with Hamiltonian K-action, U-equivariant embedding $X \hookrightarrow Z$, $[\omega_Z|_X] = [\omega]$.

Then $X \hookrightarrow G \times_U X \hookrightarrow G \times_Z \cong G/U \times Z \hookrightarrow V \times Z$ when G/U is embedded as a G-orbit in a representation V of G with flat K-invariant Kähler structure, and we can define

$$X^{ss,U}[\omega] = X \cap \{y \in G/U \times Z : \mu^{-1}(0) \cap \overline{Gy} \neq \emptyset\}$$

where $\mu = \mu_Z + \mu_V$ for moment maps $\mu_Z : Z \to \mathfrak{k}^*$ and $\mu_V : V \to \mathfrak{k}^*$.

Thm (Greb–Miebach) (i) $X^{ss,U}[\omega]$ is independent of the choice of the Hamiltonian G-extension Z (for fixed $G = K_{\mathbb{C}}$), but can depend on G and the Kähler metric on G/U;
(ii) $\exists$ geometric quotient $\pi : X^{ss,U}[\omega] \to X^{ss,U}[\omega]/U = Q$ smooth, $Q \subseteq \overline{Q}$ compact cx space, $\overline{Q} \setminus Q$ analytic, π extends to mero $X \dashrightarrow \overline{Q}$;
(iii) $\overline{Q}$ has a stratified Kähler structure restricting to a smooth Kähler form ω_Q on Q with $[\pi^* \omega_Q] = [\omega]$.

$H = U \rtimes R$ with unipotent radical U graded by $\lambda : \mathbb{C}^* \to Z(R)$.
$\hat{U} = U \rtimes \lambda(\mathbb{C}^*) \trianglelefteq H$
Adjoint action $\phi : H \to GL((\text{Lie}(\hat{U}))$ restricts to an injection
$\phi|_U : U \to SL((\text{Lie}(\hat{U})) \cong SL(d+1) = G$ where $d = \dim(U)$.
Multiplying ϕ by a character gives $\hat{\phi}$ with $\hat{\phi}|_U = \phi|_U$ and

$$\hat{\phi}(\hat{U}) \cong \{ \begin{pmatrix} a_0 & a_1 & \dots & a_d \\ 0 & (a_0)^{k_1} & \dots & p_{1d}(\underline{a}) \\ & & \dots & \\ 0 & 0 & \dots & (a_0)^{k_d} \end{pmatrix} : a_0 \in \mathbb{C}^*, a_1, \dots a_d \in \mathbb{C} \}$$

where $k_j > 1$ for $j = 1, \dots, d$ and the entries $p_{ij}(a_0, \dots, a_d)$ above the diagonal are polynomials in $a_0, \dots a_d$, homogeneous of degree i and weighted homogeneous of degree k_j.

We can use this to construct reductive envelopes/Hamiltonian G-extensions.

Lemma (Bérczi–K 2017; compare with the universal symplectic implosion's embedding in an affine space with flat Kähler metric)

$GL(d+1)/\widehat{\phi}(\widehat{U}) = (SL(d+1)/U)/(\text{finite group})$ is embedded (with good control over its boundary) in an open affine subset of

$$\mathbb{P}(V) = \mathbb{P}\Big(\bigoplus_{j=1}^{d+1} \wedge^j \Big(\bigoplus_{i=0}^{d} \mathrm{Sym}^{k_i}\mathbb{C}^{d+1}\Big)\Big)$$

as the $GL(d+1)$-orbit of $[\mathfrak{p}]$ given by

$$\mathfrak{p} = \sum_{j=0}^{d} e_0 \wedge (e_1 + (e_0)^{k_1}) \wedge \ldots \wedge (e_j + \sum_{i=1}^{j-1} p_{ij}(e_0, \ldots, e_d) + (e_0)^{k_j}) \in V$$

where $e_0, \ldots, e_d$ is the standard basis for $\mathbb{C}^{d+1}$.

Use this embedding and a large positive scalar multiple of the flat Kähler metric on V as input for the Greb–Miebach construction to realise the hope of a 'moment map' description of $X /\!\!/ H$.