Moment maps and non-reductive geometric invariant theory

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(based on joint work with G Bérczi, B Doran, V Hoskins)
Recall **symplectic reduction** (Marsden–Weinstein 1974) and **symplectic implosion** (Guillemin–Jeffrey–Sjamaar 2002).

\((X, \omega)\) compact symplectic manifold
\(K\) compact Lie group with Lie algebra \(\mathfrak{k}\) acting on \((X, \omega)\)
\(T\) maximal torus of \(K\), Lie algebra \(\mathfrak{t} \subseteq \mathfrak{k}\)
\(\mathfrak{t}^*_+ = \) positive Weyl chamber
\(K_\zeta = \{ k \in K | (Ad^* k) \zeta = \zeta \}\), when \(\zeta \in \mathfrak{t}^*_+\), with commutator subgroup \([K_\zeta, K_\zeta]\).

\(\mu : X \to \mathfrak{t}^*\) **moment map** satisfies
\[
d\mu_x(\xi).a = \omega_x(\xi, ax) \quad \forall x \in X, \xi \in T_xX, a \in \mathfrak{k}
\]
and \(\mu\) is \(K\)-equivariant (for the coadjoint action on \(\mathfrak{t}^*\)).

**Special case:** \((X, \omega)\) is Kähler and \(K\) acts holomorphically; the action extends to \(G = K_\mathbb{C}\) = complexification of \(K\).
Let $\zeta \in \mathfrak{k}^*$ be a regular value of $\mu : X \to \mathfrak{k}^*$, with stabiliser $K_\zeta$ for the coadjoint action. Then the

**Marsden-Weinstein reduction at $\zeta$**

$$\mu^{-1}(\zeta)/K_\zeta$$

is a symplectic orbifold. Often we take $\zeta = 0$ to get the ‘symplectic quotient’ $X//K = \mu^{-1}(0)/K$.

$\mu^{-1}(0)/K$ has a stratified symplectic structure with more serious singularities when 0 is not a regular value of $\mu$.

The **symplectic implosion** is also stratified symplectic, given by

$$X_{impl} = \mu^{-1}(t_+^*)/\sim$$

where $x \sim y \iff x = ky$ for some $k \in [K_\zeta, K_\zeta]$ with

$$\zeta = \mu(x) = \mu(y) \in t_+^*.$$
Example: \( K = SU(2) \) so \( t^*_+ = [0, \infty) = \{0\} \sqcup (0, \infty) \), and

\[
X_{\text{impl}} = \frac{\mu^{-1}(0)}{SU(2)} \sqcup \mu^{-1}((0, \infty))
\]

\( X_{\text{impl}} \) inherits a **symplectic structure and \( T \)-action** with moment map \( X_{\text{impl}} \to t^*_+ \subseteq t^* \) induced by the restriction of \( \mu \).

**Kähler case** (Heinzner–Huckleberry–Loose, Sjamaar 1990s): \( X//K \), and hence \( X_{\text{impl}} \), inherits a **stratified Kähler structure**, via

\[
\text{grad}_\mu(x).a = i a_x (\forall a \in \mathfrak{k}) \sim \mu^{-1}(0)_{\text{reg}}/K \cong (\text{open subset of } X)/G.
\]

\((T^*K)_{\text{impl}} \) ‘**universal imploded cross-section**’ is an affine algebraic variety over \( \mathbb{C} \), embedded as the closure of a \( K_{\mathbb{C}} \)-orbit in a representation of \( K \). In general

\[
X_{\text{impl}} \cong (X \times (T^*K)_{\text{impl}})//K
\]

which is an algebraic variety if \( X \) is algebraic.
In the Kähler case $x \in X$ is semistable for $G = K_\mathbb{C}$ iff $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$, and $x \in X$ is stable iff $Gx \cap \mu^{-1}(0)_{\text{reg}} \neq \emptyset$, defining open subsets $X^s \subseteq X^{ss}$ of $X$. If $x, y \in X^{ss}$ then $x \sim y$ iff $\overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$.

The Kähler quotient is $X \bowtie G = X^{ss}/\sim$.

The inclusion $\mu^{-1}(0) \to X^{ss}$ composed with the quotient map $X^{ss} \to X \bowtie G$ is $K$-invariant and induces a bijection

$$X//K = \mu^{-1}(0)/K \to X \bowtie G$$

to the Kähler quotient, restricting on open subsets to

$$\mu^{-1}(0)_{\text{reg}}/K \cong X^s/G \to X \bowtie G \cong \mu^{-1}(0)/K.$$  

**E.g.** Let $G = K_\mathbb{C}$ act linearly on a smooth projective variety $X \subseteq \mathbb{P}^n$ via $\rho: G \to GL(n+1)$. Assume $\rho(K) \subseteq U(n+1)$ so $K$ preserves the Fubini-Study Kähler form on $X$. Then a moment map $\mu: X \to \mathfrak{t}^*$ is given by $\mu([x]).a = \frac{\bar{x}^T \rho_*(a)x}{2\pi i |x|^2} \in \mathbb{R}$ for $a \in \mathfrak{t}$. 
**Example:** \( X = (\mathbb{P}^1)^4 \) where \( \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} = S^2 \subseteq \mathbb{R}^3 \).

\( K = SU(2) \) acting on \( X \) via rotations of \( S^2 \).

\( G = K_\mathbb{C} = SL(2; \mathbb{C}) \) Möbius transformations.

**stability:** \( (y_1, y_2, y_3, y_4) \in X^s \) iff \( y_1, y_2, y_3, y_4 \) are distinct points in \( \mathbb{P}^1 \), with \( X^s/G \cong \mathbb{P}^1 \setminus \{0, 1, \infty\} \) via the cross ratio.

**semistability:** \( (y_1, y_2, y_3, y_4) \in X^{ss} \) iff at most two of \( y_1, y_2, y_3, y_4 \) coincide in \( \mathbb{P}^1 \), with \( X \bowtie G \cong \mathbb{P}^1 \).

**moment map** \( \mu : X \to \mathfrak{k}^* \cong \mathbb{R}^3 \) is given by

\[
\mu(y_1, y_2, y_3, y_4) = y_1 + y_2 + y_3 + y_4.
\]

In this example \( X//K = \mu^{-1}(0)/K \) is represented by balanced configurations of points on \( S^2 \), and the symplectic implosion \( X_{impl} = \mu^{-1}(0)/K \sqcup \mu^{-1}((0, \infty)) \) is its union with the configurations whose centre of gravity lies on the positive \( x \)-axis.
Link with alg geom/GIT (geometric invariant theory): (Mumford, 1960s)

$G$ cx reductive group, so $G = K_C$ for maximal compact $K \leq G$; $X$ complex projective variety acted on by $G$.

We require a linearisation of the action (i.e. an ample line bundle $L$ on $X$ and a lift of the action to $L$; think of $X \subseteq \mathbb{P}^n$ and the action given by a representation $\rho: G \to GL(n+1)$).

$$
\begin{align*}
X &\twoheadrightarrow A(X) = \mathbb{C}[x_0, \ldots, x_n]/\mathcal{I}_X \\
&= \bigoplus_{k=0}^{\infty} H^0(X, L^\otimes k)
\end{align*}
$$

$X//G = A(X)^G$ algebra of invariants

$G$ reductive implies that $A(X)^G$ is a finitely generated graded complex algebra so that $X \bowtie G = \text{Proj}(A(X)^G)$ is a projective variety.
The rational map $X \dashrightarrow X \bowtie G$ fits into a diagram

\[
\begin{array}{ccc}
X & \dashrightarrow & X \bowtie G \\
\downarrow & & \downarrow \\
\text{semistable} & \text{onto} & X \bowtie G \\
\downarrow & & \downarrow \\
\text{stable} & \rightarrow & X^s/G
\end{array}
\]

where the morphism $X^{ss} \rightarrow X \bowtie G$ is $G$-invariant and surjective.

Topologically $\boxed{X \bowtie G = X^{ss} / \sim}$ where $x \sim y \iff \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$, $x \in X^{ss}$ iff $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$, $x \in X^s$ iff $Gx \cap \mu^{-1}(0)_{\text{reg}} \neq \emptyset$ and

$$X \bowtie G = \mu^{-1}(0)/K$$

for a suitable moment map $\mu$ for the action of $K$. 
**Symplectic implosion** links with (a very special case of) 
**non-reductive GIT**: 

\[ B = T \subset U_{\text{max}} \]  

Borel subgroup (maximal soluble subgp) of \( G = K \subset C \) 

such that \( G = KB \) and \( K \cap B = T \). 

\( U_{\text{max}} \) maximal unipotent subgroup of \( G \) normalised by \( T \). 

**Fact:** \( K \subset C / U_{\text{max}} \) is a quasi-affine variety whose algebra of regular functions \( \mathcal{O}(K \subset C / U_{\text{max}}) = \mathcal{O}(K \subset C)^{U_{\text{max}}} \) is finitely generated, 

so that \( K \subset C / U_{\text{max}} \) has a canonical affine completion  

\[ K \subset C \ltimes U_{\text{max}} = \text{Spec}(\mathcal{O}(K \subset C)^{U_{\text{max}}}). \]

**Thm** (GJS): \( K \subset C \ltimes U_{\text{max}} \) has a \( K \)-invariant Kähler structure which is symplectically iso to the universal implosion \((T^*K)_{\text{impl}}\). 

**Cor:** \( X \) affine or projective variety acted on linearly by \( K \subset C \) \( \Rightarrow \)  

\[ X_{\text{impl}} \cong (X \times (K \subset C \ltimes U_{\text{max}})) \ltimes K \subset C \cong X \ltimes U_{\text{max}}. \]

There is a generalisation \( X_{\text{impl}P} \) replacing \( U_{\text{max}} \) with the unipotent radical \( U_P \) of any parabolic subgroup \( P \) of \( G = K \subset C \).
What happens more generally with GIT for a non-reductive linear algebraic group $H$ over $\mathbb{C}$?

**Problem:** We can’t define a projective variety

$$X \Join H = \text{Proj}(A(X)^H)$$

because $A(X)^H$ is not necessarily finitely generated.

**Question:** Can we define a sensible ‘quotient’ variety $X \Join H$ when $H$ is not reductive? If so, can we understand it geometrically? Using moment maps?

**Partial answer:** We can define open subsets $X^s$ (‘stable points’) and $X^{ss}$ (‘semistable points’) with a geometric quotient $X^s \to X^s/H$ and an ‘enveloping quotient’ $X^{ss} \to X \Join H$. BUT $X \Join H$ is **not necessarily projective** and $X^{ss} \to X \Join H$ is **not necessarily onto**. Also the Hilbert–Mumford criteria for (semi)stability do not generalise, at least not in an obvious way.
$X$ projective variety with linear action of linear alg group $H$; $H$ has unipotent radical $U \trianglelefteq H$ with $R = H/U$ reductive.

We can try to study $X \rtimes H$ using a ‘reductive envelope’: we look for a reductive $G$ and $\phi : H \to G$ whose restriction to $U$ is injective. Then $\exists$ an induced homomorphism $H \to G \times R$ and $(G \times R)$-action on the quasi-projective variety $G \times_U X = (G \times X)/U$. We try to find a projective completion

$$\overline{G \times_U X}$$

with a $G \times R$-linearisation restricting to the given linearisation on $X$, such that $X \rtimes H$ can be identified with an open subset of the reductive GIT quotient

$$\overline{G \times_U X} \rtimes (G \times R).$$
Simple example: \( U = \mathbb{C}^+ \) and \( \tilde{U} = \mathbb{C}^+ \rtimes \mathbb{C}^* \) acting on \( \mathbb{P}^n \).

\exists \) coordinates s.t. \( \mathbb{C}^* \) acts diagonally and the generator of \( Lie(\mathbb{C}^+) \)

has Jordan normal form with blocks of size \( k_1 + 1, \ldots, k_q + 1 \). So the linear \( \mathbb{C}^+ \) action extends to \( G = SL(2) \), where

\[
\mathbb{C}^+ = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{C} \right\} \leq G,
\]

via \( \mathbb{C}^{n+1} \cong \bigoplus_{i=1}^{q} Sym^{k_i}(\mathbb{C}^2) \), and the action of (a cover of) the \( \mathbb{C}^+ \times \mathbb{C}^* \) action extends to \( GL(2) \). In this case the \( U \)-invariants are finitely generated (Weitzenböck’s theorem) so we can define

\[
\mathbb{P}^n \rtimes \mathbb{C}^+ = \text{Proj}((\mathbb{C}[x_0, \ldots, x_n])^{\mathbb{C}^+}).
\]

Note: \( G \times_{\mathbb{C}^+} \mathbb{P}^n \cong (G/\mathbb{C}^+) \times \mathbb{P}^n \cong (\mathbb{C}^2 \setminus \{0\}) \times \mathbb{P}^n \subseteq \mathbb{C}^2 \times \mathbb{P}^n \subseteq \mathbb{P}^2 \times \mathbb{P}^n \)

via \( (g, x) \mapsto (g\mathbb{C}^+, gx) \); the \( \mathbb{C}^+ \)-invariants on \( \mathbb{P}^n \) extend, and

\[
\mathbb{P}^n \rtimes \mathbb{C}^+ \cong (\mathbb{P}^2 \times \mathbb{P}^n) \rtimes SL(2) \cong (\mathbb{P}^n)_{impl}.
\]
Example when \((\mathbb{P}^n)^{ss} \to \mathbb{P}^n \bowtie \mathbb{C}^+\) is not onto:

\[\mathbb{P}^3 = \mathbb{P}(\text{Sym}^3(C^2)) = \{\text{3 unordered points on } \mathbb{P}^1\}\]

Then \((\mathbb{P}^3)^{ss} = (\mathbb{P}^3)^s = \{\text{3 points on } \mathbb{P}^1 \text{ with at most one at } \infty\}\) and its image in \(\mathbb{P}^3 \bowtie \mathbb{C}^+ = (\mathbb{P}^3)^s/\mathbb{C}^+ \sqcup \mathbb{P}^3 \bowtie \text{SL}(2)\) is the open subset \((\mathbb{P}^3)^s/\mathbb{C}^+\) which does not include the ‘boundary’ point coming from \(0 \in C^2\).

If we quotient not just by \(U = \mathbb{C}^+\) but by \(\hat{U} = \mathbb{C}^+ \rtimes \mathbb{C}^*\), where \(\mathbb{C}^*\) acts non-trivially on \(U\), then we can modify the linearisation by multiplying by a rational character of \(\hat{U}\). For some such choices of linearisation the ‘boundary’ point in the quotient by \(\mathbb{C}^+\) coming from \(0 \in C^2\) becomes unstable for the induced action on \(\mathbb{C}^*\), so we do get a surjective morphism

\[(\mathbb{P}^3)^{ss,\hat{U}} \quad \text{onto} \quad \mathbb{P}^3 \bowtie \hat{U}.\]
**Defn:** Call a unipotent linear alg group $U$ **graded unipotent** if $\exists \lambda: \mathbb{C}^* \to Aut(U)$ with all weights of the $\mathbb{C}^*$ action on $Lie(U)$ strictly positive. Then let $\hat{U} = U \rtimes \mathbb{C}^*$ be the induced semi-direct product.

Suppose that $\hat{U}$ acts linearly (with respect to an ample line bundle $L$) on a projective variety $X$. We can multiply the $\hat{U}$-linearisation by any character (or any rational character, after replacing $L$ with $L^\otimes m$ for sufficiently divisible positive positive $m$), without changing the action. If we are willing to twist by an appropriate rational character, then GIT for the $\hat{U}$ action is nearly as well behaved as in the classical case for reductive groups.

Any linear algebraic group $H$ over $\mathbb{C}$ is $U \rtimes R$ where $U \trianglelefteq H$ is its unipotent radical and $R \cong H/U$ is reductive. We say $H$ **has internally graded unipotent radical** if $R$ has a central one-parameter subgroup $\lambda: \mathbb{C}^* \to Z(R)$ which grades $U$. 
**Thm:** (Berczi, Doran, Hawes, K) Let $U$ be graded unipotent acting linearly on a projective variety $X$, and suppose that the action extends to $\hat{U} = U \rtimes \mathbb{C}^*$. Suppose also that

$$ (\ast) \quad x \in Z_{\text{min}} \Rightarrow \dim \text{Stab}_{U}(x) = 0 $$

where $Z_{\text{min}}$ is the union of connected components of $X^{\mathbb{C}^*}$ where $\mathbb{C}^*$ acts on the fibres of $L$ with minimum weight. We can twist the action of $\hat{U}$ by a (rational) character so that 0 lies just above the minimum weight for the $\mathbb{C}^*$ action on $X$, and

(i) the ring $A(X)^{\hat{U}}$ of $\hat{U}$-invariants is **finitely generated**, so that $X//\hat{U} = \text{Proj}(A(X)^{\hat{U}})$ is projective;

(ii) $X^{\geq \hat{U}}$ is a **geometric quotient** of $X_{\text{ss},\hat{U}} = X_{\text{s},\hat{U}}$ by $\hat{U}$ and $X_{\text{ss},\hat{U}}$ has a **Hilbert–Mumford** description.

Moreover, even without condition $(\ast)$ there is a projective completion of $X^{s,\hat{U}}/\hat{U}$ which is a geometric quotient by $\hat{U}$ of an open subset $\tilde{X}^{ss}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of $X$. 

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Examples of non-reductive groups $H$ with internally graded unipotent radicals:

i) $H = \text{Aut}(Y)$ where $Y$ is a complete toric variety;

ii) $H$ a parabolic subgroup of a reductive group $G$;

iii) $H = \{k$-jets of germs of biholomorphisms of $(\mathbb{C}, 0)\}$

\[
\begin{pmatrix}
  a_1 & a_2 & \cdots & a_k \\
  0 & (a_1)^2 & \cdots & p_{2k}(a) \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & (a_1)^k
\end{pmatrix}
\]

$\approx \{a_1 \in \mathbb{C}^*, a_2, \ldots a_k, \in \mathbb{C}\}$

(and similarly when we replace $(\mathbb{C}, 0)$ with $(\mathbb{C}^m, 0)$).

If $H$ acts linearly on a projective variety $X$, and the linearisation is twisted by a suitable rational character of $H$ and (*) holds, then this theorem applies to $X \triangleright H = (X \triangleright \hat{U}) \triangleright (\mathbb{R}/\mathbb{C}^*)$, which is Proj$(A(X)^H) = X^{ss}/\sim$ where the algebra of invariants $A(X)^H = (A(X)^{\hat{U}})^{R/\mathbb{C}^*}$ is finitely generated and $x \sim y$ as before.
When $G$ reductive acts linearly on a projective variety $X$, $\exists$ a stratification (\textit{\textasciitilde} Morse stratification for $\|\text{moment map}\|^2$)

$$X = \bigsqcup_{\beta \in B} S_{\beta}$$

indexed by a finite subset $B$ of a +ve Weyl chamber, with

(i) $S_0 = X^{ss}$, and for each $\beta \in B$

(ii) $S_{\beta} \cong G \times_{P_{\beta}} Y_{\beta}^{ss}$ where $P_{\beta}$ is a parabolic subgroup of $G$ and $Y_{\beta}^{ss}$ is an open subset of a projective subvariety $\overline{Y}_{\beta}$ of $X$.

$P_{\beta} = U_{\beta} \times L_{\beta}$, where its unipotent radical $U_{\beta}$ is graded by a central 1-parameter subgroup of its Levi subgroup $L_{\beta}$. To construct a quotient of (an open subset of) $S_{\beta}$ by $G$ we can study the linear action on $\overline{Y}_{\beta}$ of the parabolic subgroup $P_{\beta}$, appropriately twisted, and quotient first by $\hat{U}_{\beta}$ and then by the residual action of the reductive group $P_{\beta}/\hat{U}_{\beta} = L_{\beta}/\mathbb{C}^*$. We can use this to stratify moduli stacks and construct moduli spaces of unstable objects.
\[ H = U \rtimes R, \text{ internally graded unipotent radical } U, \; R = K_{\mathbb{C}} \]

\[ H \subseteq X \subseteq \mathbb{P}^n \text{ via } \rho : H \to GL(n+1) \text{ with } \rho(K) \subseteq U(n+1) \]

Define \( \mu_H : X \to \mathfrak{h}^* \) by \( \mu_H([x]).a = \bar{x}^T \rho_* (a)x/||x||^2 \in \mathbb{C} \quad \text{for } a \in \mathfrak{h}. \)

\( X \bowtie H = \) GIT quotient for appropriately twisted linearisation (after blowing up if need be).

When \( H = R \) is reductive \( X \bowtie H \cong X//K = \mu_H^{-1}(0)/K. \)

**Applications:** Betti numbers, intersection pairings on \( X \bowtie H \ldots \)

When \( H = P \) is a parabolic in a reductive \( G \) and the action of \( H \) extends to \( G \), then \( X \bowtie H \cong (X \bowtie U) \bowtie R \cong X_{implP}//K. \) After twisting the linearisation by a suitable character of \( H \) (or equivalently adding a suitable central constant to \( \mu_H \)), if (\( * \)) holds we have

\[ X \bowtie H \cong \mu_H^{-1}(0)/K. \]
Back to simple example:

\[ \mathbb{P}^n \cong \mathbb{C}^+ \cong \mu_{SU(2)}^{-1}(t^*_+) / \text{collapsing on the boundary } \mu_{SU(2)}^{-1}(0); \]

\[ \mathbb{P}^n \cong (\mathbb{C}^+ \times \mathbb{C}^*) \cong \mu_{SU(2)}^{-1}(t^*_+) \cap \mu_{S^1}^{-1}(\xi) / (S^1 \text{ and collapsing}). \]

Suppose \( \mu_{SU(2)}^{-1}(t) \cap \mu_{S^1}^{-1}(\xi) \subseteq \mu_{SU(2)}^{-1}((t^*_+)o) \cap \mu_{S^1}(\xi) \). Then

\[ \mathbb{P}^n \cong (\mathbb{C}^+ \times \mathbb{C}^*) \cong \mu_{SU(2)}^{-1}(t) \cap \mu_{S^1}^{-1}(\xi)/S^1 \]

where \( \mu_{t^\perp} : \mathbb{P}^n \to t^\perp \cong \text{LieC}^+ \) is projection of \( \mu_{SU(2)} \) onto \( t^\perp \). So

\[ \mathbb{P}^n \cong (\mathbb{C}^+ \times \mathbb{C}^*) \cong \mu_{S^1}^{-1}(0)/S^1 \]

where \( \mu = (\mu_{t^\perp}, \mu_{S^1} - \xi) : \mathbb{P}^n \to t^\perp \times (\text{LieS}^1)^* \).

Is there a similar description of \( X \cong H \) more generally?
**Hope:** Given (*) AND after twisting by a suitable character (add a suitable central constant to $\mu_H$), then for a suitable projective embedding of $X$:

$\mu_\hat{U}^{-1}(0)$ is a slice for the action of $U \times \mathbb{R}^*$ on the open subset

$$X^{s,\hat{U}} = \hat{U} \mu_\hat{U}^{-1}(0) \cong \hat{U} \times S^1 \mu_\hat{U}^{-1}(0)$$

of $X$, so that $\mu_\hat{U}^{-1}(0)/S^1 \cong X^{s,\hat{U}}/\hat{U} = X \rtimes \hat{U}$ and

$$\mu_H^{-1}(0)/K \cong (X \rtimes \hat{U}) \rtimes (\mathbb{R}/\mathbb{C}^*) \cong X \rtimes H.$$

**Applications:** calculating Betti numbers, generators for the cohomology ring and intersection pairings on

$$X \rtimes H = \mu_H^{-1}(0)/K,$$

via $\|\mu_H\|^2$ as an equivariantly perfect Morse function and Shaun Martin’s approach to intersection pairings by reducing to torus quotients.
**Kähler picture** following Greb–Miebach (2018): unipotent $U \leq G = K_C$ simply-connected semisimple; $U$ acting holomorphically on $(X, \omega)$ compact Kähler.

**Questions:**
(a) analogue of ‘linear action’?
(b) analogue of ‘reductive envelope’ $G \times_U X$?
(c) use of moment maps for $K$-action to construct and study quotients for $U$-action?
(d) constraints on $\omega$ to allow it to be extended to a $K$-invariant Kähler form on $G \times_U X$?
(e) link with non-reductive GIT?

**Thm** (Greb–Miebach) TFAE: (1) $G \times_U X$ is Kähler;
(2) the $U$-action on $X$ is ‘meromorphic’ (i.e. extends to meromorphic $\overline{U} \times X \to X$ for a suitable compactification $\overline{U}$);


Then $X \hookrightarrow G \times_U X \hookrightarrow G \times Z \cong G/U \times Z \hookrightarrow V \times Z$ when $G/U$ is embedded as a $G$-orbit in a representation $V$ of $G$ with flat $K$-invariant Kähler structure, and we can define

$$X^{ss,U}[\omega] = X \cap \{ y \in G/U \times Z : \mu^{-1}(0) \cap \overline{Gy} \neq \emptyset \}$$

where $\mu = \mu_Z + \mu_V$ for moment maps $\mu_Z : Z \to \mathfrak{k}^*$ and $\mu_V : V \to \mathfrak{k}^*$.

**Thm** (Greb–Miebach) (i) $X^{ss,U}[\omega]$ is independent of the choice of the Hamiltonian $G$-extension $Z$ (for fixed $G = K_{\mathbb{C}}$), but can depend on $G$ and the Kähler metric on $G/U$;
(ii) $\exists$ geometric quotient $\pi : X^{ss,U}[\omega] \to X^{ss,U}[\omega]/U = Q$ smooth, $Q \subseteq \overline{Q}$ compact cx space, $\overline{Q} \setminus Q$ analytic, $\pi$ extends to mero $X \to \overline{Q}$;
(iii) $\overline{Q}$ has a stratified Kähler structure restricting to a smooth Kähler form $\omega_Q$ on $Q$ with $[\pi^*\omega_Q] = [\omega]$. 
\( H = U \ltimes R \) with unipotent radical \( U \) graded by \( \lambda : \mathbb{C}^* \to Z(R) \).
\( \hat{U} = U \ltimes \lambda(\mathbb{C}^*) \trianglelefteq H \)

Adjoint action \( \phi : H \to GL((\text{Lie}(\hat{U}))) \) restricts to an injection \( \phi|_U : U \to SL((\text{Lie}(\hat{U}))) \cong SL(d + 1) = G \) where \( d = \dim(U) \).

Multiplying \( \phi \) by a character gives \( \hat{\phi} \) with \( \hat{\phi}|_U = \phi|_U \) and
\[
\hat{\phi}(\hat{U}) \cong \left\{ \begin{pmatrix}
  a_0 & a_1 & \cdots & a_d \\
  0 & (a_0)^{k_1} & \cdots & p_{1d}(a) \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & (a_0)^{k_d}
\end{pmatrix} : a_0 \in \mathbb{C}^*, a_1, \ldots, a_d \in \mathbb{C} \right\}
\]

where \( k_j > 1 \) for \( j = 1, \ldots, d \) and the entries \( p_{ij}(a_0, \ldots, a_d) \) above the diagonal are polynomials in \( a_0, \ldots a_d \), homogeneous of degree \( i \) and weighted homogeneous of degree \( k_j \).

We can use this to construct reductive envelopes/Hamiltonian \( G \)-extensions.
**Lemma** (Bérczi–K 2017; compare with the universal symplectic implosion’s embedding in an affine space with flat Kähler metric)

\[ \frac{GL(d+1)}{\hat{\phi}(\hat{U})} = \frac{(SL(d+1)/U)/(\text{finite group})}{\hat{\phi}(\hat{U})} \]

is embedded (with good control over its boundary) in an open affine subset of

\[ \mathbb{P}(V) = \mathbb{P}(\bigoplus_{j=1}^{d+1} \Lambda^j (\bigoplus_{i=0}^d \text{Sym}^k C^{d+1})) \]

as the \( GL(d+1) \)-orbit of \([p]\) given by

\[ p = \sum_{j=0}^{d} e_0 \wedge (e_1 + (e_0)^{k_1}) \wedge \ldots \wedge (e_j + \sum_{i=1}^{j-1} p_{ij}(e_0, \ldots, e_d) + (e_0)^{k_j}) \in V \]

where \( e_0, \ldots, e_d \) is the standard basis for \( C^{d+1} \).

Use this embedding and a large positive scalar multiple of the flat Kähler metric on \( V \) as input for the Greb–Miebach construction to realise the hope of a ‘moment map’ description of \( X \bowtie H \).