Stability of Ricci de Turck flow on Singular Spaces

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Differential Geometry in the Large
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Based on joint work with Boris Vertman

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- On the interior $C(F) = (0,1)_x \times F$ of a tubular neighborhood of F,

$$g|_{C(F)} = dx^2 + x^2 g_F + h,$$

where $|h(x)|_{\overline{g}} = O(x)$ as $x \to 0$ for $\overline{g} = dx^2 + x^2 g_F$.

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- $\mathrm{Ric}_g = 0 \Rightarrow \mathrm{Ric}_{g_F} = (n-1)g_F$.
- If $(F, g_F) = (S^n/\Gamma, g_{st})$, (M, g) is an orbifold.

Ricci de Turck flow and Ricci flow

Definition

Let (M, h_0) be a Riemannian manifold. A Ricci de Turck flow $g(t)$, $t \in I$ is a smooth family of metrics satisfying

$$\dot{g}(t) = -2\mathrm{Ric}_{g(t)} + \mathcal{L}_{W(t)}g(t) \tag{1}$$

where

$$W(t)^k = g(t)^{ij}\left(\Gamma_{ij}^k(g(t)) - \Gamma_{ij}^k(h_0)\right). \tag{2}$$

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Goal

Understand the stability of stationary points in the singular setting.

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- Vertman '16 (higher-dimensional case with tangentially stable cross section)

The Lichnerowicz Laplacian

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Let h_0 be Ricci-flat. Then, the Lichnerowicz Laplacian $\Delta_L : C^\infty(M,S) \to C^\infty(M,S)$ is

$$\Delta_L \omega = \Delta\omega - 2\mathring{R}\omega, \qquad (\mathring{R}\omega)_{ij} := R_{iklj}\omega^{kl}. \tag{4}$$

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In a neighbourhood of an isolated conical singularity,

$$\Delta_L = -\frac{\partial^2}{\partial x^2} - \frac{n}{x}\frac{\partial}{\partial x} + \frac{\Box_L}{x^2} + \mathcal{O}, \tag{6}$$

where $\Box_L$ is the tangential operator of Δ_L.

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Fact: In the orbifold case, $\Box_L \geq 0$ but $\Box_L \not> 0$.

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- tangentially stable (strictly, if not an orbifold)
- linearly stable
- integrable.

Then for every metric g_0 close enough to h_0, there exists a Ricci-de-Turck flow starting at g_0, with a change of reference metric at discrete times, which converges to a Ricci-flat metric h^* with isolated conical singularities as $t \to \infty$.

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$\Rightarrow$ The use of entropies ensures optimal results, the compact situation is now well understood.

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- No general instability assertion is known.
- Many cases are still open.

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  - S^n is tangentially stable but not strictly tangentially stable,
  - $\mathbb{CP}^p, \mathbb{HP}^p$ are tangentially unstable.

Sufficient condition for integrability

Theorem 3 (K. -Vertman '18)

Let (M, h_0) be a compact Ricci-flat Kähler manifold with an isolated conical singularity where the cross section is either strictly tangentially stable or a space form. Then h_0 is linearly stable and integrable.

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Sufficient condition for integrability

Theorem 3 (K. -Vertman '18)

Let (M, h_0) be a compact Ricci-flat Kähler manifold with an isolated conical singularity where the cross section is either strictly tangentially stable or a space form. Then h_0 is linearly stable and integrable.

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- Integrability also holds in the case of flat orbifolds.

Hybrid weighted Hölder spaces

Let (M, g) be a compact manifold with an isolated conical singularity which is strictly tangentially stable. Then,

$$\mathscr{H}^{k,\alpha}_{\gamma}(M \times [0,T], S) := \mathcal{C}^{\mathrm{k},\alpha}_{\mathsf{ie}}(M \times [0,T], S_0)_{\gamma} \;\oplus\; \mathcal{C}^{\mathrm{k},\alpha}_{\mathsf{ie}}(M \times [0,T], S_1)^{b}_{\gamma}$$

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- $\omega \in \mathcal{C}^{k,\alpha,b}_{\mathsf{ie},\gamma} \Leftrightarrow |\omega| = O(1)$ for $j, l = 0$, otherwise the same.

Properties of the spaces

The Friedrichs extension of Δ_L

Let (M, g) be a Ricci-flat manifold with an isolated conical singularity which is either strictly tangentially stable or an orbifold singularity. Then the Friedrichs extension of Δ_L

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Well-posedness

To prove short time existence and uniqueness for Ricci de Turck flow, apply the Duhamel principle and a fixed-point argument to

$$\dot{k} = -\Delta_{L,h_0} k + \nabla k * \nabla k + \nabla(k * \nabla k). \tag{7}$$

This works for $\mathscr{H}^{k,\alpha}_{\gamma}(M \times [0,T], S)$ (Vertman '16).

The integrability condition again

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Remark

- Nontrivial Ricci solitons are excluded in a recent preprint.
- We do not assume a fixed gauge condition for the metrics in $\mathscr{F}$.

A decomposition of the space of metrics

Lemma

Let $\mathscr{F}$ be as above. Then there exists a $\mathscr{H}$-neighbourhood $\mathscr{U}$ of h_0 and a smooth projection map $\Pi : cU \to \mathscr{F}$ such that

$$g - \Pi(g) \in (\ker \Delta_{L,h})^{\perp}.$$

The proof uses an implicit function argument.

Proof of the main theorem 1

Proposition

For any $N \in \mathbb{N}$, there exists $\epsilon > 0$ and $T = T(\epsilon, N)$ such that for any $g \in \mathscr{U}$ with $h = \Pi(g) \in \mathscr{U} \cap \mathscr{F}$ and $\|g - h\| \leq \epsilon$, the Ricci-de-Turck flow starting at g, with the background metric h, exists for time $T > 0$ and

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The proof uses that $g(t)$ is a fixed point of

$$\Phi_t g(t) := e^{-t\Delta_{L,h}} * Q_2(g(t)) + e^{-t\Delta_{L,h}}[g - h],$$

and mapping properties of $e^{-t\Delta_{L,h}}$.

Proof of the main theorem 2

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- Start at $g(T) \in \mathscr{U}$ the Ricci de Turck flow with $\Pi(T)$ as background
- Iterate!

Proof of the main theorem 3

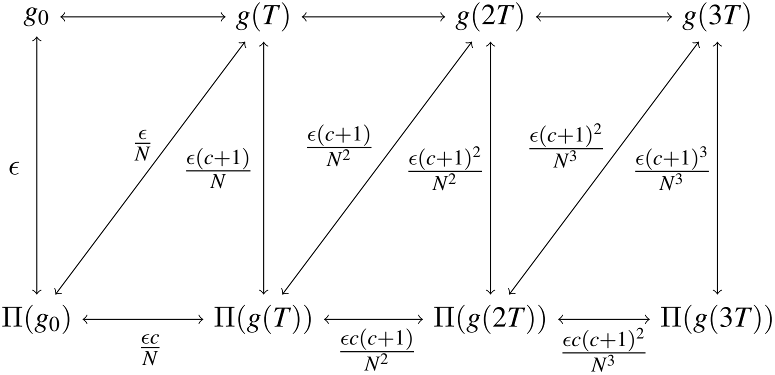

Figure: Iterative sequence of Ricci de Turck flows.

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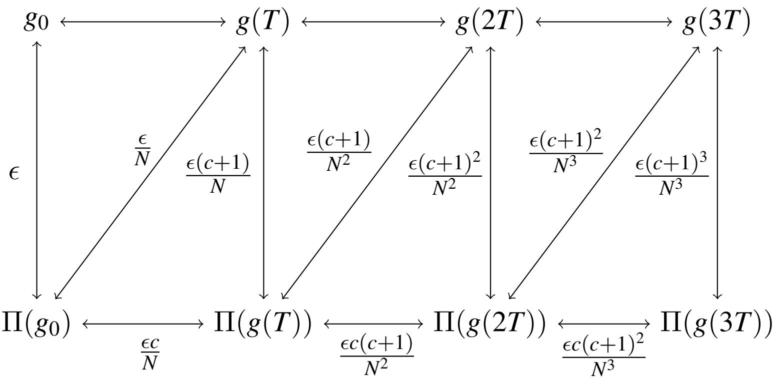

Figure: Iterative sequence of Ricci de Turck flows.

$\Rightarrow \Pi(g(kT)) \to h^* \in \mathscr{F}$ as $k \to \infty$ and $g(t) \to h^*$ as $t \to \infty$.

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- Can we use the λ-entropy to improve the stability result?
- Can we extend to more general situations (edge singularities, noncompact cones,$\ldots$)?

Thank you for your attention!