Chains in CR geometry as geodesics of a Kropina metric

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based on a joint paper with
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and on unpublished results joint with Taiji Marugame

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Main message/abstract of my talk

I connect two *a priori* unrelated subjects in differential geometry:

- CR-geometry
- Finsler geometry

I will show that for a "nondegenerate" CR structure we can construct a Finsler metric whose geodesics are "chains" of CR-structure.
This allows to use the methods of variational analysis in CR-geometry; I show their effectiveness by

- re-proving and generalising some classical results in CR- geometry (of H. Jacobowitz and J.-H. Cheng), and
- prove the existence of a closed chain on certain manifolds
- finish the talk by a list of possible continuations (the results are very new, the oldest were obtained in 2018, so everything is hot and promising)

Definition of Finsler metrics

Finsler metric is a function $F : TM \to \mathbb{R}$ such that for every $p \in M$ the restriction $F_{|T_pM}$ is a Minkowski norm, that is $\forall \xi, \nu \in T_pM, \ \forall \lambda \geq 0$

(a) $F(\lambda \cdot \xi) = \lambda \cdot F(\xi)$,

(b) $F(\xi + \nu) \leq F(\xi) + F(\nu)$,

(c) $F(\xi) = 0 \iff \xi = 0$.

Def. *(Unparameterized) Geodesics* are solutions of the extremal problem $\mathcal{L}(c) \mapsto \min$ with

$$\mathcal{L}(c) := \int_a^b F(c(t), \dot{c}(t))dt.$$

Geodesics are described by a system of $n-1$ ODE of the second order, the Euler-Langrage equations:

$$\tfrac{d}{dt}\frac{\partial F}{\partial \dot{x}_i} - \frac{\partial F}{\partial x_i} = 0.$$

History of Finsler metrics

- Finsler metrics appeared in Riemann's habilitation addressed who did not consider them interesting; were discussed in the Hermann Weyl's comment on Riemann's hablitation address who did find them interesting and suggested few directions to study.
- Were intensively studied by classics of calculus of variations – Cartheodory, Landsberg (beginning of the 20th century). They were:
  - interested in continuous optimal (variational) problems
  - impressed by description of qualitative behaviour of geodesics of Riemannian metrics with the help of Jacobi vector fields and sectional curvature
  - wanted to generalize them to the Finslerian setting.

CR-structure

Def. (M^{2n+1}, H, J) is called a CR (Cauchy-Riemann) manifold if

- $H \subset TM$ is a rank $2n$ distribution;
- $J \in \Gamma(End\, H)$ and $J^2 = -id$;
- $[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M)$, where $\mathbb{C}H = T^{1,0}M \oplus \overline{T^{1,0}M}$ is the eigenspace decomposition.

A choice of a real 1-form θ with $\ker\theta = H$ determines the *Levi form*:

$$h_\theta(X, Y) := d\theta(X, JY), \quad X, Y \in H.$$

M is called *non-degenerate* (*resp. strictly pseudoconvex*) if the Levi form is non-degenerate (resp. positive definite). Of course in this case H is a contact form.

Examples:

- $\mathbb{C}^{n+1} \supset M$ real hypersurface, $H := TM \cap J_{\mathbb{C}^{n+1}}(TM)$, $J := (J_{\mathbb{C}^{n+1}})|_H$.
- Special case (plays a role in my talk): if $M \subset \mathbb{C}^{n+1}$ is the standard sphere, the corresponding CR-structure is called **flat**
- Special case (plays no role in my talk but is a big subject in mathematics): Sasakian manifolds

Chains of a CR-structure

- *Chains* are distinguished family of curves on a CR manifold (M^{2n+1}, H, J). They satisfy a 2nd order ODE, at every point in (almost) every direction there exists a chain.

- Moser and Chern 1974 suggested that chains should play the role of geodesics in CR geometry.

The spirit of our study parallels that of classical surface theory. We list the corresponding concepts as follows:

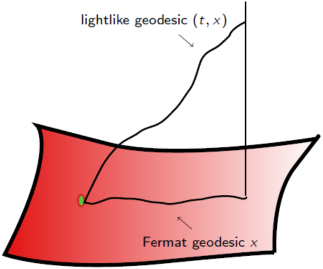
Surfaces in euclidean 3-space	Real hypersurfaces in $\mathbf{C}_{n+1}$
Group of motions	Pseudo-group of biholomorphic transformations
Immersed surface	Non-degenerate real hypersurface
Plane	Real hyperquadric
Induced riemannian structure	Induced CR-structure
Isometric imbedding	Existence of local solutions of certain systems of PDEs
Geodesics	Chains

- The flat CR-structure (coming from the standard sphere in $\mathbb{C}^{n+1}$) somehow contradicts Chern and Moser: in this case chains are intersections of complex lines (J-invariant real 2-dimensional planes) with the sphere; they may have arbitrary small radius.
- One more "countrexample" to the suggestion of Chern and Moser is the Burns-Shnider example 1977: it lives on a compact manifold, is CR-flat, but not any two points can be connected by a chain.

One of definitions of chains goes through the Fefferman metric

- Fefferman metric (of a CR-structure) is a metric on $S^1 \times M^{2n+1}$; its conformal class is canonically constructed by the CR-structure.
  - $\frac{\partial}{\partial x_0}$ a Killing vector field
  - $\frac{\partial}{\partial x_0}$ is null (=light-like).

Def. Chains are projections of the light-like geodesics to M^{2n+1} along $\frac{\partial}{\partial x_0}$.



This construction is similar to "Fermat principle" in general relativity (in mathematics big activity by E. Caponio, M. A. Javaloyes and M. Sanchez etc); were studied before by mathematical physicists including V. Perlick, G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick, M. C. Werner. The difference with our situation is that the Killing vector field is time-like. In this situation, one reduces light-like geodesics to the so-called Randers metrics.

Projection of null geodesics and Kropina (Finsler) metric

- Let $(\widetilde{M}^{n+1}, \widetilde{g})$ be a pseudo-Riemannian manifold and K a nonvanishing **null Killing** vector filed on $\widetilde{M}$.
- Let $M^n \subset \widetilde{M}$ be a hypersurface transverse to K.
- We take coordinates such that $(x^0, x^1, \dots, x^n)$ such that $M = \{x^0 = 0\}, K = \frac{\partial}{\partial x^0}$.

$$\widetilde{g} = g_{ij} dx^i dx^j + 2\omega_i dx^i dx^0$$
$$= \begin{pmatrix} 0 & \omega_j \\ \omega_i & g_{ij} \end{pmatrix}, \quad (g_{ij}, \omega_i : \ \textit{independent of } x^0)$$

Here $\omega := \omega_i dx^i = (K \lrcorner \widetilde{g})|_{TM}$, $g := g_{ij} dx^i dx^j$ on M; $\omega \neq 0$.

Projection of null geodesics and Kropina metric

Setup: $\widetilde{g} = g_{ij}dx^i dx^j + 2\omega_i dx^i dx^0 = g + 2\omega \cdot dx^0,\ M = \{x^0 = 0\}$.

Definition

The *Kropina metric* F on M associated to $\widetilde{g}$ is the (singular) Finsler metric

$$F(x,\xi) := \frac{g(\xi,\xi)}{\omega(\xi)} \Big(= \frac{\widetilde{g}(\xi,\xi)}{\widetilde{g}(K,\xi)} \Big), \quad \xi \in T_x M.$$

Definition

A curve $\gamma(t)$ on M is a (unparameterized) geodesic if γ satisfies the Euler-Lagrange equation for F and in addition $\omega(\dot{\gamma}(t)) > 0$.

- F is not defined on $\ker \omega$.

Projection of null geodesics and Kropina metric

Let $\pi : \widetilde{M} \to M,\ \pi(x^0, x^i) = (x^i)$ be the projection.

Theorem

Let $\widetilde{\gamma}(t)$ be a null geodesic of $\widetilde{g}$ with $\widetilde{g}(K, \dot{\widetilde{\gamma}}(0)) \neq 0$. Then $\gamma(t) := \pi(\widetilde{\gamma}(t))$ is a geodesic of $F = g/\omega$. Conversely, any geodesic of F is the projection of a null geodesic of $\widetilde{g}$ and the lift is uniquely determined up to translations by the flow of K.

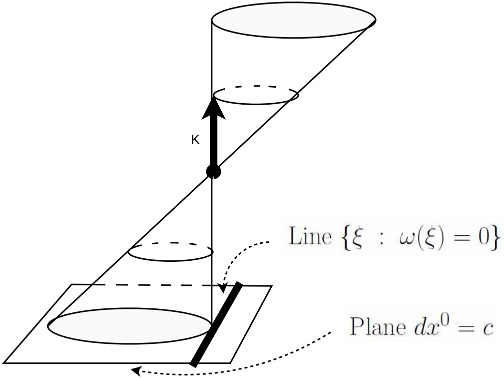
Proof is by direct calculations: geodesics of a Finsler metric are described by $n-1$ second order ordinary differential equations: they are functions on acceleration $\ddot{\gamma}(t)$, velocity $\dot{\gamma}(t)$ and the position $\gamma(t)$; the dependence on the position pops up through g, ω and their first derivatives. Projection of light-like geodesics are given by n second order ODE of Euler form. One compares and sees that (at every point $\gamma(t), \dot{\gamma}(t)$) every $\ddot{\gamma}$ which solves the second system satisfies the first one.

Let me emphasize:

- The proof is “algebraic geometric” and is done “at the second jet space at one point”: We fixed a point $p = \gamma(t)$ and considered $\underbrace{\mathbb{R}^n}_{\xi} \times \underbrace{\mathbb{R}^n}_{\nu}$. Here ξ is thought to be the velocity $\dot\gamma(t)$ and ν the acceleration $\ddot\gamma$.
- The Euler-Lagrange equation for the Kropina metric and the ODE for the projection of the light-like geodesics of the corresponding (n+1)-dimensional metric are algebraic subvarieties of $\mathbb{R}^n \times \mathbb{R}^n$. The coefficients in these subvarieties come from the entries of metric g, of ω, an of their first derivatives.
- One shows that the first contains the second.

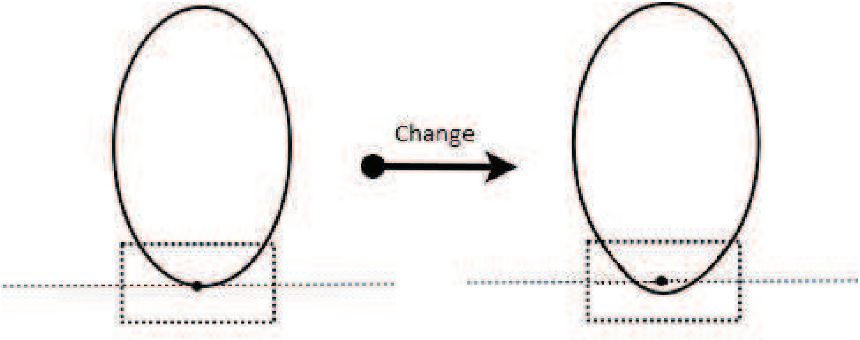
Indicatrix of Kropina metric

$\widetilde{g} = g_{ij}dx^i dx^j + 2\omega_i dx^i dx^0$, g: positive definite on $\ker\omega$.
$\{\xi \in T_xM \mid F(x,\xi) = 1\} = \{g_{ij}\xi^i\xi^j = \omega_i\xi^i\} \cong \{\xi^0 = -\frac{1}{2}\} \cap \mathcal{N} \subset T_x\widetilde{M}$.



The indicatrix (i.e., the set of $\xi \in T_xM$ such that $F(x,\xi) = 1$) of the Finsler metric is the sphere passing through origin.

For direction not lying in kernel, the geodesics are geodesics of some (nonsingular) Finsler metric; all regularity staff etc holds.



How the freedom in choosing the section $M \subset \tilde{M}$ affects the Finsler metric?

- The conformal class of Feffermann metric is canonically constructed by the CR-structure:
- The decomposition $\tilde{M} = S^2 \times M$ is not: there is a freedom of choosing the section.
- If we choose another section, the constructed Kropina metric is changed by the rule $F_{new} = F_{old} + \alpha$, where α is a closed form; this procedure does not affect geodesics of F.

First application of Finsler geometry to CR-geometry: chains determine their CR-structure

- By a classical result of J.-H. Cheng 1988, chains determine CR-structure (up to replacing J to $-J$).
- For the proof of Cheng, it was important that he considered **all** chains: he observes that if **all** chains of two CR-structures coincide, then also the contact distributions coincide. The generalisation of Čap-Žádník 2009 simply assumes that corresponding contact disctribution coincide.
- From our Finsler results, explained in the next slide, it will follow that if "sufficiently many" chains of two CR-structure coincide, then the CR-structures coincide.

Projective equivalence of Kropina metrics

Definition

Two Kropina metrics $F(x, \xi) := g(\xi, \xi)/\omega(\xi)$, $\widehat{F}(x, \xi) := \widehat{g}(\xi, \xi)/\widehat{\omega}(\xi)$ are *projectively equivalent* if there exists a sufficiently big family of curves which are (unparameterized) geodesics of both metrics.

sufficiently big family means
$\Longleftrightarrow$ For any point $p \in M$, the set of tangent vectors at p for these curves contains a nonempty open subset of T_pM.

Corollary

Sufficiently big family of chains determine the CR-structure

Proof is based on algebraic geometry in the second jet bundle

- Assume two Kropina metrics g/ω and $\widehat{g}/\widehat{\omega}$ have the same geodesics. Fix a point $p \in M$ and consider the second jet space at the point: $\underbrace{\mathbb{R}^n}_{\xi} \times \underbrace{\mathbb{R}^n}_{\nu}$.

  Here ξ is thought to be the velocity $\dot{\gamma}(t)$ and ν the acceleration $\ddot{\gamma}(t)$.
- The Euler-Lagrange equation for the Kropina metrics are algebraic subvarieties $\mathcal{E}$ and $\widehat{\mathcal{E}}$ of $\mathbb{R}^n \times \mathbb{R}^n$. These subvarieties are determined by n algebraic equations whose coefficients come from g, ω, $\frac{\partial g}{\partial x_i}$, $\frac{\partial \omega}{\partial x_i}$, $\widehat{g}, \widehat{\omega}$, $\frac{\partial \widehat{g}}{\partial x_i}$, $\frac{\partial \widehat{\omega}}{\partial x_i}$.
- If the metrics are projectively equivalent, we have $\mathcal{E} = \widehat{\mathcal{E}}$. It is sufficient to show that the condition $\mathcal{E} = \widehat{\mathcal{E}}$ implies $\omega = \mathrm{const} \cdot \widehat{\omega}$; the proof is somehow involving. It essentially uses that the problem is geometric, to be commented on the next slide. The rest of the proof (we need to show that, for projectively equivalent Kropina metrics with the same ω we have that g is also essentially the same) is straightforward.

Few remarks on the proof

- In assumptions we actually required that sufficiently big set of geodesics coincide; hence $\mathcal{E}$ and $\widehat{\mathcal{E}}$ coincide on an open subset only. But since they are algebraic, if they coincide on open subset then they coincide on a certain bigger set (which is enough for our goals). We also see that sufficiently many chains in general position is enough to determine $\mathcal{E}$
- The algebraic geometry in the proof is not that trivial – we compare two algebraic varieties in $\mathbb{R}^{2n}$ given by $n-1$ algebraic equations whose coefficients are given by nontrivial and nonlinear formulas in $g, \omega, \frac{\partial g}{\partial x_i}, \frac{\partial \omega}{\partial x_i}$. We need to show that $g, \omega, \frac{\partial g}{\partial x_i}, \frac{\partial \omega}{\partial x_i}$ are determined by the varieties. A "correct" algebraic way to do it would be to think that coefficients are unknown, so pass to dimension $2n + \frac{n(n+1)}{2} + n + \frac{n^2(n+1)}{2} + n^2$, in theory this way should work for any n, but technically it is a dead end.
- We used a lot the geometric nature of the equations: the group of diffeomorphisms of the manifold induce an action on the jet space and we used it to simplify the equations. Let us remark that it would be much more difficult to find the same proof in the context of chains, since in this case the coefficients come from 4-th order jets.

Second application of Finsler geometry in CR-geometry: local and global connectivity theorems

Theorem

Let $F = g/\omega$ be a Kropina metric on M such that ω is non-integrable and g is positive definite. Then two nearby points on M can be joined by a geodesic of F. If M is compact and connected, any two points can be joined by a geodesic of F.

Note that if g is positive definite on $\ker \omega$,
we can make g positive definite locally
by adding a closed 1-form β to F. (This does not change geodesics.)
Thus, we have:

Corollary (Jacobowitz 1985, Koch 1988)

Two nearby points on a strictly pseudoconvex CR manifold can be joined by a chain.

Local do not imply global

For strictly pseudoconvex CR manifolds one can make the corresponding Kropina metric to be positively definite locally. Globally it is not always the case, a counterexample is the Burns-Shnider example mentioned above: it is CR-flat, but not any two points can be connected.

Theorem

If a compact connect strictly pseudoconvex CR manifold admits a pseudo-Einstein contact form with positive scalar curvature, then any two points can be joined by a chain.

Remark. The condition that there exists a pseudo-Einstein contact form with positive scalar curvature ensures that the corresponding Kropina metric is “positive definite”.

Third application: the existence of closed chains

Theorem

Let $F = g/\omega$ be a Kropina metric on a compact connected M such that ω is non-integrable and g is positive definive. For any nontrivial element of $H_1(M)$ there exists a closed geodesic realizing this element.

Proof of the connectivity theorems and of the existence of closed geodesics

1. We define the (nonsymmetric) distance on M between two points p_1, p_2 as the infimum of the length of the curves γ satisfying $\omega(\dot{\gamma}) > 0$ connecting this points.
2. Since *kern*(ω) is not integrable, by the Rashevskii-Chow theorem, one can connect every two point by a geodesic with $\omega > 0$, so the distance is finite; and one can realize every nontrivial element of H_1 by a closed curve such that the Kropina metric is positive at every velocity vector.
3. We consider the sequence of admissible curves connecting the points, or the sequence of closed curves realising the element of H_1, such that the sequence of the lengths of these curves converges to infimum. By Arzela-Ascoli, the sequence of these curves has a convergent subsequence. The limit of this subsequence is the shortest curve, hence the geodesic. Of course, some technical difficulties are involved.

What to do if $H_1(M)$ is trivial (work in progress)

Recall that on any closed Finsler manifold there exists a closed geodesic (Fet 1952; special cases are due to Birkhoff and Morse).

Conjecture

Let $F = g/\omega$ be a Kropina metric on a compact connected M such that ω is non-integrable and g is positive definive. Then, there always exists a closed nontrivial geodesic.

The hope is that the proof of Fet (Birkhoff, Morse) can be generalised to this situation. The (simplified version of the) argument of Fet starts with a foliation of the manifold into circles; once we have such a foliation such that the all the circles have $\omega(\dot\gamma) > 0$ at all points the argument of Fet works and one proves the existence of a closed chain. The existence of such a foliation is what we are doing now; for 3-sphere with the standard contact structure and, more generally, for any contact structure coming from a sasakian manifold the existence of such foliation is standard so in these cases we do have a closed geodesic.

What to do next

- Finsler metrics appeared long ago and were intensively studied. The main direction of study was always geodesics: people
    - were impressed by description of qualitative behaviour of geodesics of Riemannian metrics with the help of Jacobi vector fields and sectional curvature
    - wanted to generalize them to the Finslerian setting.
- Of course there are many results in this direction:
    - Question: what do these results give us for chains?
- There are certain ‘more interesting’ classes of CR-structures: embeddable, sasakian etc. What are the properties of the corresponding Kropina metrics?
- CR-geometry is an example of the so-called “parabolic Cartan geometry” (another examples are conformal geometry, projective geometry). In all such geometries one has a family of distinguished curves
    - Are they variational? (The answer is positive for flat models)
- Thank you very much and join!

Additional material and formulas

Tanaka-Webster connection

Fix a contact form θ. An admissible (co)frame for $\mathbb{C}TM$:

$$\{Z_0 = T, Z_\alpha, Z_{\overline{\alpha}}\} \longleftrightarrow \{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\},$$

where T is the Reeb field, $\{Z_\alpha\}$ is a frame for $T^{1,0}M$, and $Z_{\overline{\alpha}} := \overline{Z_\alpha}$.

Tanaka-Webster connection

The *Tanaka-Webster connection* associated to θ is a unique linear connection ∇ on TM such that

- ∇ preserves $T^{1,0}M$ and satisfies $\nabla T = 0$, $\nabla h_\theta = 0$
- The components of the torsion tensor vanish except for

$$Tor^{0}_{\alpha\overline{\beta}} = ih_{\alpha\overline{\beta}}, \quad Tor^{\overline{\beta}}_{0\alpha} =: A_\alpha{}^{\overline{\beta}}$$

Tanaka-Webster Ricci and scalar curvatures are defined by

$$R_{\alpha\overline{\beta}} := R_\gamma{}^\gamma{}_{\alpha\overline{\beta}}, \quad R := h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}.$$

Fefferman metric

Let $\mathcal{C} := K_M/\mathbb{R}_+$ be the circle bundle over M, where

$$K_M := \{\zeta \in \wedge^{n+1}\mathbb{C}T^*M \mid \overline{Z} \lrcorner \zeta = 0, \forall Z \in T^{1,0}M\}.$$

$\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\} \rightsquigarrow \zeta = e^{is}\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \mod \mathbb{R}_+$ with $s \in \mathbb{R}/2\pi\mathbb{Z}$.
The *Fefferman metric* on $\mathcal{C}$ is defined by

$$G_\theta := h_{\alpha\overline{\beta}}\theta^\alpha \cdot \theta^{\overline{\beta}} + 2\theta \cdot \sigma,$$

where

$$\sigma := \frac{1}{n+2}\left(ds - \operatorname{Im}\omega_\alpha{}^\alpha - \frac{R}{2(n+1)}\theta\right).$$

It holds that $G_{f\theta} = fG_\theta$, so the conformal class $[G_\theta]$ is CR invariant.

$$K := (n+2)\frac{\partial}{\partial s}$$

is a null Killing vector with $K \lrcorner G_\theta = \theta$.

CR chains

Definition

A *chain* is a curve $\gamma(t)$ on M which is the projection of a null geodesic $\widetilde{\gamma}(t)$ of G_θ such that $G_\theta(K, \dot{\widetilde{\gamma}}(0)) \neq 0$.

- Since $[G_\theta]$ is CR invariant, chains are CR invariant family of curves.
- A chain satisfies $\dot\gamma(t) \notin H$ for all t.
- For any point $p \in M$ and $v \in T_pM \setminus H_p$, there exists a unique chain $\gamma(t)$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$.

Kropina metric for CR chains

We embed M into $\mathcal{C}$ by a local section $\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$.

Theorem

Locally, a chain is a geodesic of the Kropina metric

$$F := \frac{h_{\alpha\overline{\beta}}\theta^\alpha \cdot \theta^{\overline{\beta}}}{\theta} - \frac{2}{n+2}\left(\mathrm{Im}\,\omega_\alpha{}^\alpha + \frac{R}{2(n+1)}\theta\right).$$

Remarks:

- If we choose another local section, F differs by a closed 1-form.
- If M admits a pseudo-Einstein contact form ($R_{\alpha\overline{\beta}} = \frac{R}{n} h_{\alpha\overline{\beta}}$), then we have a global Kopina metric

$$F = \frac{h_{\alpha\overline{\beta}}\theta^\alpha \cdot \theta^{\overline{\beta}}}{\theta} + \frac{R}{n(n+1)}\theta.$$

Chains determine the CR structure

Well known fact in CR geometry:

Theorem (J.-H. Cheng 1988, Čap-Žádník 2009)

If chains for two non-degenerate CR structures (H, J) and (H', J') coincide, then $(H', J') = (H, \pm J)$.

In this theorem, it is assumed that *all* chains coincide, so the contact distribution H is "trivially" determined.
We can relax the assumption by using the Kropina metric:

Theorem

If a sufficiently big family of chains for two non-degenerate CR structures (H, J) and (H', J') coincide, then $(H', J') = (H, \pm J)$.

Proof of the theorem

The Kropina metrics for $(H, J), (H', J')$ are of the form

$$F = \frac{h_\theta + \theta \cdot \alpha}{\theta}, \quad F' = \frac{h'_{\theta'} + \theta' \cdot \alpha'}{\theta'}$$

with 1-forms α, α'. Since F and F' are projectively equivalent, we have $\ker\theta = \ker\theta'$, and we may assume $\theta' = \theta$.
We also have $F' = cF + \beta$ with a constant c and a (closed) 1-form β, which implies

$$d\theta(X, J'Y) = c d\theta(X, JY), \quad \textit{for all } X, Y \in H = H'.$$

Thus we have $J' = cJ$ and hence $J' = \pm J$. □

Thank you for your attention.