Volume Entropy Estimate for Integral Ricci Curvature

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$h(M, g) > 0$ iff the fundamental group $\pi_1(M)$ has exponential growth.

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  where $\min Ent(M) = \inf\{h(M,g)\,|\mathrm{Vol}(M,g)=1\}$, the minimal entropy, and $\|M\|$ is the simplicial volume

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- $$h^2(M,g) \geq 4\lambda_0,$$
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Not only interesting in differential geometry, but also in dynamical system and geometric group theory.

Volume Entropy Estimate

When $\mathrm{Ric}_M \geq (n-1)H$, Bishop volume comparison gives the upper bound

$$h(M,g) \begin{cases} = 0, & H \geq 0; \\ \leq (n-1)\sqrt{-H}, & H < 0. \end{cases} \tag{1}$$

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For M^n hyperbolic manifold, $h(M) = n-1$.

Maximal Volume Entropy Rigidity

Ledrappier-Wang 2010: If M^n compact, $\mathrm{Ric}_M \geq -(n-1)$ and $h(M, g) = n - 1$, then M is isometric to a hyperbolic manifold.

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Theorem (Connell, Dai, Núñez-Zimbrón, Perales, Suárez-Serrato and Wei, 2018)

Let $(X, \mathrm{d}, \mathfrak{m})$ *be a compact* $\mathrm{RCD}(-(N-1), N)$ *space. Then* $h(X) \leq N-1$ *and equal to* $N-1$ *iff* X *is isometric to a hyperbolic space.*

Quantitative Maximal Volume Entropy Rigidity

Chen-Rong-Xu 2016: If M^n has $\mathrm{diam} M \leq D$, $\mathrm{Ric}_M \geq -(n-1)$, $h(M,g) \geq n-1-\epsilon(n,D)$, then M is diffeomorphic ($\Psi(\epsilon|n,D)$-isometric) to a hyperbolic manifold.

Minimal Volume Entropy Rigidity I

Besson-Courtois-Gallot 1995: If $f : M^n \to Y^n$ is a degree k map, where M compact, Y compact hyperbolic, then

$$h(M)^n \mathrm{vol}\,(M) \geq k(n-1)^n \mathrm{vol}\,(Y)$$

and "$=$" iff f is homotopic to a Riemannian covering.

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and "=" iff f is homotopic to a Riemannian covering.

In particular, if M has $\mathrm{Ric}_M \geq -(n-1)$ and f is degree 1 map, then $\mathrm{vol}\,(M) \geq \mathrm{vol}\,(Y)$ and " = " $\Leftrightarrow M \overset{isom}{\cong} Y$.

Minimal Volume Entropy Rigidity II

Connell, Dai, Núñez-Zimbrón, Perales, Suárez-Serrato and Wei 2019: Let Y^N be a closed hyperbolic manifold, and (X, d, m) an RCD$(-(N-1), N)$ space. If X and Y are homotopic, then

$$m(X) \geq \mathrm{Vol}(Y).$$

Moreover, equality occurs if and only if X is isometric to Y.

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CDNPSW2019: also a general version with degree.

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CDNPSW 2019: extend this to RCD spaces.

Integral Curvature Lower Bounds

The following quantity measure the amount of Ricci curvature lying below $(n-1)H$ in L^p norm.

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$$k(p,H) = \left(\frac{1}{\mathrm{vol}(M)} \int_M \rho_H^p dv\right)^{\frac{1}{p}} = \left(\fint_M \rho_H^p dv\right)^{\frac{1}{p}},$$

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Clearly $k(p,H) = 0$ iff $\mathrm{Ric}_M \geq (n-1)H$.

Goal

Problem: Study the geomerty of manifolds with integral Ricci lower bounds as compared to the geometry of manifolds with pointwise Ricci lower bounds.

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Focus of Today: Get good volume entropy estimates for integral Ricci curvature!

Upper Bound of Volume Entropy

Theorem (Chen-Wei, 2018)

M^n, compact, diam$(M) \leq D$, $p > \frac{n}{2}$, $H \leq 0$. $\exists\, \delta(n, D, p, H) > 0$, s.t., if $k(H, p) \leq \delta \leq \delta(n, D, p, H)$, then

$$h(M) \leq (n-1)\sqrt{-H} + c(n, D, p, H)\delta^{\frac{1}{2}}.$$

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Remark:
Aubry2009, $h(M) \le c'(n,p)\left(⨍_M \max\{-\operatorname{Ric}(x), 0\}^p\right)^{\frac{1}{2p}}$,
$c'(n,p) \to \sqrt{n-1}$, $p \to \infty$.

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$c'(n,p) \to \sqrt{n-1}$, $p \to \infty$.

When $\delta \to 0$, Aubry's estimate approaches $(n-1)\sqrt{-H}$ only when $p \to \infty$, while our holds for any fixed $p > \frac{n}{2}$.

Application

By Švarc 1955, for M, compact,

$$h(M) \leq h(\pi_1(M), S_0) \leq (2\mathrm{diam}\,(M) + 1)h(M)$$

where the algebraic entropy $h(G,S) = \lim_{R\to\infty} \frac{\ln \#\{\gamma \in G, |\gamma|_S \leq R\}}{R}$ of a finitely generated group G and a symmetric generating set S. And let $h(G) = \inf_S h(G,S)$. Then a direct appliaction is

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Corollary

M^n, compact. If M is almost non-negative in integral sense, i.e., $\exists\, g_i$ on M s.t., $k_{g_i}(0,p) diam^2(M, g_i) \to 0$ as $i \to \infty$, then $h(\pi_1(M)) = 0$.

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In fact, by Petersen-Wei1997 and Breuillard-Green-Tao 2012, $\pi_1(M)$ is virtually nilpotent.

Quantitative Rigidity for Integral Ricci curvature

Theorem (Chen-Wei 2018)

Given $n, p > \frac{n}{2}, v, D$, $\exists\, \epsilon(n,p,D,v), \delta(n,p,D,v)$, s.t., for M^n compact, $x \in M$ satisfying

$$\left.\begin{array}{c} diam(M) \le D \\ k(-1,p) \le \delta \\ h(M) \ge n-1-\epsilon \\ \mathrm{vol}\,(B_1(x)) \ge v \end{array}\right\} \Rightarrow \begin{array}{c} \exists\, f : M \overset{diffeo}{\cong} \mathbb{H}^n/\Gamma, \\ f \text{ is } \Psi(\delta,\epsilon|n,p,D,v)\text{-isometry.} \end{array}$$

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This extends Chen-Rong-Xu2016 to integral Ricci curvature in the noncollapsing case.

Theorem (Chen-Wei 2018)

Given $n \geq 3, p > \frac{n}{2}, D > 0$, $\exists\, \epsilon(n,p,D) > 0$, $\delta(n,p,D) > 0$, s.t., for Y^n compact hyperbolic manifold with diam$(Y) \leq D$, and for M^n compact with

$$\left.\begin{array}{c} \exists\, f : M \to Y, \textit{degree } 1 \textit{ map} \\ k(-1,p) \leq \delta \\ \mathrm{vol}\,(M) \leq (1+\epsilon)\mathrm{vol}\,(Y) \end{array}\right\} \Rightarrow \begin{array}{c} \exists\, f : M \overset{\textit{diffeo}}{\cong} \mathbb{H}^n/\Gamma, \\ f \textit{ is } \Psi(\delta,\epsilon|n,p,D)\textit{-isometry.} \end{array}$$

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This generalizes Bessières-Besson-Courtois-Gallot 2010 quantitative minimal volume entropy rigidity to integral Ricci curvature.

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- By Bishop volume comparison

$$h(M) = \lim_{R\to\infty} \frac{\ln \mathrm{vol}\,(B_R(\tilde{x}))}{R} \leq \lim_{R\to\infty} \frac{\ln \mathrm{vol}\,(\underline{B}_R^{-1})}{R} = n-1.$$

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- Volume comparison loss effective when $R \to \infty$.

Integral curvature bound of $\tilde{M}$

Theorem (Aubry, 2009)

Given $n, p > \frac{n}{2}, H \leq 0, D$, $\exists\, \delta = \delta(n, p, H, D)$, $c = c(n, H, D)$, *s.t., for a compact manifold* M^n *with*

$$k_M(H, p) \leq \delta, \quad diam(M) \leq D,$$

$$\Rightarrow c^{-1} k_M(H, p) \leq k_{\tilde{M}}(H, p, R) \leq c\, k_M(H, p)$$

for all $R \geq 3D$.

Volume comparison on large annulus

Theorem (Chen-Wei 2018)

Given $n, p > \frac{n}{2}, H \leq 0, R > 1, \exists\, \delta_0 = \delta(n, p, H, R)$ s.t., for a complete manifold M^n with $k(H, p, 1) \leq \delta < \delta_0$ and for $L > 1$,

$$\frac{\mathrm{vol}\,(\partial B_{L+R}(x))}{\mathrm{vol}\,(\partial B_L(x))} \leq \frac{\mathrm{vol}\,(\partial \underline{B}^H_{L+R})}{\mathrm{vol}\,(\partial \underline{B}^H_L)} \left(1 + c(n, p, H, R)\delta^{\frac{1}{2}}\right).$$

Note the constant c is independent of L and.

$$\lim_{L\to\infty} \frac{\mathrm{vol}\,(\partial \underline{B}^H_{L+R})}{\mathrm{vol}\,(\partial \underline{B}^H_L)} = e^{(n-1)\sqrt{-H}R}.$$

Proof Sketch of Uppper Bound Estimate

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Now for r large $(H = -1)$

$$\mathrm{vol}\,(\partial B_r(\tilde{x})) \leq \mathrm{vol}\,(\partial B_{r-5D}(\tilde{x}))(1 + c\delta^{\frac{1}{2}})(1 + \Psi(r^{-1}))e^{5(n-1)D}$$

$$\leq \mathrm{vol}\,(\partial B_{r_0}(\tilde{x}))\left((1 + c\delta^{\frac{1}{2}})(1 + \Psi(r_0^{-1}))e^{5(n-1)D}\right)^{\frac{r-r_0}{5D}}$$

$$\lessapprox \mathrm{vol}\,(\partial \underline{B}_{r_0}^{-1})(1 + \Psi(\delta|n, p, r_0))$$
$$\left((1 + c\delta^{\frac{1}{2}})(1 + \Psi(r_0^{-1}))e^{5(n-1)D}\right)^{\frac{r-r_0}{5D}}.$$

□

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With volume entropy upper bound for integral curvature we can follow the steps in the pointwise case.

Almost Maximal Case

Consider M_i^n satisfying

$$\begin{cases} \bar{k}(-1,p) \le \delta_i \to 0 \\ d \ge \operatorname{diam}(M_i) \\ h(M_i) \ge n-1-\epsilon_i \to n-1 \\ \operatorname{vol}(B_1(x_i)) \ge v \end{cases}$$

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By the precompactness [Petersen-Wei1997, Aubry2009] we have

$$\begin{array}{ccc} (\tilde{M}_i, \tilde{p}_i, \Gamma_i) & \xrightarrow{GH} & (\tilde{X}, \tilde{x}, G) \\ \downarrow\pi_i & & \downarrow\pi \\ (M_i, x_i) & \xrightarrow{GH} & (X = \tilde{X}/G, x) \end{array}$$

where $\Gamma_i = \pi_1(M_i, p_i)$.

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Almost Minimal Case

Consider $(M_i, x_i) \to (X, x)$ and hyperbolic manifold Y_i with 1-degree map $f_i : M_i \to Y_i$ and

$$\bar{k}_{M_i}(-1, p) \leq \delta_i \to 0, \quad \frac{\mathrm{vol}\,(M_i)}{\mathrm{vol}\,(Y_i)} \leq 1 + \epsilon_i \to 1, \quad \mathrm{diam}\,(Y_i) \leq d.$$

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- Step 2: Assume $F_i \to F : X \to Y$, which is 1-Lipschitz. F is an isometry.

Thank you