A model flow for ideal submanifolds

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3-15 February 2019
Australian-German workshop on differential geometry in the large, MATRIX

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Motivation: CMC hypersurfaces and PMC submanifolds

Joint with Ben Andrews, James McCoy, Glen Wheeler.

Take home message from this talk. In this talk we propose a new way to think about CMC hypersurfaces and PMC submanifolds: as *ideal submanifolds*.

The main theorem in this talk is a pilot case: *planar ideal curves*.

Minimal submanifolds

Let us take $X : M^n \to \mathbb{R}^{n+m}$ to be a smooth closed immersed submanifold of Euclidean space. (Or more generally in a Riemannian manifold.)
Minimal submanifolds have a variational characterisation in terms of the L^2-gradient of the area functional

$$A[X] = \int_{M^n} d\mu$$

where $d\mu$ is the area measure induced by X. (This means that we take $g = X^* g^{\mathbb{R}^{n+m}}$ so that (M^n, g) is a Riemannian manifold, and then $d\mu = \sqrt{\det g}\, d\mathcal{L}$.)

CMC and PMC submanifolds

For hypersurfaces with constant mean curvature, there are two main variational approaches:

1. Minimise $A(X)$ subject to a constraint on enclosed volume. This leads to the *volume-preserving mean curvature flow* (VPMCF).

2. Take a volume-preserving gradient of A. For example the H^{-1}-gradient. This gives rise to the *surface diffusion flow* (SDF).

The equilibrium set for both flows are CMC hypersurfaces, but the operators are very different. For (VPMCF), the velocity is

$$H - \frac{1}{A[X]} \int_{M^n} H \, d\mu$$

whereas for (SDF) the velocity is

$$-\Delta H \, .$$

Both have zero average, which is why both preserve enclosed volume. However one is non-local and second order, and the other is of fourth order.

Here we propose a third option. For (VPMCF), we have a minimisation problem with a constraint. This results in a non-local operator. The (SDF) gives a local operator, but the variational problem is not in L^2; rather, it is in H^{-1}. Our proposal is to consider the functional

$$E[X] = \frac{1}{2}\int_{M^n} |\nabla H|^2 \, d\mu$$

and minimise E in L^2. We call critical points of E *ideal submanifolds*.

This is a variational problem in L^2 and the operator is local. Of course, nothing is free, and the resultant operator is now of sixth order.

There doesn't seem to be a clear 'winner' (yet). The energy E is the *Dirichlet energy for the mean curvature*. We feel that, intuitively at least, the ideal submanifold approach is at least as good as (VPMCF) and (SDF).

To test this intuition we need to check a few essential points:

1. That ideal submanifolds are CMC (PMC)
2. That the L^2-gradient flow around CMC hypersurfaces (PMC submanifolds) is stable.

The main result of this talk is to confirm both of these points in the simplest case of $n = m = 1$.

Planar ideal curves

Suppose $\gamma_0 : \mathbb{S} \to \mathbb{R}^2$ is a circle immersed regularly in the plane, and consider the energy

$$E[\gamma_0] = \frac{1}{2}\int_\gamma k_s^2\, ds\,,$$

where s denotes the Eulidean arc-length, $k = \langle \gamma_{ss}, \nu \rangle$ is the scalar curvature, $\tau = \gamma_s$ is the tangent vector and $\nu = (-\tau_2, \tau_1)$ the normal vector along γ.

We will take the first variation of E.

Evolution equations

Consider now a one-parameter family of curves $\gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2$ evolving with velocity

$$\partial_t \gamma = \vec{V} = V\nu + W\tau \,.$$

We find

$$\frac{d}{dt}\frac{1}{2}\int_\gamma k_s^2\,ds = \int_\gamma k_s(V_{s^3} + V_s k^2 + 3Vkk_s + k_{ss}W)\,ds + \frac{1}{2}\int_\gamma (W_s - Vk)k_s^2\,ds$$

Integration by parts yields...

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_\gamma k_s^2\,ds &= \int_\gamma V\Big[-k_{s^4} - k_{ss}k^2 - 2kk_s^2 + 3kk_s^2 - \frac{1}{2}kk_s^2\Big]\,ds\\ &= \int_\gamma V\Big[-k_{s^4} - k_{ss}k^2 + \frac{1}{2}k_s^2 k\Big]\,ds\,.\end{aligned}$$

We see that the tangential term has vanished, and can now write down the L^2-gradient.

For γ to be the steepest descent gradient flow of E in L^2, we must have $E' = -\|\gamma_t\|_2^2$, that is, we require

$$V = k_{s^4} + k_{ss}k^2 - \frac{1}{2}k_s^2 k\,.$$

We therefore have the flow:

$$\partial_t \gamma = \left(k_{s^4} + k_{ss}k^2 - \frac{1}{2}k_s^2 k\right)\nu\,. \qquad (1)$$

Note that the tangential motion is *arbitrary*.

Local existence and uniqueness

Theorem

Let $\gamma_0 : \mathbb{S} \to \mathbb{R}^2$ be a closed immersed curve of class $C^{6,\alpha}$. There exists a unique smooth maximal family $\gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2$, $T \in (0, \infty]$, of immersed curves such that $\gamma(s, 0) = \gamma_0(s)$ and $\partial_t \gamma = \left(k_{s^4} + k_{ss} k^2 - \frac{1}{2} k_s^2 k \right) \nu$. Furthermore, if $T < \infty$, then the quantity

$$Q[\gamma_t] = L[\gamma_t] + \int_\gamma k_{s^5}^2 \, ds$$

is unbounded as $t \to T$.

Closed ideal curves are standard ω-circles

Let γ be a closed curve satisfying

$$\mathcal{K}[\gamma] = k_{s^4} + k_{ss}k^2 - \frac{1}{2}k_s^2 k = 0\,, \tag{2}$$

a critical point for the energy E: an ideal curve. Let us prove:

Theorem

Suppose $\gamma : [0, L] \to \mathbb{R}^2$ is a closed curve satisfying (2). Then $\gamma([0, L]) = \mathbb{S}_r(x)$, that is, γ is a standard round ω-circle.

Proof.

Integrating the equation. Since $\mathcal{K}[\gamma] = 0$ we find that

$$\left(k_{s^3}^2 + k_{ss}^2 k^2 + \frac{1}{4} k_s^4 - k_{ss} k_s^2 k\right)_s = 2 k_{s^3} \mathcal{K} = 0\,.$$

Therefore there exists a $C \in \mathbb{R}$ such that for each $s \in [0, L]$,

$$Q(s) = k_{s^3}^2 + k_{ss}^2 k^2 + \frac{1}{4} k_s^4 - k_{ss} k_s^2 k = C\,.$$

Let $s_0 \in [0, L]$ be a point where $k_s(s_0) = 0$ (note that k is a periodic function). Then

$$Q(s_0) = (k_{s^3}^2 + k_{ss}^2 k^2)(s_0) = C \geq 0\,. \tag{3}$$

Proof ctd.

We have two cases.

Case 1: $C = 0$. In this case intergation yields

$$\int_\gamma \left[k_{s^3}^2 + k_{ss}^2 k^2 + \frac{7}{12} k_s^4\right] ds = 0\,.$$

We conclude that k_s is constant, which, together with the closedness of γ, implies the result.

Case 2: $C \neq 0$. In this case, as noted earlier $C > 0$ and we consider the rescaling $\eta = \rho\gamma$. We calculate on η

$$Q(s) = k_{s^3}^2 + k_{ss}^2 k^2 + \frac{1}{4} k_s^4 - k_{ss} k_s^2 k = C\rho^{-8}\,.$$

Proof ctd.

Choosing $\rho = C^{\frac{1}{8}}$ we find

$$Q(s) = k_{s^3}^2 + k_{ss}^2 k^2 + \frac{1}{4} k_s^4 - k_{ss} k_s^2 k = 1\,. \tag{4}$$

Now we write $Q(s) = M^2(s) + N^2(s)$ where

$$M(s) = k_{s^3} \quad \text{and} \quad N(s) = k_{ss} k - \frac{1}{2} k_s^2\,.$$

Equation (4) implies that there exists a $\phi : [0, L] \to \mathbb{R}$ such that

$$M(s) = \cos\phi(s) \quad \text{and} \quad N(s) = \sin\phi(s)\,.$$

Proof ctd.

Since

$$N_s = \phi_s \cos\phi = \phi_s M = k_{s^3} k = kM$$

we have $\phi_s = k$. Otherwise, analyticity and closedness of γ imply that k is constant, and so γ is a standard round ω-circle. Therefore, let us proceed with $\phi_s = k$. Integration yields

$$\phi(s + nL) = \phi(s) + 2\omega n\pi\,,$$

where ω is the winding number of γ.

Proof ctd.

Let $\tau(s) = (x_s, y_s)$ where $\gamma = (x, y)$. Then $x_s^2 + y_s^2 = 1$ and again we find a function $\theta : [0, L] \to \mathbb{R}$ such that

$$\tau(s) = (\cos\theta(s), \sin\theta(s)) .$$

This function θ is (up to translation) the standard notion of tangential angle. As $\tau_s = k\nu = \theta_s(s)(-\sin\theta, \cos\theta)$ and $\nu(s) = (-\sin\theta, \cos\theta)$, we must also have that $\theta_s = k$ and so

$$\theta(s) = \phi(s) + \theta_0$$

for some $\theta_0 \in \mathbb{R}$.

Proof ctd.

This implies that

$$\begin{aligned}\tau(s) &= (\cos(\phi(s)+\theta_0),\ \sin(\phi(s)+\theta_0)) \\ &= (\cos\phi(s)\cos\theta_0 - \sin\phi(s)\sin\theta_0, \\ &\qquad \sin\phi(s)\cos\theta_0 + \cos\phi(s)\sin\theta_0)\,.\end{aligned}$$

In particular

$$y_s = N(s)\cos\theta_0 + M(s)\sin\theta_0\,.$$

By closedness, we have

$$0 = \int_\gamma y_s\,ds = \cos\theta_0\int_\gamma N(s)\,ds + \sin\theta_0\int_\gamma M(s)\,ds\,.$$

Proof ctd.

Now

$$\int_\gamma M(s)\,ds = \int_\gamma k_{s^3}\,ds = 0$$

and

$$\int_\gamma N(s)\,ds = -\frac{3}{2}\int_\gamma k_s^2\,ds\,.$$

Therefore we find

$$\cos\theta_0 \int_\gamma k_s^2\,ds = 0\,.$$

Either $k_s = 0$ and we are in Case 1, or $\cos\theta_0 = 0$.

Proof ctd.

If $\cos\theta_0 = 0$, then $\sin\theta_0 \neq 0$ and we calculate

$$0 = \int_\gamma x_s\, ds = \sin\theta_0 \frac{3}{2} \int_\gamma k_s^2\, ds\,.$$

This again implies $k_s = 0$ and so, finally, we are finished. □

Main results for the planar flow

An obstacle to compactness?

The gradient flow tends to increase length. This is natural, since for $\gamma : [0, L] \to \mathbb{R}^2$ any smooth curve, the scaled curve $\gamma_\rho := \rho\gamma$ has energy

$$E[\gamma_\rho] = \frac{1}{2}\int_{\gamma_\rho} (k^\rho_{s_\rho})^2\, ds_\rho = \frac{1}{2}\int_\gamma (\rho^{-2}k_s)^2\, \rho ds = \rho^{-3}E[\gamma]\,.$$

Therefore the energy may be decreased simply by enlarging the curve.

Elastic and Ideal

This is a situation similar to that of the *elastic flow*, the L^2-gradient flow for the energy

$$\frac{1}{2}\int_{\gamma} k^2\, ds\,.$$

For that flow, there is in fact no convergence: the flow enlarges any initial curve to infinity.

The key difference with the ideal curve flow is that we just proved the flow is stationary only at *circles*, and circles have *zero energy*, so the rescaling of a circle doesn't actually change the energy at all!

Our Plan

1. The single most important thing to prove is an *a-priori length estimate*. This is because an estimate on length is the only thing needed to conclude global existence and convergence.
The claim is: We always have global existence, and convergence follows so long as we have a uniform length estimate.
The first step is to prove this claim.

2.

Our Plan

1. *We always have global existence, and convergence follows so long as we have a uniform length estimate.*

2. When can we guarantee length is bounded? We study the flow in a (geometric) neighbourhood of an ω-circle and find a length bound for this initial data.

Our Plan

1. *We always have global existence, and convergence follows so long as we have a uniform length estimate.*

2. *A-priori estimate for length under an appropriate small neighbourhood condition.*

How we establish global existence for arbitrary smooth initial data

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How we establish global existence for arbitrary smooth initial data

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$\Rightarrow$ Prove that all derivatives of curvature do not become unbounded in finite time.

$\Rightarrow$ Prove that $T = \infty$.

$\Rightarrow$ Prove that there exists a $t_0 = t_0(\gamma_0)$ such that for $t > t_0$ the curve $\gamma(\cdot, t)$ has scale-invariant energy smaller than ε_2.

How we preserve scale-invariant smallness and establish exponentially fast smooth convergence

2. *A-priori estimate for length under an appropriate small neighbourhood condition.* ... and convergence for scale-invariant energy smaller than a universal $\varepsilon_2 = \varepsilon_2(\gamma_0)$.

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2. *A-priori estimate for length under an appropriate small neighbourhood condition.* ... and convergence for scale-invariant energy smaller than a universal $\varepsilon_2 = \varepsilon_2(\gamma_0)$.

$\Rightarrow$ Prove an estimate for the Euler-Lagrange operator in L^2 from below in terms of the scale-invariant energy.

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$\Rightarrow$ Prove that smallness of the scale-invariant energy is preserved along the flow.

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$\Rightarrow$ Prove that length is bounded a-priori.

$\Rightarrow$ Prove **exponential decay** of energy.

$\Rightarrow$ Prove uniform lower bound for length, decay of derivatives of curvature, and full convergence to an ω-circle.

Theorem (AMWW '18)

Let $\gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2$ be the steepest descent L^2-gradient flow for the functional $E[\gamma] = \frac{1}{2}\int_\gamma k_s^2\, ds$, where $\gamma(\cdot, 0) = \gamma_0(\cdot)$ is smooth and T is maximal, $T \in (0, \infty]$. Then:

(a) *The maximal time of existence is infinite ($T = \infty$); and*

(b) *If the length $L[\gamma_t]$ is uniformly bounded along the flow, then γ converges exponentially fast in the C^∞-topology to a standard round ω-circle, where $\omega = \frac{1}{2\pi}\int_{\gamma_0} k_0\, ds_0$.*

Theorem (AMWW '18 ctd)

Furthermore, there exists a universal constant $\varepsilon_2 > 0$ *such that if* $\gamma_0 : \mathbb{S} \to \mathbb{R}^2$ *satisfies*

$$(L^3 E)[\gamma_0] < \varepsilon_2 \,,$$

then length is uniformly bounded along the flow, and the convergence statement from part (b) above holds.

What's next?

Our main goal is the next most simple case: surfaces in $\mathbb{R}^3$.

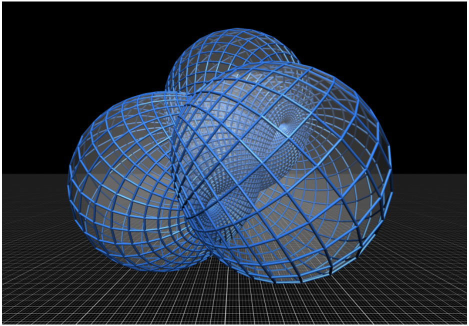

Image from Felix Knöppel.

Thank you for your attention!