Stability, Finiteness and Dimension 4

Joint work in progress with Curtis Pro

February 5, 2019

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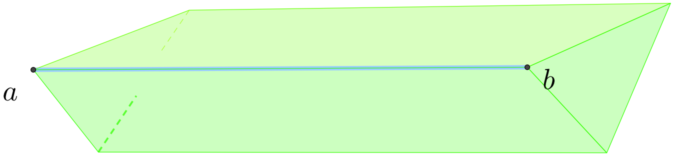
In contrast, the limit spaces of our class can have singular sets of dimensions 0, 1 and 2.

Moreover, the 1 and 2 dimensional singular sets can bifurcate.

Singular Set



In the double of this convex subset, the interior of the segment *ab* is a 1–dimensional singular set (blue points).

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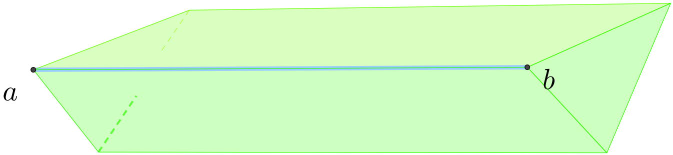

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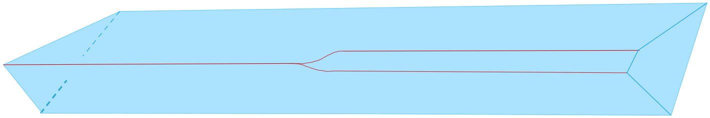
Bifurcating Singular Set



The red points in the double of this convex set are a bifurcated 1–dimensional singular set.

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- **Perelman's Framed Cover —> Handle Stability Lemma**.

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Perelman's Covering Lemma: Given $\varkappa > 0$ there is a $\varkappa$-framed cover $\mathcal{P} = \mathcal{P}^0 \cup \mathcal{P}^1 \cup \mathcal{P}^2$ of $X \setminus X^4_\varkappa$ so that the elements of $\mathcal{P}^0$ are disjoint.

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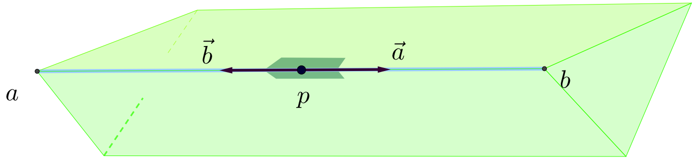
$X^4_\varkappa$ is the set of points in X that have $(4, \varkappa)$–framed neighs.

1-Framed Neighborhood



$\mathrm{dist}_a(\cdot)$ gives a 1–framing of the dark green neigh. of p.

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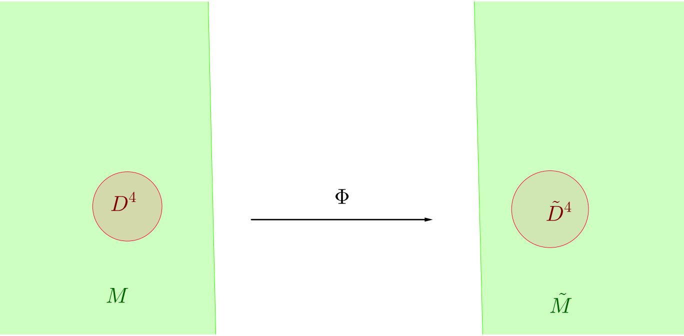
The Ugly–the way that Perelman's framed sets can intersect each other.

Cerf's Extension Theorem



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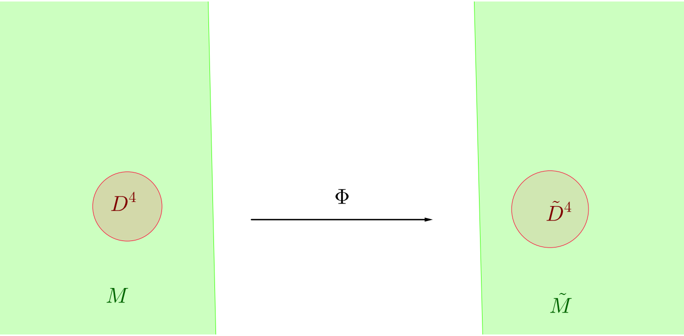

Figure: Cerf's Extension Theorem

If Φ : *closure* $\left(M^4 \setminus D^4\right) \longrightarrow$ *closure* $\left(\tilde{M}^4 \setminus \tilde{D}^4\right)$ is a diffeo so that $\Phi\left(\partial D^4\right) = \partial \tilde{D}^4$,

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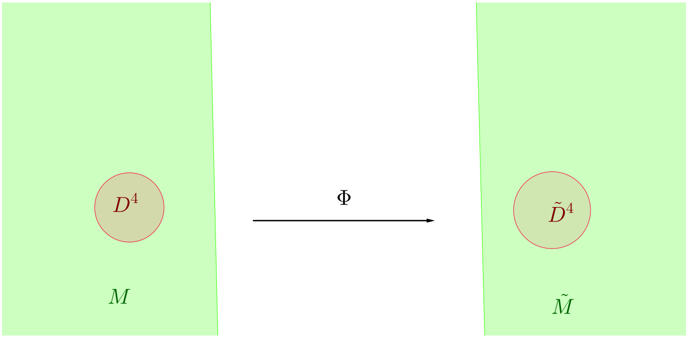
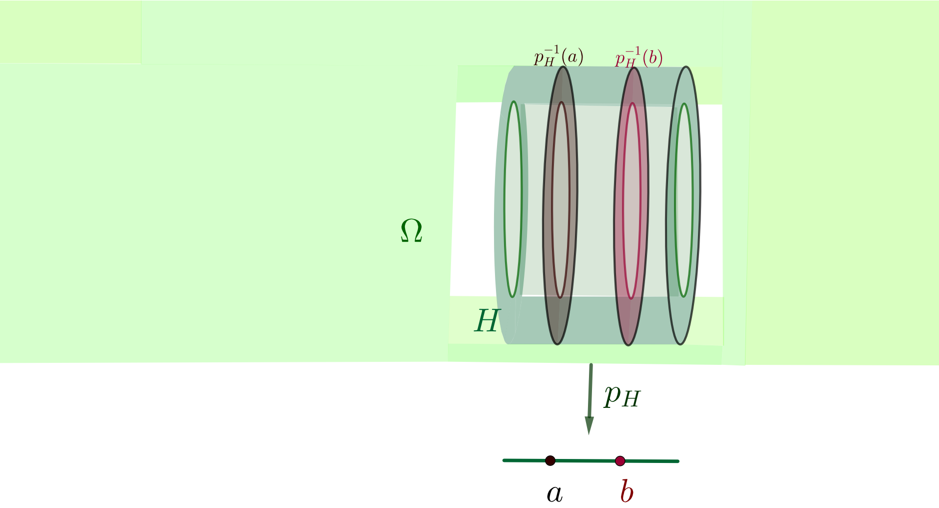


Figure: Cerf's Extension Theorem

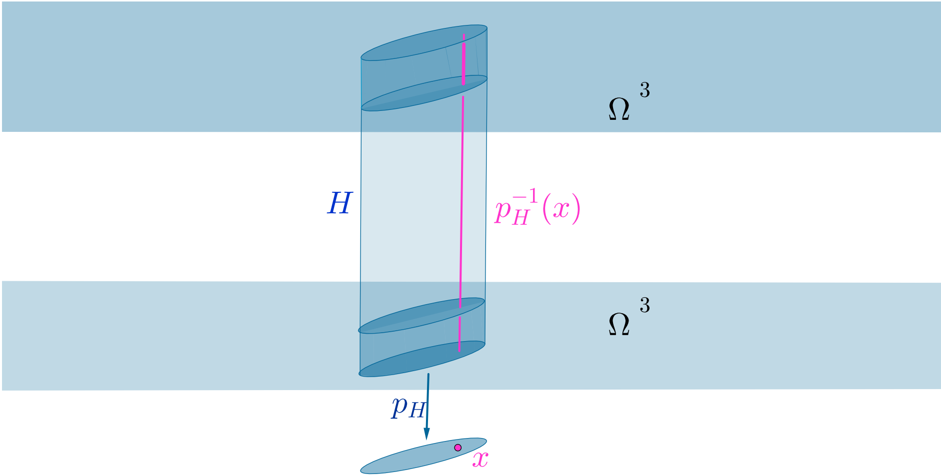
If $\Phi : closure\left(M^4 \setminus D^4\right) \longrightarrow closure\left(\tilde{M}^4 \setminus \tilde{D}^4\right)$ is a diffeo so that $\Phi\left(\partial D^4\right) = \partial \tilde{D}^4$, then Φ extends to a diffeomorphism $M \longrightarrow \tilde{M}$.

A 2–handle in a three dim space



Attached along fiber boundaries.

A 1–handle in a three dim Space



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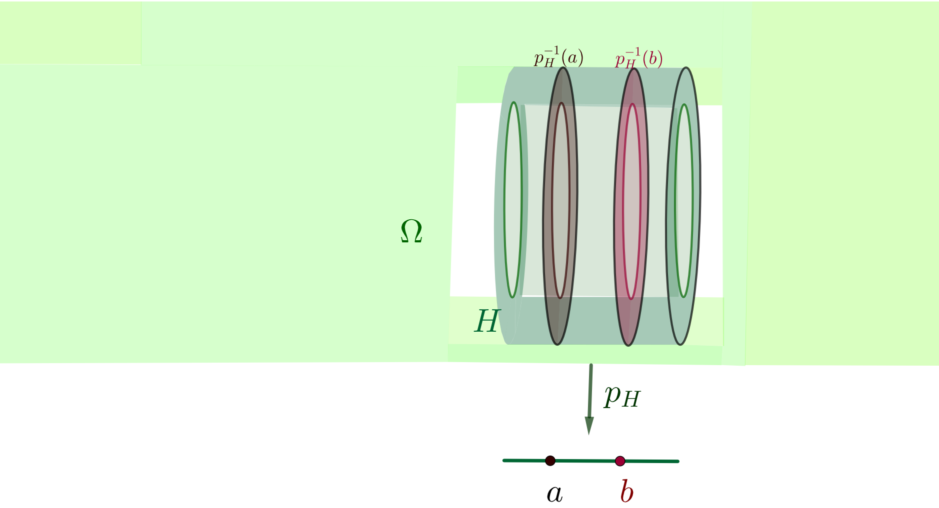

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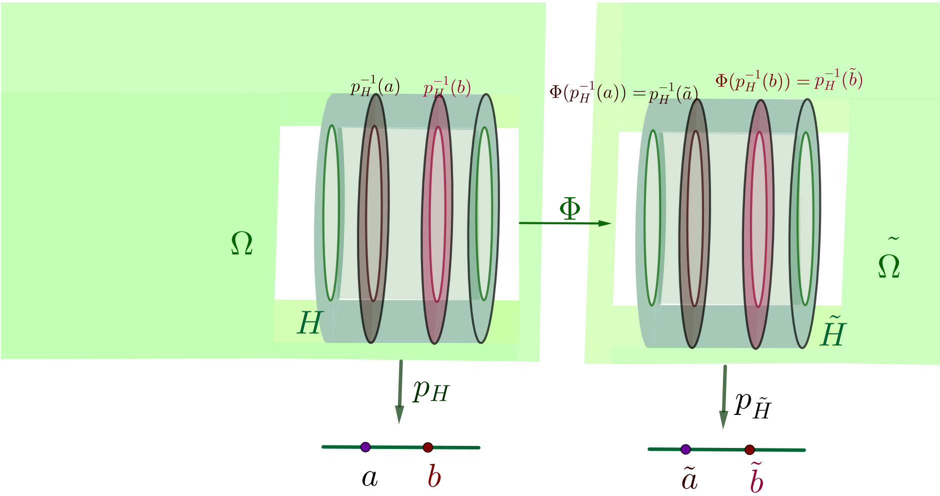
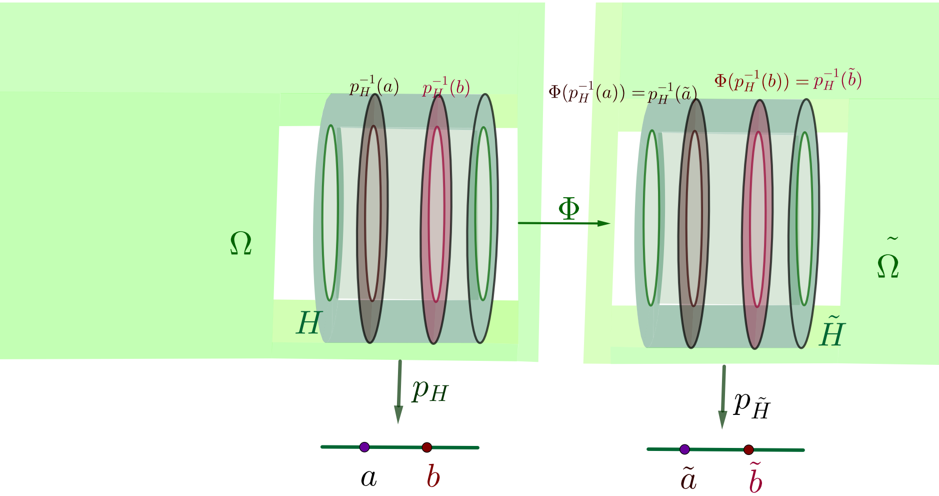


Smale's Extension Theorem: M^4, $\tilde{M}^4$ are $\Omega^4, \tilde{\Omega}^4$ with k–handles (H, p_H), $\left(\tilde{H}, p_{\tilde{H}}\right)$ attached, $k = 2, 3$.



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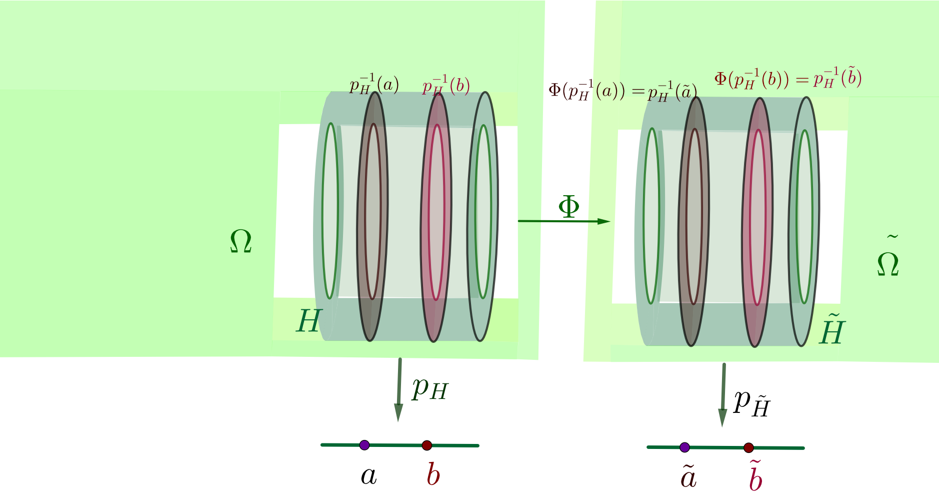

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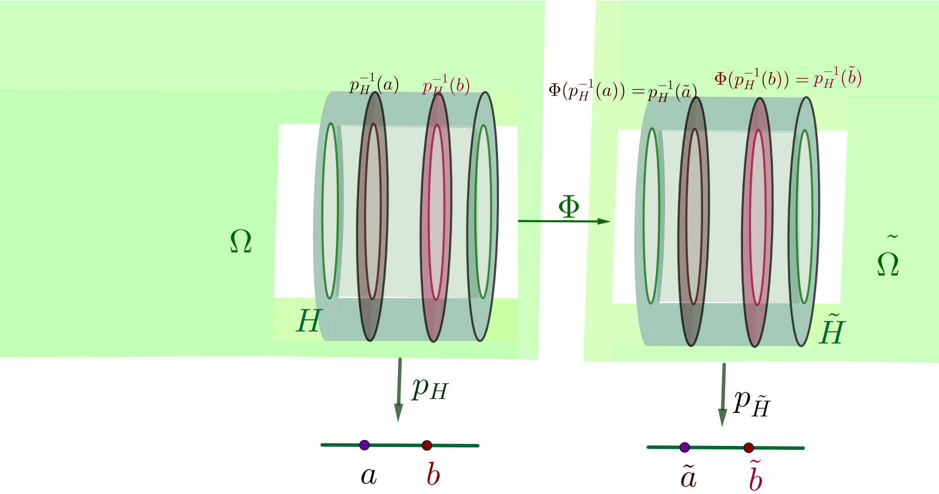

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E.g. if $(H, p_H) \in \mathcal{H}^2$, then $\partial p_H^{-1}(p_H(x)) \subset X_{\varkappa}^4$ for all $x \in H$.

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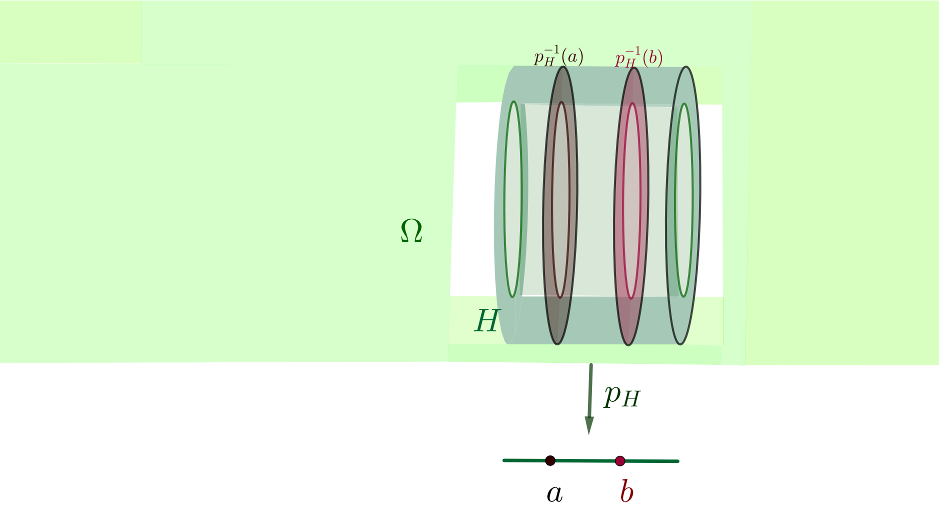
Proper Frame



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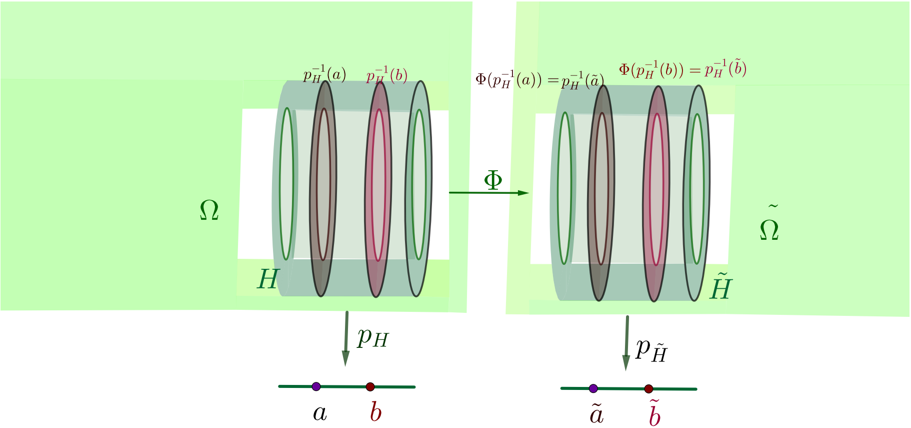
$$p_{H_\beta} \circ \Phi_{\beta,\alpha} = p_{H_\alpha},$$

and

$$r_{H_\beta} \circ \Phi_{\beta,\alpha} = r_{H_\alpha},$$

where r_{H_α} and r_{H_β} are fiber exhaustion functions for H_α and H_β, respectively.

Extension to Two Framed Sets



Smale Ext. thm, **proper,** and **KMS Diffeo respects** $\mathcal{H}$

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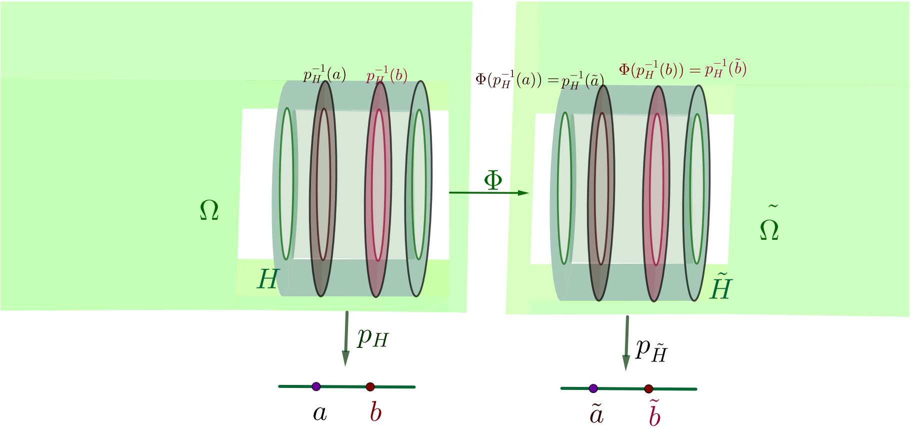

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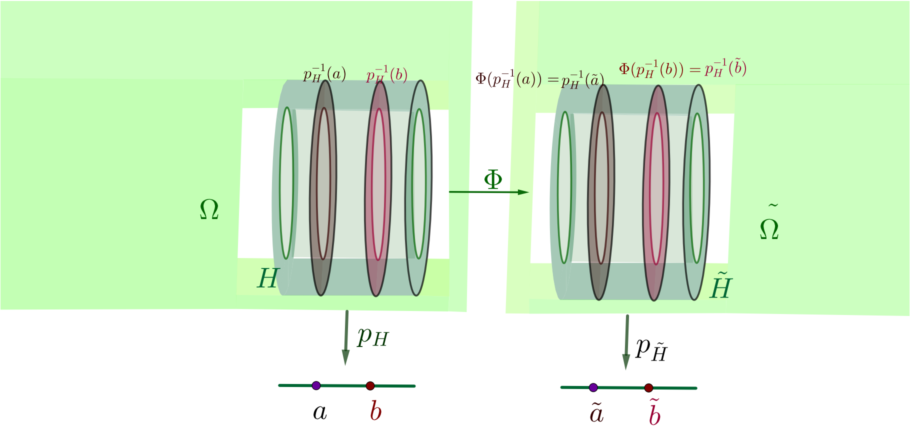

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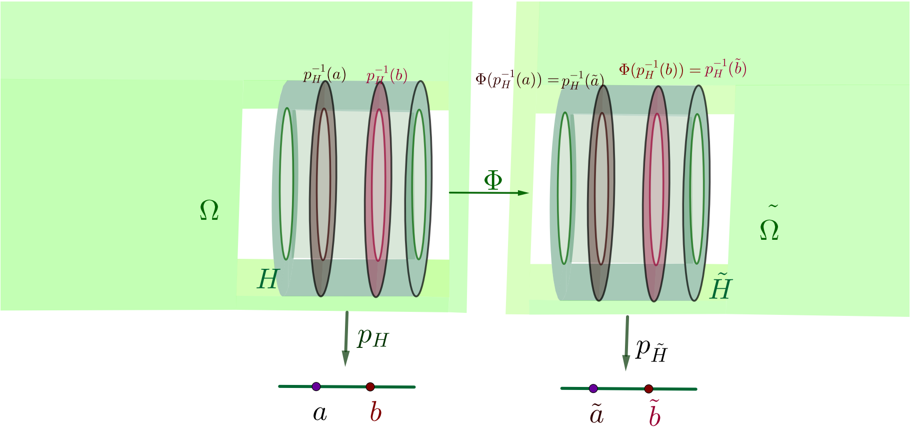

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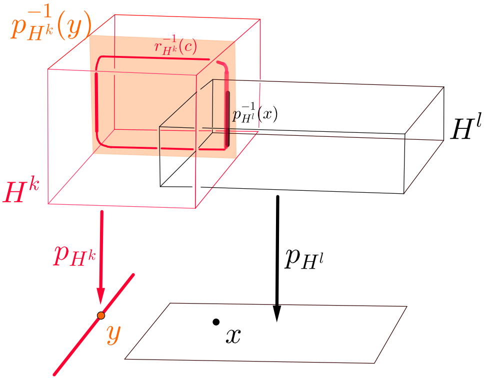
Extension to 1 and 0 Framed Sets

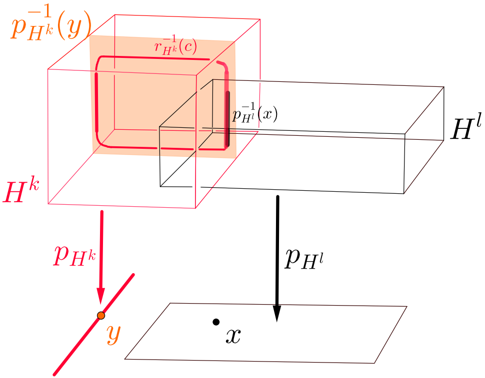
To continue the extension process to the 1 and 0 framed sets we establish one extra condition: Our framed cover is **respectful**.

Respectful Framed Sets



The framed sets H^k and H^l intersect respectfully,

Respectful Framed Sets



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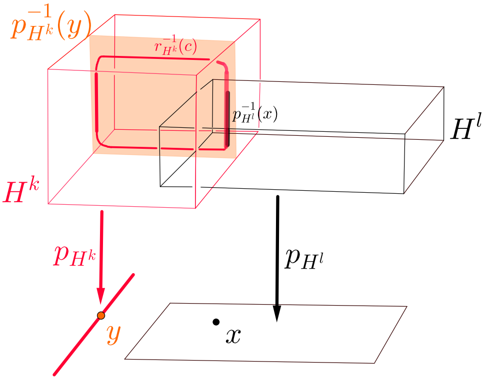

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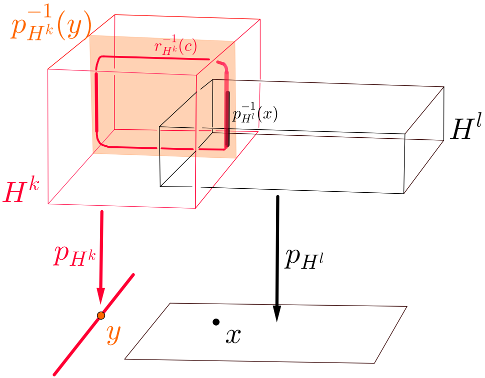

The framed sets H^k and H^l intersect respectfully, because the fibers of p_{H^l} that intersect H^k are contained in both the fibers of p_{H^k} and level sets of the fiber exhaustion function r_{H^k} of H^k.

Recap

Perelman's Covering Lemma: Given $\varkappa > 0$ there is a proper $\varkappa$-framed cover $\mathcal{P} = \mathcal{P}^0 \cup \mathcal{P}^1 \cup \mathcal{P}^2$ of $X \setminus X^4_\varkappa$ so that the elements of $\mathcal{P}^0$ are disjoint.

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Hard Task 2: The handle respect condition.

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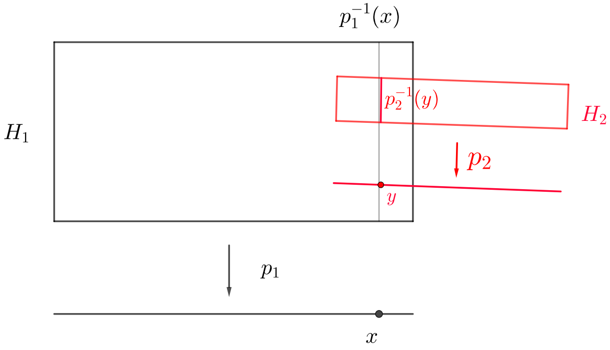
Improving Perelman's Cover: Kappa-Lined Up



Framed sets are κ–lined up when their fibers are almost parallel.

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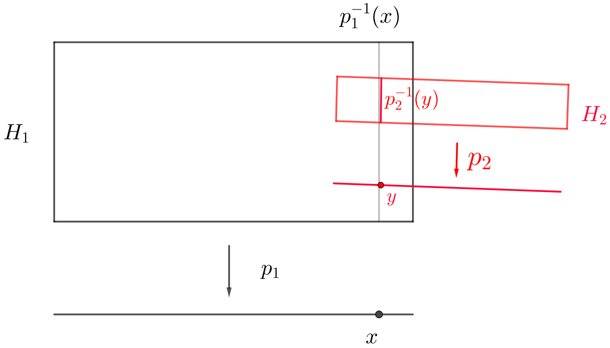
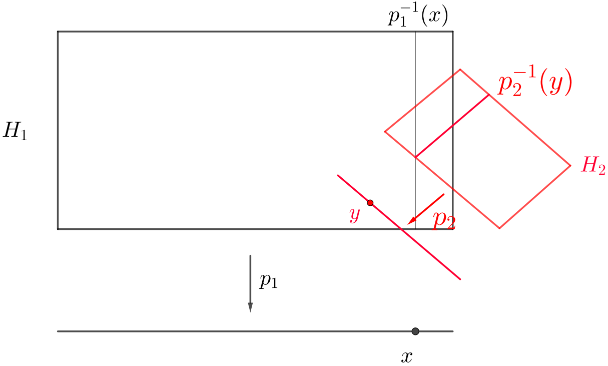


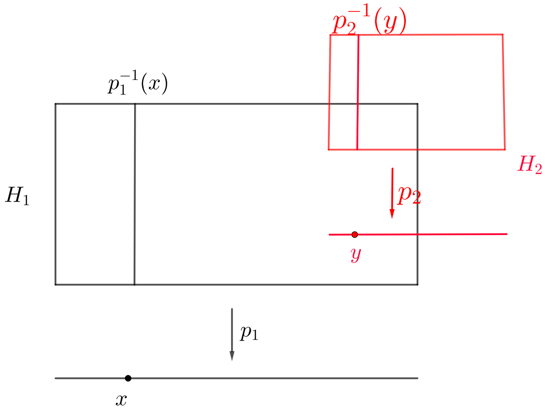
Framed sets are κ–lined up when their fibers are almost parallel. Formally: $|dp_1 - I \circ dp_2| < \kappa$, where $I : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ is an isometry.

Not Kappa-Lined Up



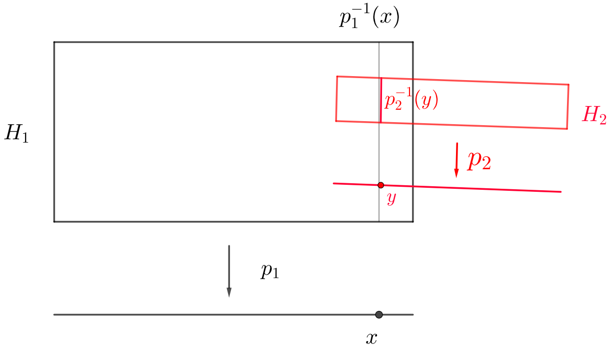
These framed sets are NOT lined up since their fibers are not almost parallel.

Not Fiber Swallowing



These framed sets are κ–lined up, but not fiber swallowing. The fibers of the little set that intersect the big set are not almost contained in the big set.

Kappa-Lined Up and Fiber Swallowing



This intersection is κ–lined up and fiber swallowing.

Lined Up and Fiber Swallowing Lemma

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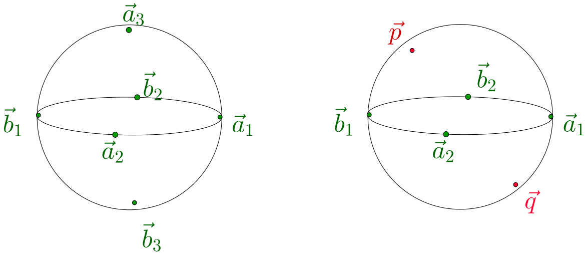
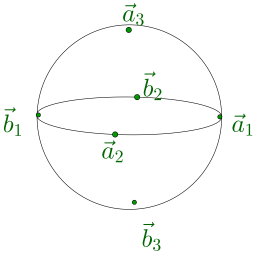
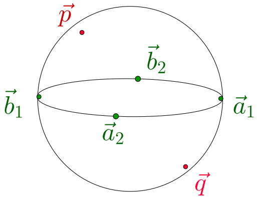
Tool for Lining: Up Plaut's Characteriz. of Rnd. Sphs.



$\left\{ \left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right), \left(\vec{a}_3, \vec{b}_3\right) \right\}$ is a Global Strainer.





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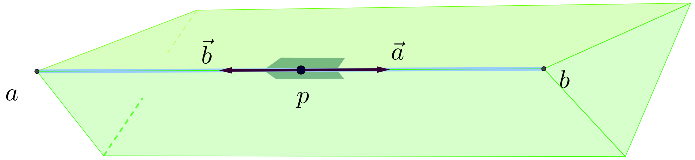
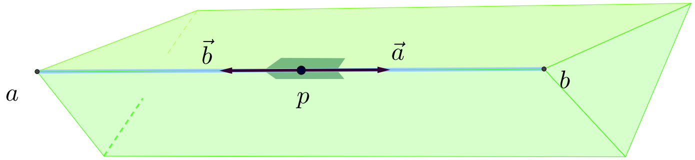
$\left\{ \left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right), (\vec{p}, \vec{q}) \right\}$ satisfies Plaut's condition.

1-Framed Neighborhood



$\mathrm{dist}_a(\cdot)$ gives a 1–framing of the dark green neigh. of p.

1-Framed Neighborhood



$\mathrm{dist}_a(\cdot)$ gives a 1–framing of the dark green neigh. of p. $\left\{\vec{a}, \vec{b}\right\}$ is a pair of vectors in Σ_p with angle π.

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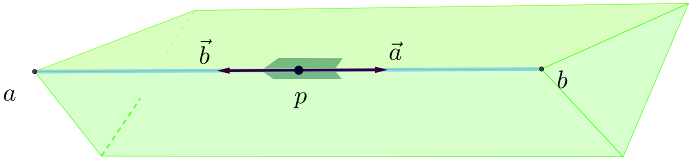
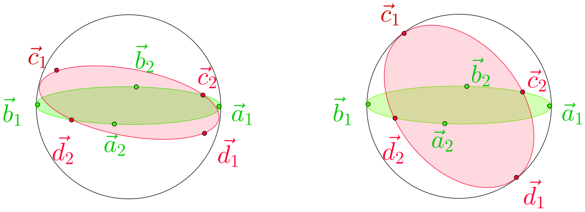
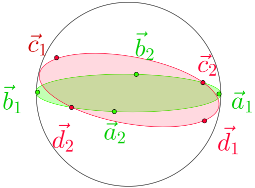
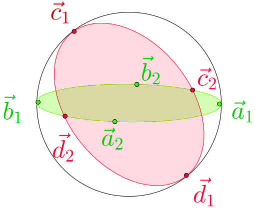


$\operatorname{dist}_a(\cdot)$ gives a 1–framing of the dark green neigh. of p. $\left\{\vec{a}, \vec{b}\right\}$ is a pair of vectors in Σ_p with angle π. $\vec{a} = \nabla \operatorname{dist}_a(\cdot)\big|_p$.



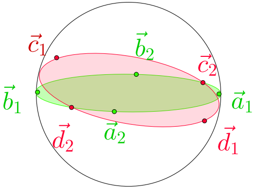
Directions for 2 sets of strainers, $\left\{\left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right)\right\}$ and $\left\{\left(\vec{c}_1, \vec{d}_1\right), \left(\vec{c}_2, \vec{d}_2\right)\right\}$, in an unknown space of directions Σ_p.

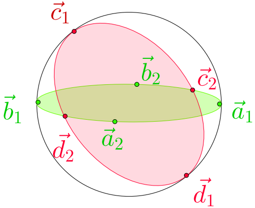




Directions for 2 sets of strainers, $\left\{\left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right)\right\}$ and $\left\{\left(\vec{c}_1, \vec{d}_1\right), \left(\vec{c}_2, \vec{d}_2\right)\right\}$, in an unknown space of directions Σ_p. On the left, the corresponding submersions are κ–lined up.

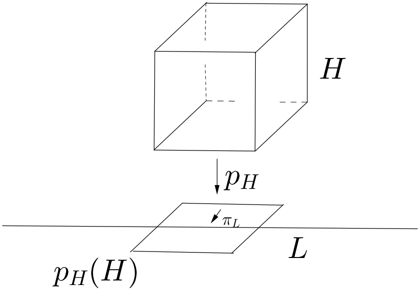
Tool for Lining: Up Plaut's Characteriz. of Rnd. Sphs





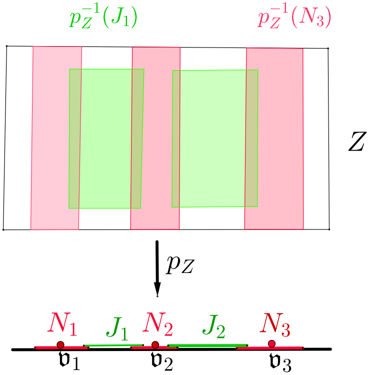
Directions for 2 sets of strainers, $\left\{\left(\vec{a}_1, \vec{b}_1\right), \left(\vec{a}_2, \vec{b}_2\right)\right\}$ and $\left\{\left(\vec{c}_1, \vec{d}_1\right), \left(\vec{c}_2, \vec{d}_2\right)\right\}$, in an unknown space of directions Σ_p. On the left, the corresponding submersions are κ–lined up. On the right, the submersions are not κ–lined up so p is actually 3—strained.

Tool for Disjointness: Artificial Framing

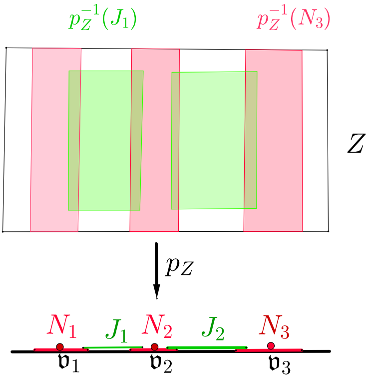


(H, p_H) is 2–framed. $(H, \pi_L \circ p_H)$ is **artificially** 1–framed.

Disjoint 1–Framed sets via extra 0–framed sets

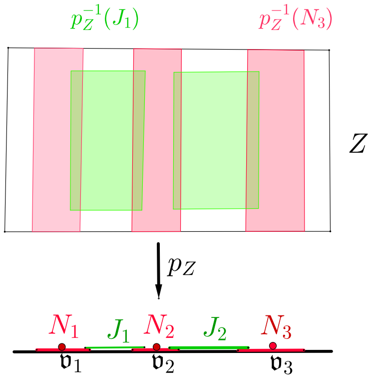


Graph —> Graph Cover —> Cograph



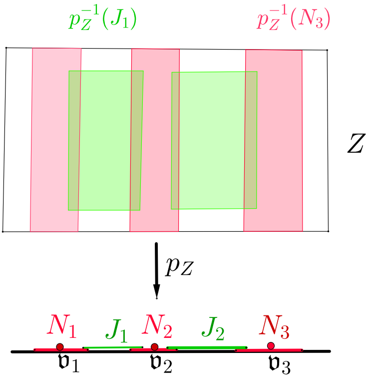
(Z, p_Z) is 1–framed.

Graph —> Graph Cover —> Cograph



(Z, p_Z) is 1–framed. View $\{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3\} \subset p_Z(Z)$ as vertices of a graph Γ.

Graph —> Graph Cover —> Cograph



(Z, p_Z) is 1–framed. View $\{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3\} \subset p_Z(Z)$ as vertices of a graph Γ. $\{N_i, J_j\}_{i,j}$ is a graph cover of Γ, and

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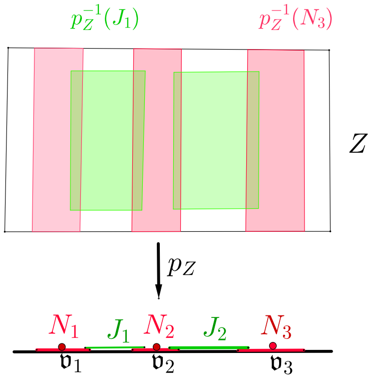
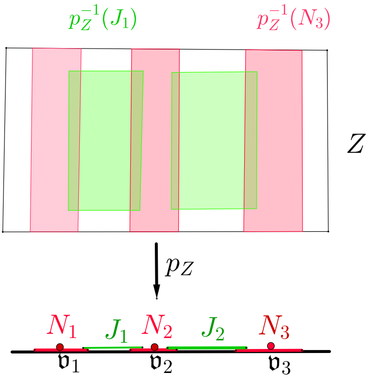


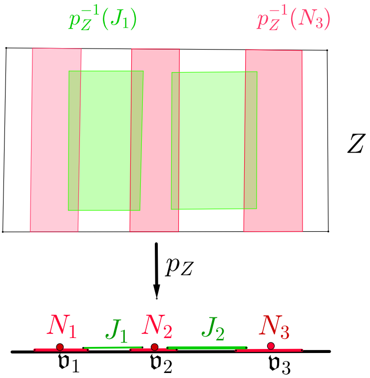
(Z, p_Z) is 1–framed. View $\{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3\} \subset p_Z(Z)$ as vertices of a graph Γ. $\{N_i, J_j\}_{i,j}$ is a graph cover of Γ, and $\{p^{-1}(N_i), p^{-1}(J_j)\}_{i,j}$ is a cograph.

Cograph Separates 1–Framed with extra 0–framed sets



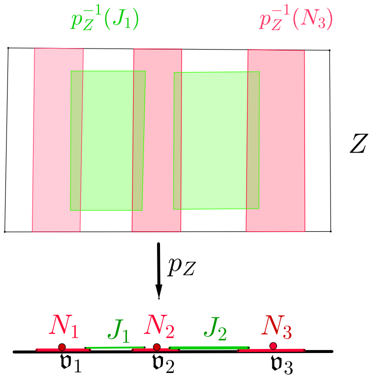
The $\left\{p^{-1}\left(N_i\right),\right\}_i$ and $\left\{p^{-1}\left(J_j\right)\right\}_j$ are each disjoint collections.

Cograph Separates 1–Framed with extra 0–framed sets



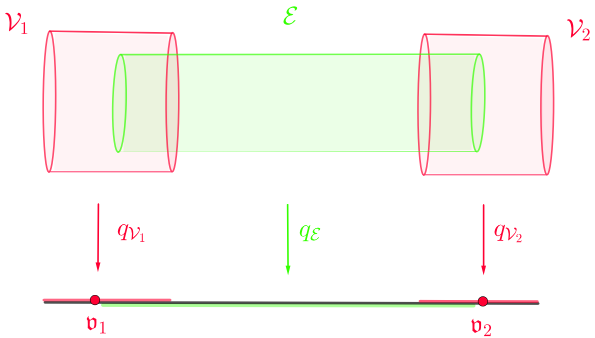
The $\left\{p^{-1}\left(N_i\right),\right\}_i$ and $\left\{p^{-1}\left(J_j\right)\right\}_j$ are each disjoint collections. View the $p^{-1}\left(N_i\right)$ as artificially 0–framed and

Cograph Separates 1–Framed with extra 0–framed sets



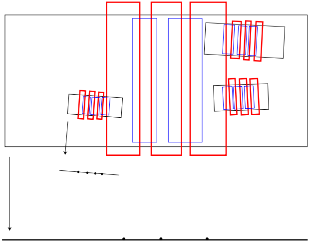
The $\{p^{-1}(N_i),\}_i$ and $\{p^{-1}(J_j)\}_j$ are each disjoint collections. View the $p^{-1}(N_i)$ as artificially 0–framed and the $p^{-1}(J_j)$ as artificially 1–framed.

Disjoint 1–Framed sets via extra 0–framed sets



Simplest possible cograph in a dim 3. Graph has two vertices $\mathfrak{v}_1$ and $\mathfrak{v}_2$. Green cylinder is called a coedge. Red cylinders are called covertices. The 3 submersions, $q_{\mathcal{V}_1}$, $q_{\mathcal{E},}$, $q_{\mathcal{V}_2}$, are κ–lined up, so on this scale their fibers appear to parallel.

Fitting The Local Cographs Together



Three 1–framed sets with subordinate cographs.

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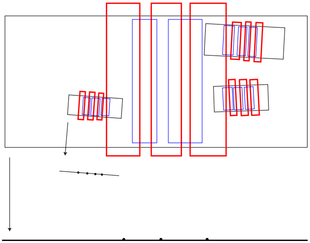
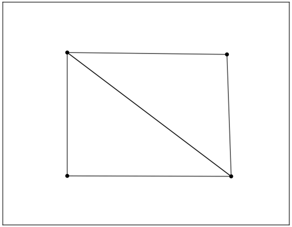
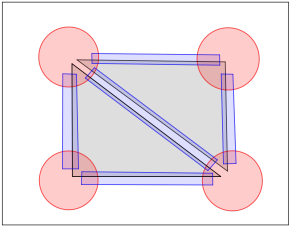


Three 1–framed sets with subordinate cographs. We must fit together cographs from 1–framed sets.

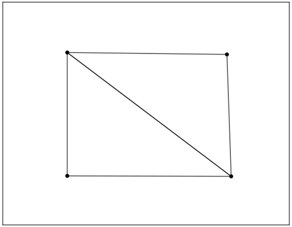
Triangulation —> Triangulation Cover

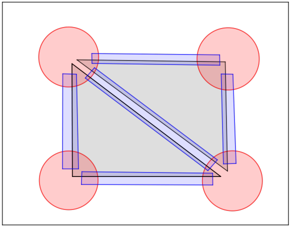




Disjoint neighs. of vertices (red),

Triangulation —> Triangulation Cover





Disjoint neighs. of vertices (red),
Disj. neighs of shrunken edges (blue),

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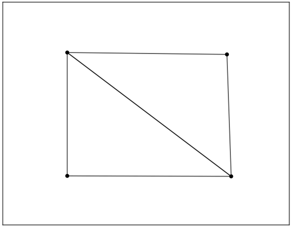
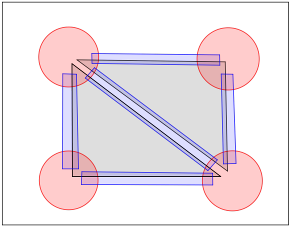
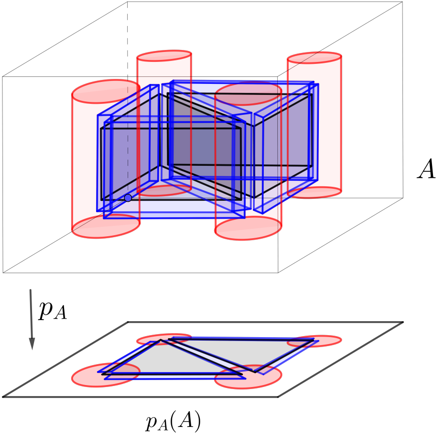




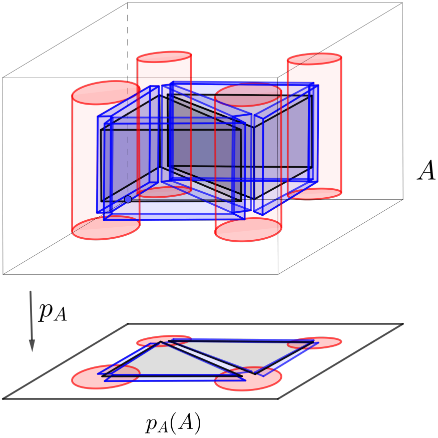
Disjoint neighs. of vertices (red),
Disj. neighs of shrunken edges (blue),
Disj shrunken faces (black)

Triangulation Cover —> Cotriangulation



Vertices —> Disj. Covertices (red)

Triangulation Cover —> Cotriangulation



Vertices —> Disj. Covertices (red)
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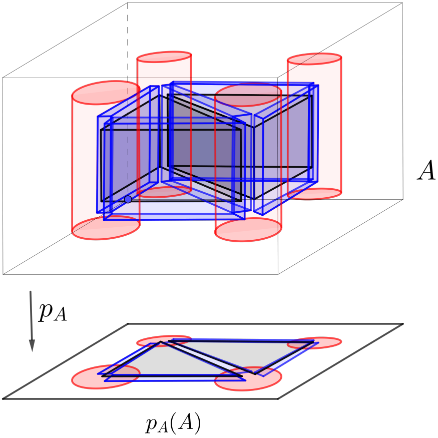
Triangulation Cover —> Cotriangulation



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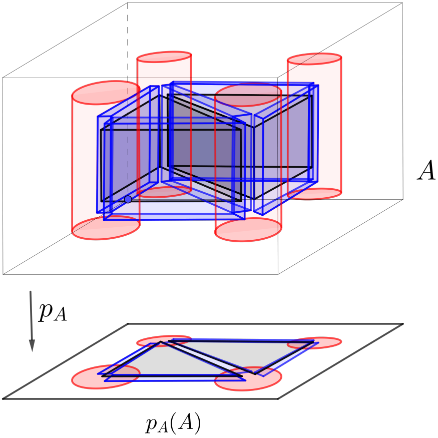

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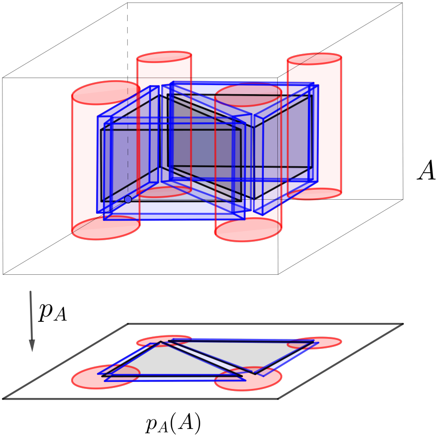

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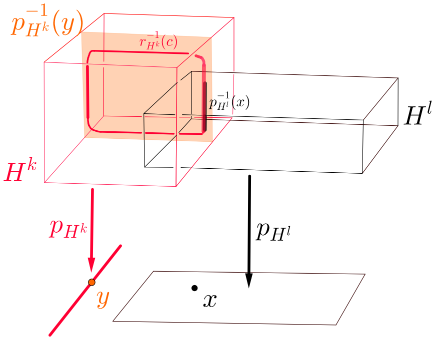
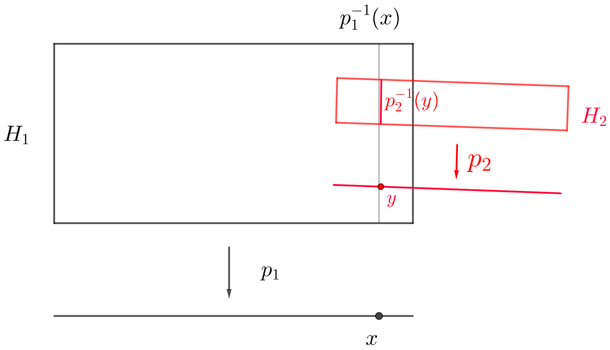
Establishing the Handle Respect Condition



The framed sets H^k and H^l intersect respectfully, because the fibers of p_{H^l} that intersect H^k are contained in both the fibers of p_{H^k} and level sets of the fiber exhaustion function

Strategy to get Handle Respect Condition



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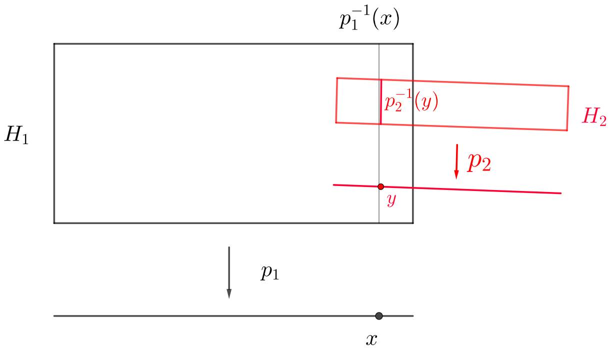
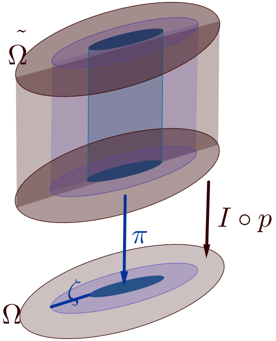


Combine the κ–lined up condition with the fact that a space of submersions is locally contractible in the C^1–topology to glue the submersions of neighboring framed sets

Gluing Kappa Lined Up Submersions



If p and and π are κ–lined up, they can be glued, if $\kappa \left(1 + \frac{\mathrm{diam}(\tilde{\Omega})}{\zeta}\right)$ is small.

Submersion Deformation Lemma

Let $W \Subset U \Subset \Omega \subset \mathbb{R}^k$ be three nonempty, open, pre-compact s.t.

$$\mathrm{dist}(W, \Omega \setminus U) > \zeta.$$

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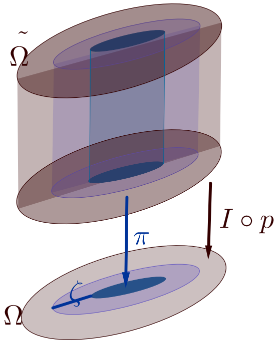

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$$\psi = \begin{cases} \pi & \text{on } \pi^{-1}(W) \\ I \circ p & \text{on } \tilde{\Omega} \setminus \pi^{-1}(U). \end{cases}$$

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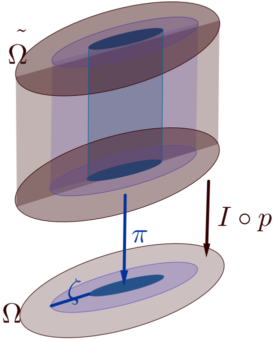
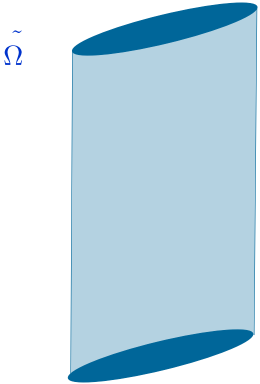
Moreover, ψ is $\kappa\left(1 + 2\frac{\mathrm{diam}(\tilde{\Omega})}{\zeta}\right)$–lined up with p, π

Gluing Kappa Lined Up Submersions



If p and and π are κ–lined up, they can be glued, if $\kappa\left(1+\frac{\operatorname{diam}(\tilde{\Omega})}{\zeta}\right)$ is small.

No control on Diam(~Omega)/zeta



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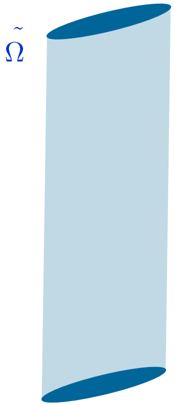

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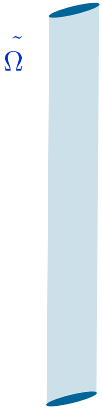

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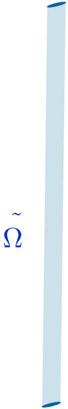
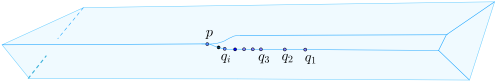
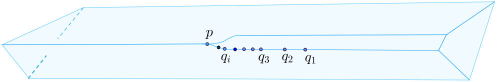


Tricky Analysis Because of Bifurcations



Imagine constructing a Perelman Framed set at $q_i \longrightarrow p$.

Tricky Analysis Because of Bifurcations



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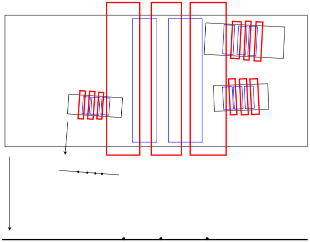

Bifurcating 1–strata create configurations like this one. That can iterate an unknown number of times at unknown scales.

Tricky Analysis Because of Bifurcations

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Tricky Analysis Because of Bifurcations

There are several reasons why the analysis required to do this is tricky:

1. While the quantity κ of "κ–lined up" can be arbitrarily small, it is a priori in the sense that we must choose it before choosing the Perelman cover.
2. We do not know how to control the relative sizes of the Perelman handles, or on the height to width ratios of our handles.

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4. Instead we impose geometric constraints on the geometry of the triangulations that generate our cotriangulations.
5. We call these special Triangulations CDGs, for Chew–Delaunay–Gromov.

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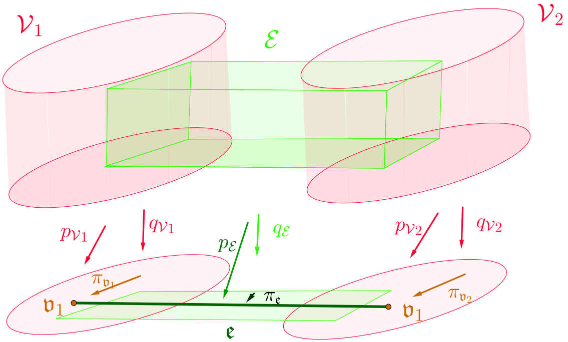
3. CDGs can always be extended.
4. (Pro-W.) There is always a way to subdivide a CDG and get a CDG.

Advantages of Chew's Constraints



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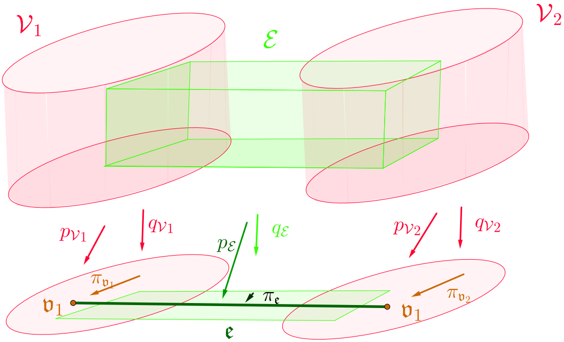

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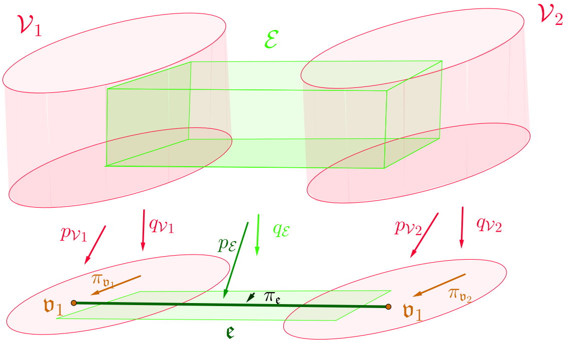

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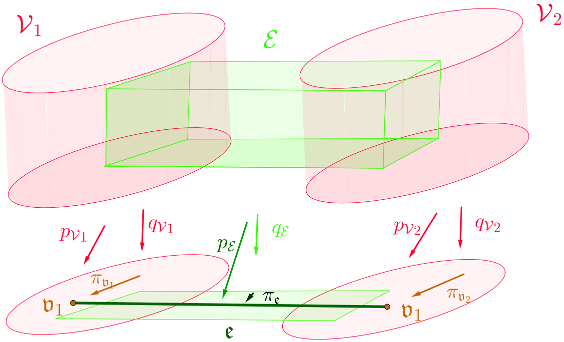

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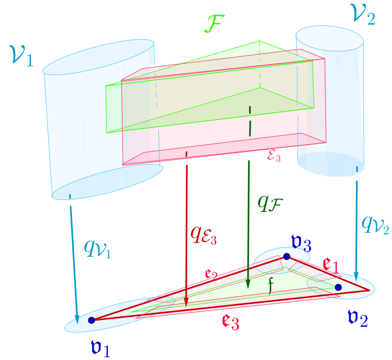

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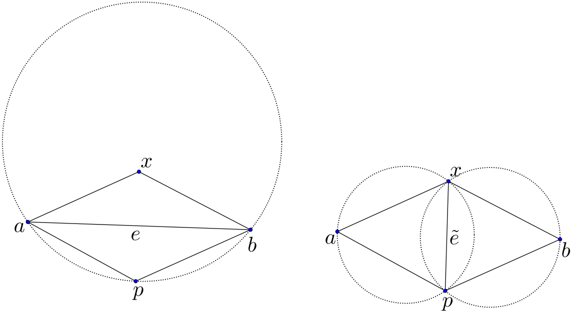

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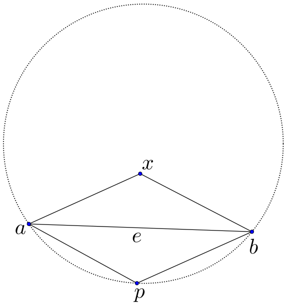

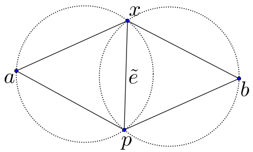

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