Subdivided Claws and the Clique-Stable Set Separation Property

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Abstract Let $\mathscr{C}$ be a class of graphs closed under taking induced subgraphs. We say that $\mathscr{C}$ has the *clique-stable set separation property* if there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathscr{C}$ there is a collection $\mathscr{P}$ of partitions (X,Y) of the vertex set of G with $|\mathscr{P}| \leq |V(G)|^c$ and with the following property: if K is a clique of G, and S is a stable set of G, and $K \cap S = \emptyset$, then there is $(X,Y) \in \mathscr{P}$ with $K \subseteq X$ and $S \subseteq Y$. In 1991 M. Yannakakis conjectured that the class of all graphs has the clique-stable set separation property, but this conjecture was disproved by M. Göös in 2014. Therefore it is now of interest to understand for which classes of graphs such a constant c exists. In this paper we define two infinite families $\mathscr{S}, \mathscr{K}$ of graphs and show that for every $S \in \mathscr{S}$ and $K \in \mathscr{K}$, the class of graphs with no induced subgraph isomorphic to S or K has the clique-stable set separation property.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. A *clique* in G is a set of pairwise adjacent vertices, and a *stable set* is a set of pairwise non-adjacent vertices. Let $\mathscr{C}$ be a class of graphs closed under taking induced subgraphs. We say that $\mathscr{C}$ has the *clique-stable set separation property* if there exists $c \in \mathbb{N}$ such that for every graph $G \in \mathscr{C}$ there is a collection $\mathscr{P}$ of partitions (X,Y) of the vertex set of G with $|\mathscr{P}| \leq |V(G)|^c$ and with the following property: if K is a clique of G, and S is a stable set of G, and $K \cap S = \emptyset$, then there is $(X,Y) \in \mathscr{P}$ with $K \subseteq$

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X and $S \subseteq Y$. This property plays an important role in a large variety of fields: communication complexity, combinatorial optimization, constraint satisfaction and others (for a comprehensive survey of these connections see [3]).

In 1991 Mihalis Yannakakis conjectured that the class of all graphs has the clique-stable set separation property [5], but this conjecture was disproved by Mika Göös in 2014 [2]. Therefore it is now of interest to understand for which classes of graphs such a constant c exists; our main result falls into that category.

Let G be a graph and let X,Y be disjoint subsets of $V(G)$. We denote by $G[X]$ the subgraph of G induced by X, by $N(X)$ the set of all vertices of $V(G) \setminus X$ with a neighbor in X, and by $N[X]$ the set $N(X) \cup X$. We say that X is *complete* to Y if every vertex of X is adjacent to every vertex of Y, and that X is *anticomplete* to Y if every vertex of X is non-adjacent to every vertex of Y. We say that X and Y are *matched* if every vertex of X has exactly one neighbor in Y, and every vertex of Y has exactly one neighbor in X (and therefore $|X| = |Y|$). For a graph H, we say that G is H*-free* if no induced subgraph of G is isomorphic to H.

Next we define two types of graphs. Let $p,q \in \mathbb{N}$. We define the graph $F_S^{p,q}$ as follows:

- $V(F_S^{p,q}) = K \cup S_1 \cup S_2 \cup S_3$ where K is a clique, S_1, S_2, S_3 are stable sets, and the sets K, S_1, S_2, S_3 are pairwise disjoint;
- $|K| = |S_1| = p$, and K and S_1 are matched;
- $|S_2| = |S_3| = q$, and S_2 and S_3 are matched;
- K is complete to S_2;
- there are no other edges in $F_S^{p,q}$.

The graph $F_K^{p,q}$ is obtained from $F_S^{p,q}$ by making all pairs of vertices of S_3 adjacent.

Fig. 1 The graphs $F_S^{3,3}$ and $F_K^{3,3}$

Let $\mathscr{F}^{p,q}$ be the class of all graphs that are both $F_S^{p,q}$-free and $F_K^{p,q}$-free. We can now state our main result:

Theorem 1. *For all $p,q > 0$ the class $\mathscr{F}^{p,q}$ has the clique-stable set separation property.*

Since the clique-stable set separation property is preserved under taking complements, we immediately deduce:

Theorem 2. *For all $p,q>0$ the class of graphs whose complements are in $\mathscr{F}^{p,q}$ has the clique-stable set separation property.*

2 The Proof

In this section we prove 1. The idea of the proof comes from [1]. Let $G \in \mathscr{F}^{p,q}$. Define $\mathscr{P}_1$ to be the set of all partitions $(N[X], V(G) \setminus N[X])$ and $(N(X), V(G) \setminus N(X))$ where X is a subset of $V(G)$ with $|X| < p$. Clearly $|\mathscr{P}_1| \leq 2|V(G)|^p$.

Write $R = R(q,q)$ to mean the smallest positive integer R such that every 2-coloring of the edges of the complete graph on R vertices contains a monochromatic complete graph on q vertices. Ramsey's Theorem [4] implies:

Theorem 3. $R(q,q) \leq 2^{2q}$.

For $a,b \in \mathbb{N}$ let the graph $F_{a,b}$ be defined as follows:

- $V(F_{a,b}) = K_1 \cup S_1 \cup S_2 \cup W$ where K_1 is a clique, S_1, S_2 are stable sets, and the sets K_1, S_1, S_2, W are pairwise disjoint;
- $|K_1| = |S_1| = a$, and K_1 and S_1 are matched;
- $|S_2| = |W| = b$, and S_2 and W are matched;
- K_1 is complete to S_2;
- there is no restriction on the adjacency of pairs of vertices of W;
- there are no other edges in $F_{a,b}$.

From the definition of R we immediately deduce:

Theorem 4. *G is $F_{p,R}$-free.*

For every triple $X = (K_1, S_1, S_2)$ of pairwise disjoint non-emtpy subsets of $V(G)$ such that $|K_1| = |S_1| = p$ and $|S_2| < R$ we define the partition P_X of $V(G)$ as follows. Let Z be the set of all vertices of G that are anticomplete to $K_1 \cup S_1$. Let A_X be the set of all vertices v of G such that

- either $v \in K_1$, or v is complete to K_1, and
- either v has a neighbor in S_1, or v has a neighbor in $Z \setminus N(S_2)$.

Note that, since S_1 is a stable set and Z is anticomplete to S_1, A_X is disjoint from $S_1 \cup Z$. Define $P_X = (A_X, V(G) \setminus A_X)$, and let $\mathscr{P}_2$ be the set of all such partitions P_X. Since $|K_1 \cup S_1 \cup S_2| \leq 2p + R - 1$, and since by 3 $R \leq 2^{2q}$, we deduce that $|\mathscr{P}_2| < |V(G)|^{2p+2^{2q}}$.

In order to complete the proof of 1 we will prove the following:

Theorem 5. *For every clique K and stable set S of G such that $K \cap S = \emptyset$, there exists $(X,Y) \in \mathscr{P}_1 \cup \mathscr{P}_2$ with $K \subseteq X$ and $S \subseteq Y$.*

Proof. Let K and S be as in the statement of 5.

(1) *We may assume that K is a maximal clique of G, and S is a maximal stable set of G.*

Let K' be a maximal clique of G with $K \subseteq K'$, and let S' be a maximal stable set of G with $S \subseteq S'$. If $K' \cap S' = \emptyset$, then the existence of the desired partition for K,S follows from the existence of such a partition for K',S'; thus we may assume that $K' \cap S' \neq \emptyset$. Since K' is a clique and S' is a stable set, it follows that $|K' \cap S'| = 1$, say $K' \cap S' = \{v\}$. But now the partitions $(N[\{v\}], V(G) \setminus N[\{v\}])$ and $(N(\{v\}), V(G) \setminus N(\{v\}])$ are both in $\mathscr{P}_1$, and at least one of them has the desired property. This proves (1).

In view of (1) from now on we assume that K is a maximal clique of G, and S is a maximal stable set of G. Consequently every vertex of K has a neighbor in S. Let $S'_1 \subseteq S$ be a minimal subset of S such that every vertex of K has a neighbor in S'_1. It follows from the minimality of S'_1 that there is a subset K'_1 of K such that S'_1 and K'_1 are matched. If $|S'_1| < p$, then the partition $(N(S'_1), V(G) \setminus N(S'_1)) \in \mathscr{P}_1$ has the desired property, so we may assume that $|S'_1| \geq p$.

Let S_1 be a subset of S'_1 with $|S_1| = p$, and let $K_1 = N(S_1) \cap K'_1$. Then S_1 and K_1 are matched, and so $|K_1| = p$. Let Z be the set of vertices of G that are anticomplete to $S_1 \cup K_1$. Then $S'_1 \setminus S_1 \subseteq Z \cap S$, and in particular every vertex of K has a neighbor either in S_1 or in $Z \cap S$. Let S' be the subset of vertices of $S \setminus S_1$ that are complete to K_1. Note that $S' \cap Z = \emptyset$. Let S_2 be a minimal subset of S' such that $N(S_2) \cap Z = N(S') \cap Z$. It follows from the minimality of S_2 that there is a subset $W \subseteq Z \cap N(S')$ such that W and S_2 are matched. Observe that $G[K_1 \cup S_1 \cup S_2 \cup W]$ is isomorphic to $F_{p,|S_2|}$ (with K_1, S_1, S_2, W as in the definition of $F_{a,b}$). It follows from 4 that $|S_2| < R$.

Let $X = (K_1, S_1, S_2)$. We claim that the partition $P_X \in \mathscr{P}_2$ has the desired property for the pair K,S. Recall that $P_X = (A_X, V(G) \setminus A_X)$, where A_X is the set of all vertices v of G such that

- either $v \in K_1$, or v is complete to K_1, and
- either v has a neighbor in S_1, or v has a neighbor in $Z \setminus N(S_2)$.

We need to show that $K \subseteq A_X$, and $S \cap A_X = \emptyset$.

(2) $K \subseteq A_X$.

Let $k \in K$. Clearly either $k \in K_1$ or k is complete to K_1. Moreover, k has a neighbor in S'_1, and $S'_1 \subseteq S_1 \cup (Z \cap S)$. Since S is a stable set, it follows that $Z \cap S \subseteq Z \setminus N(S_2)$, and thus k has a neighbor either in S_1, or in $Z \setminus N(S_2)$. This proves (2).

(3) $S \cap A_X = \emptyset$.

Suppose that $s \in S \cap A_X$. Then $s \notin K_1$; therefore s is complete to K_1, and so $s \in S'$. Since S is a stable set, it follows that s is anticomplete to S_1, and therefore s has a neighbor in $Z \setminus N(S_2)$. But $N(S') \cap Z = N(S_2) \cap Z$, a contradiction. This proves (3).

Now 5 follows from (2) and (3). $\square$

This completes the proof of 1.

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