Notes on tree- and path-chromatic number

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Abstract *Tree-chromatic number* is a chromatic version of treewidth, where the cost of a bag in a tree-decomposition is measured by its chromatic number rather than its size. *Path-chromatic number* is defined analogously. These parameters were introduced by Seymour [JCTB 2016]. In this paper, we survey all the known results on tree- and path-chromatic number and then present some new results and conjectures. In particular, we propose a version of Hadwiger's Conjecture for tree-chromatic number. As evidence that our conjecture may be more tractable than Hadwiger's Conjecture, we give a short proof that every K_5-minor-free graph has tree-chromatic number at most 4, which avoids the Four Colour Theorem. We also present some hardness results and conjectures for computing tree- and path-chromatic number.

1 Introduction

Tree-chromatic number is a hybrid of the graph parameters treewidth and chromatic number, recently introduced by Seymour [17]. Here is the definition.

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A *tree-decomposition* of a graph G is a pair $(T,\mathscr{B})$ where T is a tree and $\mathscr{B} := \{B_t \mid t \in V(T)\}$ is a collection of subsets of vertices of G, called *bags*, satisfying:

- for each $uv \in E(G)$, there exists $t \in V(T)$ such that $u,v \in B_t$, and
- for each $v \in V(G)$, the set of all $t \in V(T)$ such that $v \in B_t$ induces a non-empty subtree of T.

A graph G is *k-colourable* if each vertex of G can be assigned one of k colours, such that adjacent vertices are assigned distinct colours. The *chromatic number* of a graph G is the minimum integer k such that G is k-colourable.

For a tree-decomposition $(T,\mathscr{B})$ of G, the *chromatic number* of $(T,\mathscr{B})$ is $\max\{\chi(G[B_t]) \mid t \in V(T)\}$. The *tree-chromatic number* of G, denoted $\mathsf{tree}\text{-}\chi(G)$, is the minimum chromatic number taken over all tree-decompositions of G. The *path-chromatic number* of G, denoted $\mathsf{path}\text{-}\chi(G)$, is defined analogously, where we insist that T is a path instead of an arbitrary tree. Henceforth, for a subset $B \subseteq V(G)$, we will abbreviate $\chi(G[B])$ by $\chi(B)$. For $v \in V(G)$, let $N_G(v)$ be the set of neighbours of v and $N_G[v] := N_G(v) \cup \{v\}$.

The purpose of this paper is to survey the known results on tree- and path-chromatic number, and to present some new results and conjectures.

Clearly, $\mathsf{tree}\text{-}\chi$ and $\mathsf{path}\text{-}\chi$ are monotone under the subgraph relation, but unlike treewidth, they are not monotone under the minor relation. For example, $\mathsf{tree}\text{-}\chi(K_n) = n$, but the graph G obtained by subdividing each edge of K_n is bipartite and so $\mathsf{tree}\text{-}\chi(G) \leq \chi(G) = 2$.

By definition, for every graph G,

$$\mathsf{tree}\text{-}\chi(G) \leq \mathsf{path}\text{-}\chi(G) \leq \chi(G).$$

Section 2 reviews results that show that each of these inequalities can be strict and in fact, both of the pairs $(\mathsf{tree}\text{-}\chi(G), \mathsf{path}\text{-}\chi(G))$ and $(\mathsf{path}\text{-}\chi(G), \chi(G))$ can be arbitrarily far apart.

We present our new results and conjectures in Sections 3-5. In Section 3, we propose a version of Hadwiger's Conjecture for tree-chromatic number and show how it is related to a 'local' version of Hadwiger's Conjecture. In Section 4, we prove that K_5-minor-free graphs have tree-chromatic number at most 4, without using the Four Colour Theorem. We finish in Section 5, by presenting some hardness results and conjectures for computing $\mathsf{path}\text{-}\chi$ and $\mathsf{tree}\text{-}\chi$.

2 Separating χ, $\mathsf{path}\text{-}\chi$ and $\mathsf{tree}\text{-}\chi$

Complete graphs are a class of graphs with unbounded tree-chromatic number. Are there more interesting examples? The following lemma of Seymour [17] leads to an answer. A *separation* (A,B) of a graph G is a pair of edge-disjoint subgraphs whose union is G.

Lemma 1. *For every graph G, there is a separation (A,B) of G such that $\chi(A\cap B) \leq \mathsf{tree}\text{-}\chi(G)$ and*

$$\chi(A-V(B)), \chi(B-V(A)) \geq \chi(G) - \mathsf{tree}\text{-}\chi(G).$$

Seymour [17] noted that Lemma 1 shows that the random construction of Erdős [6] of graphs with large girth and large chromatic number also have large tree-chromatic number with high probability.

Interestingly, it is unclear if the known *explicit* constructions of large girth, large chromatic graphs also have large tree-chromatic number. For example, *shift graphs* are one of the classic constructions of triangle-free graphs with unbounded chromatic number, as first noted in [7]. The vertices of the n-th shift graph S_n are all intervals of the form $[a,b]$, where a and b are integers satisfying $1 \leq a < b \leq n$. Two intervals $[a,b]$ and $[c,d]$ are adjacent if and only if $b=c$ or $d=a$. The following lemma (first noted in [17]) shows that the gap between χ and $\mathsf{path}\text{-}\chi$ is unbounded on the class of shift graphs.

Lemma 2. *For all $n \in \mathbb{N}$, $\mathsf{path}\text{-}\chi(S_n) = 2$ and $\chi(S_n) \geq \lceil \log_2 n \rceil$.*

Proof. The fact that $\chi(S_n) \geq \lceil \log_2 n \rceil$ is well-known; we include the proof for completeness. Let $\ell = \chi(S_n)$ and $\phi : V(S_n) \to [\ell]$ be a proper ℓ-colouring of S_n. For each $j \in [n]$ let $C_j = \{\phi([i,j]) \mid i < j\}$. We claim that for all $j<k$, $C_j \neq C_k$. By definition, $\phi([j,k]) \in C_k$. If $C_j = C_k$, then $\phi([i,j]) = \phi([j,k])$ for some $i<j$. But this is a contradiction, since $[i,j]$ and $[j,k]$ are adjacent in S_n. Since there are 2^ℓ subsets of $[\ell]$, $2^\ell \geq n$, as required.

We now show that $\mathsf{path}\text{-}\chi(S_n) = 2$. For each $i \in [n]$, let $B_i = \{[a,b] \in V(S_n) \mid a \leq i \leq b\}$. Let P_n be the path with vertex set $[n]$ (labelled in the obvious way). We claim that $(P_n, \{B_i \mid i \in [n]\})$ is a path-decomposition of S_n. First observe that $[a,b] \in B_i$ if and only if $a \leq i \leq b$. Next, for each edge $[a,b][b,c] \in E(S_n)$, $[a,b],[b,c] \in B_b$. Finally, observe that for all $i \in [n]$, $X_i = \{[a,b] \in B_i \mid b = i\}$ and $Y_i = \{[a,b] \in B_i \mid b > i\}$ is a bipartition of $S_n[B_i]$. Therefore, S_n has path-chromatic number 2, as required.

Given that shift graphs contain large complete bipartite subgraphs, the following question naturally arises.

Open Problem 1 *Does there exist a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $s \in \mathbb{N}$ and all $K_{s,s}$-free graphs G, $\chi(G) \leq f(s, \mathsf{tree}\text{-}\chi(G))$?*

It is not obvious that the parameters $\mathsf{path}\text{-}\chi$ and $\mathsf{tree}\text{-}\chi$ are actually different. Indeed, Seymour [17] asked if $\mathsf{path}\text{-}\chi(G) = \mathsf{tree}\text{-}\chi(G)$ for all graphs G? Huynh and Kim [10] answered the question in the negative by exhibiting for each $k \in \mathbb{N}$, an infinite family of k-connected graphs for which $\mathsf{tree}\text{-}\chi(G)+1 = \mathsf{path}\text{-}\chi(G)$. They also prove that the Mycielski graphs [14] have unbounded path-chromatic number.

However, can $\mathsf{tree}\text{-}\chi(G)$ and $\mathsf{path}\text{-}\chi(G)$ be arbitrarily far apart? Seymour [17] suggested the following family as a potential candidate. Let T_n be the complete binary rooted tree with 2^n leaves. A path P in T_n is called a V if the vertex of P

closest to the root (which we call the *low point* of the V) is an internal vertex of P. Let G_n be the graph whose vertices are the Vs of T_n, where two Vs are adjacent if the low point of one is an endpoint of the other.

Lemma 3 ([17]). *For all* $n \in \mathbb{N}$, $\mathsf{tree}\text{-}\chi(G_n) = 2$ *and* $\chi(G_n) \geq \lceil \log_2 n \rceil$.

Proof. For each $t \in V(T_n)$, let B_t be the set of Vs in T_n which contain t. We claim that $(T_n, \{B_t \mid t \in V(T_n)\})$ is a tree-decomposition of G_n with chromatic number 2. First observe that if P is a V, then $\{t \in V(T_n) \mid P \in B_t\} = V(P)$, which induces a non-empty subtree of T_n. Next, if P_1 and P_2 are adjacent Vs with $V(P_1) \cap V(P_2) = \{t\}$, then $P_1, P_2 \in B_t$. Finally, for each $t \in B_t$, let X_t be the elements of B_t whose low point is t and let $Y_t := B_t \setminus X_t$. Then (X_t, Y_t) is a bipartition of $G_n[B_t]$, implying that $\mathsf{tree}\text{-}\chi(G_n) = 2$.

For the second claim, it is easy to see that G_n contains a subgraph isomorphic to the n-th shift graph S_n. Thus, $\chi(G_n) \geq \chi(S_n) \geq \lceil \log_2 n \rceil$, by Lemma 2.

Barrera-Cruz, Felsner, Mészáros, Micek, Smith, Taylor, and Trotter [1] subsequently proved that $\mathsf{path}\text{-}\chi(G_n) = 2$ for all $n \in \mathbb{N}$. However, with a slight modification of the definition of G_n, they were able to construct a family of graphs with tree-chromatic number 2 and unbounded path-chromatic number.

Theorem 2 ([1]). *For each integer* $n \geq 2$*, there exists a graph* H_n *with* $\mathsf{tree}\text{-}\chi(H_n) = 2$ *and* $\mathsf{path}\text{-}\chi(H_n) = n$.

The definition of H_n is as follows. A subtree of the complete binary tree T_n is called a Y if it has three leaves and the vertex of the Y closest to the root of T_n is one of its three leaves. The vertices of H_n are the Vs and Ys of T_n. Two Vs are adjacent if the low point of one is an endpoint of the other. Two Ys are adjacent if the lowest leaf of one is an upper leaf of the other. A V is adjacent to a Y if the low point of the V is an upper leaf of the Y. The proof that $\mathsf{path}\text{-}\chi(H_n) = n$ uses Ramsey theoretical methods for trees developed by Milliken [13].

3 Hadwiger's Conjecture for tree-χ and path-χ

One could hope that difficult conjectures involving χ might become tractable for tree-χ or path-χ, thereby providing insightful intermediate results. Indeed, the original motivation for introducing tree-χ was a conjecture of Gyárfás [8] from 1985, on χ-boundedness of triangle-free graphs without long holes [1].

Conjecture 1 (Gyárfás's Conjecture [8]). For every integer ℓ, there exists c such that every triangle-free graph with no hole of length greater than ℓ has chromatic number at most c.

Seymour [17] proved that Conjecture 1 holds with χ replaced by tree-χ.

[1] A *hole* in a graph is an induced cycle of length at least 4.

Theorem 3 ([17]). *For all integers $d \geq 1$ and $\ell \geq 4$, if G is a graph with no hole of length greater than ℓ and $\chi(N_G(v)) \leq d$ for all $v \in V(G)$, then* $\text{tree-}\chi(G) \leq d(\ell-2)$.

Note that Theorem 3 with $d = 1$ implies that $\text{tree-}\chi(G) \leq \ell - 2$ for every triangle-free graph G with no hole of length greater than ℓ. A proof of Gyárfás's Conjecture [8] (among other results) was subsequently given by Chudnovsky, Scott, and Seymour [3].

The following is another famous conjectured upper bound on χ, due to Hadwiger [9]; see [16] for a survey.

Conjecture 2 ([9]). If G is a graph without a K_{t+1}-minor, then $\chi(G) \leq t$.

We propose the following weakenings of Hadwiger's Conjecture.

Conjecture 3. If G is a graph without a K_{t+1}-minor, then $\text{tree-}\chi(G) \leq t$.

Conjecture 4. If G is a graph without a K_{t+1}-minor, then $\text{path-}\chi(G) \leq t$.

By Theorem 2, $\text{tree-}\chi(G)$ and $\text{path-}\chi(G)$ can be arbitrarily far apart, so Conjecture 3 may be easier to prove than Conjecture 4. By Theorem 3, χ and $\text{tree-}\chi$ can be arbitrarily far apart, so Conjecture 3 may be easier to prove than Hadwiger's Conjecture. We give further evidence of this in the next section, by proving Conjecture 3 for $t = 5$, without using the Four Colour Theorem.

Robertson, Seymour, and Thomas [15] proved that every K_6-minor-free graph is 5-colourable. Their proof uses the Four Colour Theorem and is 83 pages long. Thus, even if we are allowed to use the Four Colour Theorem, it would be interesting to find a short proof that every K_6-minor-free graph has tree-chromatic number at most 5.

Conjectures 3 and 4 are also related to a 'local' version of Hadwiger's Conjecture via the following lemma.

Lemma 4. *Let $(T, \{B_t \mid t \in V(T)\})$ be a* $\text{tree-}\chi$*-optimal tree-decomposition of G, with $|V(T)|$ minimal. Then there are vertices $v \in V(G)$ and $\ell \in V(T)$ such that $N_G[v] \subseteq B_\ell$.*

Proof. Let ℓ be a leaf of T and u be the unique neighbour of ℓ in T. If $B_\ell \subseteq B_u$, then $T - \ell$ contradicts the minimality of T. Therefore, there is a vertex $v \in B_\ell$ such that $v \notin B_t$ for all $t \neq \ell$. It follows that $N_G[v] \subseteq B_\ell$, as required.

Lemma 4 immediately implies that the following 'local version' of Hadwiger's Conjecture follows from Conjecture 3.

Conjecture 5. If G is a graph without a K_{t+1}-minor, then there exists $v \in V(G)$ such that $\chi(N_G[v]) \leq t$.

It is even open whether Conjectures 3, 4, or 5 hold with an upper bound of $10^{100}t$ instead of t. Finally, the following apparent weakening of Hadwiger's Conjecture (and strengthening of Conjecture 5) is actually equivalent to Hadwiger's Conjecture.

Conjecture 6. If G is a graph without a K_{t+1}-minor, then $\chi(N_G[v]) \leq t$ for all $v \in V(G)$.

Proof (Proof of equivalence to Hadwiger's Conjecture). Clearly, Hadwiger's Conjecture implies Conjecture 6. For the converse, let G be a graph without a K_{t+1}-minor. Let G^+ be the graph obtained from G by adding a new vertex v adjacent to all vertices of G. Since G^+ has no K_{t+2}-minor, Conjecture 6 yields $\chi(N_{G^+}[v]) \leq t+1$. Since $\chi(N_{G^+}[v]) = \chi(G)+1$, we have $\chi(G) \leq t$, as required.

4 K_5-minor-free graphs

As evidence that Conjecture 3 may be more tractable than Hadwiger's Conjecture, we now prove it for K_5-minor-free graphs without using the Four Colour Theorem. We begin with the planar case.

Theorem 4. *For every planar graph G,* tree-$\chi(G) \leq 4$.

Proof. We use the same tree-decomposition previously used by Eppstein [5] and Dujmović, Morin, and Wood [4].

Say G has n vertices. We may assume that $n \geq 3$ and that G is a plane triangulation. Let $F(G)$ be the set of faces of G. By Euler's formula, $|F(G)| = 2n-4$ and $|E(G)| = 3n-6$. Let r be a vertex of G. Let $(V_0, V_1, \dots, V_t)$ be the bfs layering of G starting from r. Let T be a bfs tree of G rooted at r. Let T^* be the subgraph of the dual G^* with vertex set $F(G)$, where two vertices are adjacent if the corresponding faces share an edge not in T. Thus

$$|E(T^*)| = |E(G)| - |E(T)| = (3n-6)-(n-1) = 2n-5 = |F(G)|-1 = |V(T^*)|-1.$$

By the Jordan Curve Theorem, T^* is connected. Thus T^* is a tree.

For each vertex u of T^*, if u corresponds to the face xyz of G, let $C_u := P_x \cup P_y \cup P_z$, where P_v is the vertex set of the vr-path in T, for each $v \in V(G)$. See [5, 4] for a proof that $(T^*, \{C_u : u \in V(T^*)\})$ is a tree-decomposition of G.

We now prove that $G[C_u]$ is 4-colourable. Let ℓ be the largest index such that $\{x,y,z\} \cap V_\ell \neq \emptyset$. For each $k \in \{0,\dots,\ell\}$, let $G_k = G[C_u \cap (\bigcup_{j=0}^{k} V_j)]$. Note that $G_\ell = G[C_u]$. We prove by induction on k that G_k is 4-colourable. This clearly holds for $k \in \{0,1\}$, since $|V(G_1)| \leq 4$.

For the inductive step, let $k \geq 2$. For each $i \in \{0,\dots,\ell\}$, let $W_i = C_u \cap V_i$. Since W_i contains at most one vertex from each of P_x, P_y, and P_z, $|W_i| \leq 3$.

First suppose $|W_i| \leq 2$ for all $i \leq k$. Since all edges of G are between consecutive layers or within a layer, we can 4-colour G_k by using the colours $\{1,2\}$ on the even layers and $\{3,4\}$ on the odd layers.

Next suppose $|W_k| \leq 2$. We are done by the previous case unless $k = \ell, |W_\ell| \in \{1,2\}$, and $|W_{\ell-1}| = 3$. By induction, let $\phi' : V(G_{\ell-2}) \to [4]$ and $\phi : V(G_{\ell-1}) \to [4]$ be 4-colourings of $G_{\ell-2}$ and $G_{\ell-1}$, respectively. If $|W_\ell| = 1$, then clearly we can extend ϕ to a 4-colouring of G_ℓ. So, we may assume $|W_\ell| = 2$.

Note that ϕ extends to a 4-colouring of G_ℓ unless every vertex of $W_{\ell-1}$ is adjacent to every vertex of W_ℓ and the two vertices of W_ℓ are adjacent. If $G[W_{\ell-1}]$ is a triangle, then $G[W_{\ell-1} \cup W_\ell] = K_5$, which contradicts planarity. If $G[W_{\ell-1}]$ is a path, say abc, then we obtain a K_5-minor in G by contracting all but one edge of the a–c path in T. If $W_{\ell-1}$ is a stable set, then ϕ' can be extended to a 4-colouring of $G_{\ell-1}$ such that all vertices in $W_{\ell-1}$ are the same colour. This colouring can clearly be extended to a 4-colouring of G_ℓ. The remaining case is if $G[W_{\ell-1}]$ is an edge ab together with an isolated vertex c. It suffices to show that there is a colouring of $G_{\ell-1}$ that uses at most two colours on $W_{\ell-1}$, since such a colouring can be extended to a 4-colouring of G_ℓ. Note that ϕ' can be extended to such a colouring unless ϕ' uses three colours on $W_{\ell-2}$ and a and b are adjacent to all vertices of $W_{\ell-2}$. Since ϕ is a 4-colouring, this implies that ϕ uses at most two colours on $W_{\ell-2}$. Thus we may recolour ϕ so that only two colours are used on $W_{\ell-1}$, as required.

Henceforth, we may assume $|W_k| = 3$. By induction, let $\phi : V(G_{k-1}) \to [4]$ be a 4-colouring of G_{k-1}. Let $\phi_{k-1} = \phi(W_{k-1})$.

If $|\phi_{k-1}| = 1$, then we can extend ϕ to a 4-colouring of G_k by using $[4] \setminus \phi_{k-1}$ to 3-colour W_k.

Suppose $|\phi_{k-1}| = 2$. By induction, G_{k-2} has a 4-colouring ϕ'. If W_{k-1} is a stable set, then we can extend ϕ' to a 4-colouring of G_{k-1} such that all vertices of W_{k-1} are the same colour. Thus, $|\phi'_{k-1}| = 1$, and we are done by the previous case. Let $a, b \in W_{k-1}$ such that $ab \in E(G_{k-1})$. Let c be the other vertex of W_{k-1} (if it exists). By relabeling, we may assume that $\phi(a) = 1, \phi(b) = 2$, and $\phi(c) = 2$. Let $N(a)$ be the set of neighbours of a in W_k and $N(b,c)$ be the set of neighbours of $\{b,c\}$ in W_k. Observe that ϕ extends to a 4-colouring of G_k unless $N(a) = N(b,c) = W_k$. However, if, $N(a) = N(b,c) = W_k$, then we obtain a K_5-minor in G by using T to contract W_k onto $\{x,y,z\}$ and c onto b (if c exists). This contradicts planarity.

The remaining case is $|\phi_{k-1}| = 3$. In this case, ϕ extends to a 4-colouring of G_k, unless there exist distinct vertices $a, b \in W_{k-1}$ such that a and b are both adjacent to all vertices of W_k. Again we obtain a K_5-minor in G by using T to contract W_k onto $\{x,y,z\}$ and contracting all but one edge of the a–b path in T.

We finish the proof by using Wagner's characterization of K_5-minor-free graphs [19], which we now describe. Let G_1 and G_2 be two graphs with $V(G_1) \cap V(G_2) = K$, where K is a clique of size k in both G_1 and G_2. The *k-sum* of G_1 and G_2 (along K) is the graph obtained by gluing G_1 and G_2 together along K (and keeping all edges of K). The *Wagner graph* V_8 is the graph obtained from an 8-cycle by adding an edge between each pair of antipodal vertices.

Theorem 5 (Wagner's Theorem [19]). *Every edge-maximal K_5-minor-free graph can be obtained from* 1-, 2-, *and* 3-*sums of planar graphs and* V_8.

Theorem 6. *For every K_5-minor-free graph G,* $\text{tree-}\chi(G) \leq 4$.

Proof. Let G be a K_5-minor-free graph. We proceed by induction on $|V(G)|$. We may assume that G is edge-maximal. First note that if $G = V_8$, then $\text{tree-}\chi(G) \leq \chi(G) = 4$. Next, if G is planar, then $\text{tree-}\chi(G) \leq 4$ by Theorem 4 (whose proof avoids the Four Colour Theorem). By Theorem 5, we may assume that G is a k-sum of two graphs G_1 and G_2, for some $k \in [3]$. Let K be the clique in $V(G_1) \cap V(G_2)$ along which the k-sum is performed. Since G_1 and G_2 are both K_5-minor-free graphs with $|V(G_1)|, |V(G_2)| < |V(G)|$, we have $\mathsf{tree}\text{-}\chi(G_1) \leq 4$ and $\mathsf{tree}\text{-}\chi(G_2) \leq 4$ by induction. For $i \in [2]$, let $(T^i, \{B^i_t \mid t \in V(T^i)\})$ be a tree-decomposition of G_i with chromatic number at most 4. Since K is a clique in G_i, $K \subseteq B^1_x \cap B^2_y$ for some $x \in V(T^1)$ and $y \in V(T^2)$. Let T be the tree obtained from the disjoint union of T^1 and T^2 by adding an edge between x and y. Then $(T, \{B^1_t \mid t \in V(T^1)\} \cup \{B^2_t \mid t \in V(T^2)\})$ is a tree-decomposition of G with chromatic number at most 4.

5 Computing $\mathsf{tree}\text{-}\chi$ and $\mathsf{path}\text{-}\chi$

We finish by showing some hardness results for computing $\mathsf{tree}\text{-}\chi$ and $\mathsf{path}\text{-}\chi$. We need some preliminary results. For a graph G, let K^G_t be the graph consisting of t disjoint copies of G and all edges between distinct copies of G.

Lemma 5. *For all $t \in \mathbb{N}$ and all graphs G without isolated vertices,*

$$(t-1)\chi(G) + 2 \leq \mathsf{tree}\text{-}\chi(K^G_t) \leq \mathsf{path}\text{-}\chi(K^G_t) \leq t\,\chi(G).$$

Proof. Let $(T, \{B_t \mid t \in V(T)\})$ be a $\mathsf{tree}\text{-}\chi$-optimal tree-decomposition of $K := K^G_t$, with $|V(T)|$ minimal. By Lemma 4, there exists $\ell \in V(T)$ and $v \in V(K)$ such that $N_K[v] \subseteq B_\ell$. Since G has no isolated vertices, v has a neighbour in the same copy of G in which it belongs. Therefore,

$$\mathsf{tree}\text{-}\chi(K) \geq \chi(B_\ell) \geq \chi(N_K[v]) \geq 2 + (t-1)\chi(G).$$

For the other inequalities, $\mathsf{tree}\text{-}\chi(K) \leq \mathsf{path}\text{-}\chi(K) \leq \chi(K) = t\,\chi(G)$.

We also require the following hardness result of Lund and Yannakakis [12].

Theorem 7 ([12]). *There exists $\varepsilon > 0$, such that it is NP-hard to correctly determine $\chi(G)$ within a multiplicative factor of n^ε for every n-vertex graph G.*

Our first theorem is a hardness result for approximating $\mathsf{tree}\text{-}\chi$ and $\mathsf{path}\text{-}\chi$.

Theorem 8. *There exists $\varepsilon' > 0$, such that it is NP-hard to correctly determine $\mathsf{tree}\text{-}\chi(G)$ within a multiplicative factor of $n^{\varepsilon'}$ for every n-vertex graph G. The same hardness result holds for $\mathsf{path}\text{-}\chi$ with the same ε'.*

Proof. We show the proof for $\mathsf{tree}\text{-}\chi$. The proof for $\mathsf{path}\text{-}\chi$ is identical. Let $\varepsilon' = \frac{\varepsilon}{3}$, where ε is the constant from Theorem 7. Let G be an n-vertex graph.

Note that K^G_n has n^2 vertices, and $(n^2)^{\varepsilon'} = n^{\frac{2\varepsilon}{3}}$. If $k \in [\frac{\mathsf{tree}\text{-}\chi(K^G_n)}{n^{\frac{2\varepsilon}{3}}}, n^{\frac{2\varepsilon}{3}}\,\mathsf{tree}\text{-}\chi(K^G_n)]$, then $\frac{k}{n} \in [\frac{\chi(G)}{n^\varepsilon}, n^\varepsilon \chi(G)]$ by Lemma 5. Therefore, if we can approximate $\mathsf{tree}\text{-}\chi(K^G_n)$ within a factor of $(n^2)^{\varepsilon'}$, then we can approximate $\chi(G)$ within a factor of n^ε.

For the decision problem, we use the following hardness result of Khanna, Linial, and Safra [11].

Theorem 9 ([11]). *Given an input graph G with $\chi(G) \neq 4$, it is NP-complete to decide if $\chi(G) \leq 3$ or $\chi(G) \geq 5$.*

As a corollary of Theorem 9, we obtain the following.

Theorem 10. *It is NP-complete to decide if* $\mathsf{tree}\text{-}\chi(G) \leq 6$. *It is also NP-complete to decide if* $\mathsf{path}\text{-}\chi(G) \leq 6$.

Proof. Let G be a graph without isolated vertices and $\chi(G) \neq 4$. By Lemma 5, if $\mathsf{tree}\text{-}\chi(K_2^G) \leq 6$, then $\chi(G) \leq 3$ and if $\mathsf{tree}\text{-}\chi(K_2^G) \geq 7$, then $\chi(G) \geq 5$. Same for $\mathsf{path}\text{-}\chi$. Finally, a tree- or path-decomposition and a 6-colouring of each bag is a certificate that $\mathsf{tree}\text{-}\chi(G) \leq 6$ or $\mathsf{path}\text{-}\chi(G) \leq 6$.

Combining the standard $O(2^n)$-time dynamic programming for computing pathwidth exactly (see Section 3 of [18]) and the $2^n n^{O(1)}$-time algorithm of Björklund, Husfeldt, and Koivisto [2] for deciding if $\chi(G) \leq k$, yields a $4^n n^{O(1)}$-time algorithm to decide to $\mathsf{path}\text{-}\chi(G) \leq k$. As far as we know, there is no faster algorithm for deciding $\mathsf{path}\text{-}\chi(G) \leq k$ (except for small values of k, where faster algorithms for deciding k-colourability can be used instead of [2]).

Finally, unlike for $\chi(G)$, we conjecture that it is still NP-complete to decide if $\mathsf{tree}\text{-}\chi(G) \leq 2$.

Conjecture 7. It is NP-complete to decide if $\mathsf{tree}\text{-}\chi(G) \leq 2$. It is also NP-complete to decide if $\mathsf{path}\text{-}\chi(G) \leq 2$.

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