Note on Hedetniemi's conjecture and the Poljak-Rödl function

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Abstract Hedetniemi conjectured in 1966 that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$ for any graphs G and H. Here $G \times H$ is the graph with vertex set $V(G) \times V(H)$ defined by putting (x,y) and (x',y') adjacent if and only if $xx' \in E(G)$ and $yy' \in V(H)$. This conjecture received a lot of attention in the past half century. It was disproved recently by Shitov. The Poljak-Rödl function is defined as $f(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}$. Hedetniemi's conjecture is equivalent to saying $f(n) = n$ for every integer n. Shitov's result shows that $f(n) < n$ when n is sufficiently large. Using Shitov's result, Tardif and Zhu showed that $f(n) \le n - (\log n)^{1/4-o(1)}$ for sufficiently large n. Using Shitov's method, He and Wigderson showed that for $\varepsilon \approx 10^{-9}$ and n sufficiently large, $f(n) \le (1-\varepsilon)n$. In this note we observe that a slight modification of the proof in the paper of Zhu and Tardif shows that $f(n) \le (\frac{1}{2} + o(1))n$ for sufficiently large n. On the other hand, it is unknown whether $f(n)$ is bounded by a constant. However, we do know that if $f(n)$ is bounded by a constant, then the smallest such constant is at most 9. This note gives self-contained proofs of the above mentioned results.

1 Introduction

The *product* $G \times H$ of graphs G and H has vertex set $V(G) \times V(H)$ and has (x,y) adjacent to (x',y') if and only if $xx' \in E(G)$ and $yy' \in E(H)$. Many names for this product are used in the literature, including the *categorical product*, the *tensor product* and the *direct product*. It is the most important product in this note. We just call it *the product*. We may write $x \sim y$ (in G) to denote $xy \in E(G)$.

A proper colouring ϕ of G induces a proper colouring Φ of $G \times H$ defined as $\Phi(x,y) = \phi(x)$. So $\chi(G \times H) \le \chi(G)$. Symmetrically, we also have $\chi(G \times H) \le$

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$\chi(H)$. Therefore $\chi(G\times H)\leq \min\{\chi(G),\chi(H)\}$. In 1966, Hedetniemi conjectured in [5] that $\chi(G\times H)=\min\{\chi(G),\chi(H)\}$ for all graphs G and H. This conjecture received a lot of attention in the past half century (see [1, 6, 10, 13, 18, 19]). Some special cases are confirmed. In particular, it is known that if $\min\{\chi(G),\chi(H)\}\leq 4$, then the conjecture holds [1]. Also, a fractional version of Hedetniemi's conjecture is true [19]. However, Shitov recently refuted Hedetniemi's conjecture [11]. He proved that for sufficiently large n, there are n-chromatic graphs G and H with $\chi(G\times H)<n$.

The Poljak-Rödl function [9] is defined as

$$f(n)=\min\{\chi(G\times H):\chi(G)=\chi(H)=n\}.$$

Hedetniemi's conjecture is equivalent to saying $f(n)=n$ for all positive integer n. Shitov's result shows that $f(n)<n$ for sufficiently large n. Right after Shitov put his result on arxiv, using his result, Tardif and Zhu [16] showed that the difference $n-f(n)$ can be arbitrarily large. Indeed, they proved that $f(n)\leq n-(\log n)^{1/4-o(1)}$ for sufficiently large n. It is also shown in [16] that if a special case of Stahl's conjecture in [12] on the multi-chromatic number of Kneser graphs is true, then $\lim_{n\to\infty} f(n)/n\leq 1/2$. He and Wigderson, using Shitov's method, proved that $f(n)\leq(1-\varepsilon)n$ for $\varepsilon\approx 10^{-9}$ and suffciently large n. Very recently, Zhu observed that the conclusion $\lim_{n\to\infty} f(n)/n\leq 1/2$ holds without assuming Stahl's conjecture.

2 Exponential graph

One of the standard tools used in the study of Hedetniemi's conjecture is the concept of *exponential graphs*. Let c be a positive integer. We denote by $[c]$ the set $\{1,2,\ldots,c\}$. For a graph G, the exponential graph K_c^G has vertex set

$$\{f: f \text{ is a mapping from } V(G)\to[c]\},$$

with $fg\in E(K_c^G)$ if and only if for any edge $xy\in E(G)$, $f(x)\neq g(y)$. In particular, $f\sim f$ is a loop in K_c^G if and only if f is a proper c-colouring of G. So if $\chi(G)>c$, then K_c^G has no loop.

For convenience, when we study properties of K_c^G, vertices in K_c^G will be called *maps*. The term "vertices" is reserved for vertices of G. That is, to refer to a vertex of K_c^G, we will say that it is a map in K_c^G or a map from G to $[c]$.

For two graphs G and H, a *homomorphism from G to H* is a mapping $\phi: V(G)\to V(H)$ that preserves edges, i.e., for every edge xy of G, $\phi(x)\phi(y)$ is an edge of H. We say G is *homomorphic* to H, and write $G\to H$, if there is a homomorphism from G to H. The "homomorphic" relation "$\to$" is a quasi-order. It is reflexive and transitive: if $G\to H$ and $H\to Q$ then $G\to Q$. The composition $\psi\circ\phi$ of a homomorphism ϕ from G to H and a homomorphism ψ from H to Q is a homomorphism from G to Q.

Note that a homomorphism from a graph G to K_c is equivalent to a proper c-colouring of G. Thus if $G \to H$, then $\chi(G) \leq \chi(H)$.

Lemma 1. *For any graph F, $\chi(G \times F) \leq c$ if and only if F is homomorphic to K_c^G.*

Proof. Assume $\chi(G \times F) \leq c$ and $\Psi : V(G \times F) \to [c]$ is a proper colouring of $G \times F$. For any vertex $u \in V(F)$, let $f_u \in K_c^G$ be defined as $f_u(v) = \Psi(u,v)$. Then the mapping sending u to f_u is a homomorphism from F to K_c^G. Indeed, if $uv \in E(F)$, then for any edge $xy \in E(G)$, $(u,x) \sim (v,y)$ in $G \times F$. Therefore $f_u(x) = \Psi(u,x) \neq \Psi(v,y) = f_v(y)$. Thus $f_u \sim f_v$ in K_c^G.

Conversely, the mapping $\Psi : V(G \times K_c^G) \to [c]$ defined as $\Psi(x,f) = f(x)$ is a proper colouring of $G \times K_c^G$. Indeed, if $(x,f) \sim (y,g)$ in $G \times K_c^G$, then $xy \in E(G)$ and $fg \in E(K_c^G)$. Therefore $\Psi(x,f) = f(x) \neq g(y) = \Psi(g,y)$.

If F is homomorphic to K_c^G, then $G \times F$ is homomorphic to $G \times K_c^G$ and hence $\chi(G \times F) \leq c$.

In this sense, K_c^G is the largest graph H in the order of homomorphism with the property that $\chi(G \times H) \leq c$. Thus Hedetniemi's conjecture is equivalent to the following statement:

If $\chi(G) > c$, then $\chi(K_c^G) = c$.

The concept of exponential graphs was first used by El-Zahar and Sauer in [1], where it is shown that if $\chi(G) \geq 4$, then K_3^G is 3-colourable. Hence the product of two 4-chromatic graphs has chromatic number 4.

The result of El-Zahar and Sauer is still the best result in the positive direction of Hedetniemi's conjecture. We do not know whether or not the product of two 5-chromatic graphs equals 5. On the other hand, there is a nice strengthening of this result by Tardif [14] in the study of multiplicative graphs. We say a graph Q is *multiplicative* if for any two graphs G, H, $G \not\to Q$ and $H \not\to Q$ implies that $G \times H \not\to Q$. Hedetniemi's conjecture is equiavelnt to say that K_n is multiplicative for any positive integer n. El-Zahar and Sauer proved that K_3 is multiplicative. Häggkvist, Hell, Miller and Neumann Lara [3] proved that odd cycles are multiplicative and Tardif [14] proved that circular cliques $K_{p/q}$ for $p/q < 4$ are multiplicative, where $K_{p/q}$ has vertex set $[p]$ with $i \sim j$ if and only if $q \leq |i-j| \leq p-q$. (So $K_{p/1} = K_p$ and $K_{(2k+1)/k} = C_{2k+1}$).

3 Shitov's Theorem

To disprove Hedetniemi's conjecture, it suffices to find a graph G and a positive integer c so that $\chi(G) > c$ and $\chi(K_c^G) > c$.

For a map $f \in K_c^G$, the image set of f is $Im(f) = \{f(v) : v \in V(G)\}$. Note that for $f, g \in K_c^G$, if $Im(f) \cap Im(g) = \emptyset$, then $f \sim g$. For $i \in [c]$, we denote by $g_i \in V(K_c^G)$ the constant map $g_i(v) = i$ for all $v \in V(G)$. So $Im(g_i) = \{i\}$. Thus for any graph G and any positive integer c, $\{g_i : i \in [c]\}$ induces a c-clique in K_c^G and $\chi(K_c^G) \geq c$.

We denote by $G[K_q]$ the graph obtained from G by *blowing up* each vertex of G into a q-clique. The vertices of $G[K_q]$ are denoted by (x,i), where $x \in V(G)$ and $i \in [q]$. So (x,i) and (y,j) are adjacent in $G[K_q]$ if and only if either $x \sim y$ or $x = y$ and $i \neq j$. For a graph G, the *independence number* $\alpha(G)$ of G is the size of a largest independent set in G. This section proves the following result of Shitov:

Theorem 1 (Shitov). *Let G be a graph with $|V(G)| = p$, $\alpha(G) \leq \frac{p}{4.1}$ and* $\mathrm{girth}(G) \geq 6$. *Let $q \geq 2^{p-1}p^2$ and $c = 4q+2$. Then $\chi(G[K_q]) > c$ and $\chi\left(K_c^{G[K_q]}\right) > c$.*

The above formulation of the theorem is slightly different from the formulation in [11]. The proof also seems different. But all the claims and lemmas are either stated in [11] or hidden in the text in [11].

It is a classical result of Erdős [2] that there are graphs of arbtirary large girth and large chromatic number. This result is included in most graph theory textbooks (see [17]). The probabilistic proof of this result actually shows that there are graphs G of arbitrary large girth and arbtirary small independence ratio $\alpha(G)/|V(G)|$. What we need here is a graph of girth 6 and with $\alpha(G) \leq |V(G)|/4.1$.

Proof of Theorem 1 Since $G[K_q]$ has the same independence number as G, we have

$$\chi(G[K_q]) \geq \frac{|V(G[K_q])|}{\alpha(G[K_q])} = \frac{|V(G)|q}{\alpha(G)} \geq 4.1q > c.$$

It remains to show that $\chi(K_c^{G[K_q]}) > c$.

Assume to the contrary that $\chi(K_c^{G[K_q]}) = c$ (recall that $K_c^{G[K_q]}$ has a c-clique and hence has chromatic number at least c), and Ψ is c-colouring of K_c^G. We may assume that the constant map g_i is coloured by colour i. Thus for any map $\phi \in K_c^{G[K_q]}$, if $i \notin Im(\phi)$, then $\phi \sim g_i$ and hence $\Psi(\phi) \neq i$. Thus we have the following lemma.

Lemma 2. *For any map $\phi \in K_c^{G[K_q]}$, $\Psi(\phi) \in Im(\phi)$.*

Definition 1. A map $\phi \in K_c^{G[K_q]}$ is called *simple* if ϕ is constant on each copy of K_q that is a blow-up of a vertex of G, i.e., for any $x \in V(G), i, j \in [q]$, $\phi(x,i) = \phi(x,j)$.

For simplicity, we shall write $\phi(x)$ for $\phi(x,i)$ when ϕ is a simple map.

Note that in $K_c^{G[K_q]}$, two simple maps ϕ and ψ are adjacent if and only if for each edge xy of G, $\phi(x) \neq \psi(y)$, and moreover, for each vertex x, $\phi(x) \neq \psi(x)$. This is so, because for $i \neq j \in [q]$, $(x,i)(x,j)$ is an edge of $G[K_q]$ and $\phi(x)$ is a shorthand for $\phi(x,i)$ and $\psi(x)$ is a shorthand for $\psi(x,j)$.

In this sense, the subgraph of $K_c^{G[K_q]}$ induced by simple maps is isomorphic to $K_c^{G^o}$, where G^o is obtained from G by adding a loop to each vertex of G. We shall just treat $K_c^{G^o}$ as an induced subgraph of $K_c^{G[K_q]}$ and write $\phi \in V(K_c^{G^o})$ to mean that ϕ is a simple map in $K_c^{G[K_q]}$. Most of our argument is about properties of the subgraph $K_c^{G^o}$ of $K_c^{G[K_q]}$.

The graph $K_c^{G[K_q]}$ is a huge graph. As G has girth 6 and fractional chromatic number at least 4.1, $p = |V(G)|$ is probably about 200. The number in $K_c^{G[K_q]}$ is c^{pq}, which is roughly $(2^{200})^{2^{200}}$. The subgraph $K_c^{G^o}$ has c^p vertices, which is roughly $(2^{200})^{200}$. So $K_c^{G^o}$ is huge, but it is a very tiny fraction of $K_c^{G[K_q]}$.

Definition 2. For $v \in V(G)$ and $b \in [c]$, let

$$I(v,b) = \{\phi \in K_c^{G^o} : \Psi(\phi) = b = \phi(v)\}.$$

By Observation 2, $\Psi(\phi) \in Im(\phi)$ for any $\phi \in K_c^{G^o}$. Therefore

$$V(K_c^{G^o}) = \bigcup_{v \in V(G), b \in [c]} I(v,b).$$

As $K_c^{G^o}$ has c^p vertices, the average size of $I(v,b)$ is

$$\frac{c^p}{pc} = \frac{c^{p-1}}{p}.$$

Definition 3. We say $I(v,b)$ is *large* if $|I(v,b)| \geq 2pc^{p-2}$.

Observe that, by hypothesis, c is much larger than p. The power of c is the dominating factor. So $2pc^{p-2}$ is much smaller than the average size of $I(v,b)$. Thus intuitively, "most" of the $I(v,b)$'s should be large. So the next lemma is not a surprise.

Lemma 3. *There exists a vertex v of G such that*

$$|\{b \in [c] : I(v,b) \text{ is large }\}| > c/2.$$

Proof. For each vertex v of G, let $S(v) = \{b : I(v,b) \text{ is small}\}$. Assume to the contrary that for each v, $|S(v)| \geq c/2$. Let

$$\mathscr{L} = \{\phi \in K_c^{G^o} : \forall v \in V(G), \phi(v) \in S(v)\}.$$

Then

$$|\mathscr{L}| = \prod_{v \in V(G)} |S(v)| \geq \left(\frac{c}{2}\right)^p.$$

For any $\phi \in \mathscr{L}$, if $\phi \in I(v,b)$, then $I(v,b)$ is small. Thus

$$\mathscr{L} \subset \bigcup_{v \in V(G), b \in [c], I(v,b) \text{ is small}} I(v,b).$$

Therefore $|\mathscr{L}| < p \cdot c \cdot 2pc^{p-2} = 2p^2c^{p-1}$. But then

$$\left(\frac{c}{2}\right)^p < 2p^2c^{p-1}$$

which implies that $c < 2^{p+1}p^2$. But by our choice of c, we have $c = 4q+2 > 4q \geq 2^{p+1}p^2$, a contradiction. □

For two vertices x,y of G, denote by $d_G(x,y)$ the distance between x and y. Let v be a vertex of G for which $|\{b \in [c] : I(v,b)$ is large $\}| > c/2$. For $t \in \{2q+1, 2q+2, \ldots, 4q+2\}$, let $\mu_t \in K_c^{G[K_q]}$ be defined as

$$\mu_t(x,i) = \begin{cases} i, & \text{if } d_G(x,v) = 0,2, \\ q+i, & \text{if } d_G(x,v) = 1, \\ t, & \text{if } d_G(x,v) \geq 3. \end{cases}$$

Observe that μ_t are not simple maps. These will be the only non-simple maps used in the proof.

Claim. The set of maps $\{\mu_t : t \in \{2q+1, 2q+2, \ldots, 4q+2\}\}$ induces a clique in $K_c^{G[K_q]}$.

Proof. Assume to the contrary that for some $t \neq t'$, $\mu_t \not\sim \mu_{t'}$. Then there is an edge $(x,i)(y,j)$ of $G[K_q]$ such that $\mu_t(x,i) = \mu_{t'}(y,j)$. Let $\alpha = \mu_t(x,i) = \mu_{t'}(y,j)$.

Then $\alpha \in Im(\mu_t) \cap Im(\mu_{t'}) \subseteq \{i, q+i, t\} \cap \{j, q+j, t'\}$. As $t \neq t'$, we conclude that $i = j$ and $\alpha = i$ or $q+i$. Since $(x,i),(y,i)$ are distinct adjacent vertices, we conclude that $x \neq y$ and $xy \in E(G)$. If $\alpha = i$, then $d_G(x,v), d_G(y,v) \in \{0,2\}$ implies that G has a 3-cycle or a 5-cycle, contrary to the assumption that G has girth 6. If $\alpha = q+i$, then $d_G(v,x) = d_G(v,y) = 1$, and G has a 3-cycle, again a contradiction. This completes the proof of Claim 3.

So maps $\{\mu_t : t = 2q+1, 2q+2, \ldots, 4q+2\}$ are coloured by distinct colours, and hence there exists t such that $\Psi(\mu_t) \notin \{1,2,\ldots,2q\}$. As $\Psi(\mu_t) \in Im(\mu_t) = \{1,2,\ldots,q,t\}$, we have $\Psi(\mu_t) = t$.

Since $|\{b \in [c] : I(v,b)$ is large $\}| > c/2 = 2q+1$, there is a colour $b \in [c] - \{1,2,\ldots,2q,t\}$ such that $I(v,b)$ is large. Let $\theta \in K_c^{G^o}$ be defined as follows:

$$\theta(x) = \begin{cases} b, & \text{if } d_G(x,v) \geq 2, \\ t, & \text{if } d_G(x,v) \leq 1. \end{cases}$$

Claim. For $t \in \{2q+1, 2q+1, \ldots, 4q+2\}$, $\theta \sim \mu_t$.

Proof. Assume to the contrary that $\theta \not\sim \mu_t$. Then there is an edge $(x,i)(y,j) \in E(G[K_q])$ such that $\theta(x) = \theta(x,i) = \mu_t(y,j)$. (Note that $\theta(x,i) = \theta(x)$ as θ is a simple map). As $Im(\theta) \cap Im(\mu_t) = \{t\}$, we conclude that $\theta(x) = \mu_t(y,j) = t$. But then $d_G(x,v) \leq 1$ and $d_G(y,v) \geq 3$, and hence $x \neq y$ and $xy \notin E(G)$, contrary to the assumption that $(x,i)(y,j) \in E(G[K_q])$.

Thus $\Psi(\theta) \neq \Psi(\mu_t) = t$. As $\Psi(\theta) \in Im(\theta)$, we conclude that $\Psi(\theta) = b$.

Claim. For any $\phi \in I(v,b)$, there exists a vertex $x \neq v$ such that $\phi(x) \in \{b,t\}$.

Proof. Let $\phi \in I(v,b)$. By definition $\Psi(\phi) = b = \phi(v)$. So $\Psi(\phi) = \Psi(\theta)$. Hence $\phi \not\sim \theta$. So there is an edge $xy \in E(G^o)$ such that $\phi(x) = \theta(y)$. If $x = v$, then $\theta(y) = \phi(v) = b$. By definition of θ, we have $d_G(y,v) \geq 2$. Hence xy cannot be an edge in G^o, a contradiction. So $x \neq v$. As $\phi(x) = \theta(y) \in \{b,t\}$, this completes the proof of the claim.

For each $x \neq v$, let

$$J_x = \{\phi \in I(v,b) : \phi(x) \in \{b,t\}\}.$$

For a map $\phi \in J_x$, the image $\phi(v)$ of v is fixed, i.e., $\phi(v) = b$. The image $\phi(x)$ of x has two choices: b and t. For each of other $n-2$ vertices y of G, $\phi(y)$ has c choices. Therefore $|J_x| \leq 2c^{n-2}$. By Claim 3, $I(v,b) = \cup_{x \in V(G)-\{v\}} J_x$. So $|I(v,b)| \leq 2(n-1)c^{n-2}$, contrary to the assumption that $I(v,b)$ is large. This completes the proof of Theorem 1.

Remark 1. The key part of the proof of Theorem 1 is to show that $K_c^{G[K_q]}$ is not c-colourable. For each vertex v of G, for $t \in \{2q+1, 2q+2, \ldots, 4q+2\}$, let

$$\mu_{v,t}(x,i) = \begin{cases} i, & \text{if } d_G(x,v) = 0,2, \\ q+i, & \text{if } d_G(x,v) = 1, \\ t, & \text{if } d_G(x,v) \geq 3; \end{cases}$$

Let H be the subgraph of $K_c^{G[K_q]}$ induced by

$$V(K_c^{G^o}) \cup \{\mu_{v,t} : v \in V(G), t \in \{2q+1, 2q+2, \ldots, 4q+2\}\}.$$

What we have proved is that the subgraph H of $K_c^{G[K_q]}$ is not c-colourable. Note that H is a very tiny fraction of $K_c^{G[K_q]}$, although H by itself is a huge graph.

The reviewer of this note asks if there is an intuition as to why this subgraph H is the right thing to be thinking about. Also, once you have the intuition that this subgraph should have high chromatic number, why are the sets $I(b,v)$ the right things to look at to analyse this?

This is also a question in my mind. Reading Shitov's paper, one naturally wonders how did he come up with this proof? I am not the right person to answer this question. However, since one main purpose of this note is to explain Shitov's proof, I will give it a try.

The maps $\{g_i : i \in [c]\}$ is already a c-clique. So all the c colours are used by these maps in a proper c-colouring Ψ of $K_c^{G[K_q]}$, where we assume that $\Phi(g_i) = i$ for $i \in [c]$. To derive a contradiction, it is natural to consider maps that have many neighbors in this c-clique, namely, maps ϕ with a small image set $Im(\phi)$. The smallest image set has size 2 (for otherwise it is one of these constant maps). For each vertex v of G, for any two colors $b,t \in [c]$, let $\theta_{v,b,t} \in K_c^{G^o}$ be defined as

$$\theta_{v,b,t}(x) = \begin{cases} b, & \text{if } d_G(x,v) \geq 2, \\ t, & \text{if } d_G(x,v) \leq 1. \end{cases}$$

Now $\Psi(\theta_{v,b,t}) = b$ or t, as it is adjacent to every g_i with $i \neq b,t$. If we can somehow fix the colour of $\theta_{v,b,t}$ to be b, that is very useful. The maps $\mu_{v,t}$ are used to forces $\theta_{v,b,t}$ to be colored by b.

Now it remains to find a map ϕ with $\Psi(\phi) = b$ which is adjacent to $\theta_{v,b,t}$, so that we obtain a contradiction to the assumption that Ψ is a proper colouring of $K_c^{G[K_q]}$. The candidates are those maps $\phi \in K_c^{G^o}$ such that $\Psi(\phi) = \phi(v) = b$, because if $\Psi(\phi) = b$, then there is a vertex $x \in V(G)$ such that $\phi(x) = b$. If $x \neq v$, then ϕ is not adjacent to $\theta_{v,b,t}$. Indeed, the definition of $\theta_{v,b,t}$ is chosen in such a way that its neighors coloured with colour b has a simple structure.

This is why we have the definition of $I(v,b)$.

Can we find such a map in $I(v,b)$? Intuitively, this is promising: For a map $\phi \in I(v,b)$ to be a neighbor of $\theta_{v,b,t}$, one just need to avoid assigning color b to any other vertex, and avoid assigning color t to the neighbours of v. If $I(v,b)$ is large enough, then such a map shall exists. Once we have shown that for approriate v,b,t, $I(v,b)$ is large enough and by using maps $\mu_{v,t}$, we can force $\theta_{v,b,t}$ be coloured by b, we arrive at a contradiction.

4 The Poljak-Rödl function

The Poljak-Rödl function is defined in [9]:

$$f(n) = \min\{\chi(G \times H) : \chi(G), \chi(H) \geq n\}.$$

Hedetniemi's conjecture is equivalent to saying that $f(n) = n$ for all positive integer n. Shitov's Theorem says that for sufficiently large n, $f(n) \leq n-1$. Using Shitov's result, Tardif and Zhu [16] proved that $f(n) \leq n - (\log n)^{1/4-o(1)}$. Tardif and Zhu asked in [16] if there is a positive constant ε such that $f(n) \leq (1-\varepsilon)n$ for sufficiently large n. This question was answered in affirmative by He and Wigderson [4] with $\varepsilon \approx 10^{-9}$. On the other hand, in [16], Tardif and Zhu proved that if a special case of a conjecture of Stahl [12] conserning the multi-chromatic number of Kneser graph is true, then we have $\limsup_{n\to\infty} \frac{f(n)}{n} \leq \frac{1}{2}$.

Recently, I proved in [20] that the conclusion $\limsup_{n\to\infty} \frac{f(n)}{n} \leq \frac{1}{2}$ holds without assuming Stahl's conjecture.

Theorem 2. *For $d \geq 1$, let G be a p-vertex graph of girth 6 and with $\alpha(G) \leq \frac{p}{8.1d}$. Let $q \geq 2^{p-1}p^2$ and $c = 4q+2$. Then $\chi(G[K_q]) \geq 2dc - 2c + 2$ and $\chi(K_{dc}^{G[K_q]}) \geq 2dc - 2c + 2$. Consequently, $f(2dc - 2c + 2) \leq dc$.*

Proof. As explained before, the existence of a graph G described above was proved by Erdős. Similarly as in the proof of Theorem 1, $\chi(G[K_q]) \geq \frac{|V(G[K_q])}{\alpha(G[K_q])} \geq \frac{pq}{p/8.1d} = 8.1dq \geq 2dc > 2dc-2c+2$. Now we show that $\chi(K_{dc}^{G[K_q]}) \geq 2dc-2c+2$.

Assume Ψ is a $(dc+t)$-colouring of $K_{dc}^{G[K_q]}$ with colour set $[dc+t]$. We shall show that $dc+t \geq 2dc-2c+2$, i.e., $t \geq dc-2c+2$. Let $S=[dc+t]-[dc]$. The colours in $[dc]$ are called *primary colours* and colours in S are called *secondary colours*. So we have $t=|S|$ secondary colours.

Similarly as before, we may assume that $\Psi(g_i)=i$ for $i \in [dc]$. Then for any map $\phi \in K_{dc}^{G[K_q]}$, if $i \notin Im(\phi)$, then $\phi \sim g_i$ and $\Psi(\phi) \neq i$. Thus for any $\phi \in K_{dc}^{G[K_q]}$, $\Psi(\phi) \in Im(\phi) \cup S$.

For positive integers $m \geq 2k$, let $K(m,k)$ be the Kneser graph whose vertices are k-susbets of $[m]$, and for two k-subsets A,B of $[m]$, $A \sim B$ if $A \cap B = \emptyset$. It was proved by Lovász in [7] that $\chi(K(m,k)) = m-2k+2$.

For a c-subset A of $[cd]$, let H_A be the subgraph of $K_{cd}^{G[K_q]}$ induced by

$$\{\phi \in V(K_{cd}^{G[K_q]}) : Im(\phi) \subseteq A\}.$$

Then H_A is isomorphic to $K_c^{G[K_q]}$. By Theorem 1, $|\Psi(H_A)| \geq c+1$. As $Im(\phi) \subseteq A$ and $|A|=c$, $\Phi(H_A)$ contains at least one secondary colour. Let $\tau(A)$ be an arbitrary secondary colour contained in $\Psi(H_A)$.

If A,B are c-subsets of $[dc]$ and $A \cap B = \emptyset$, then every vertex in H_A is adjacent to every vertex in H_B. Hence $\Psi(H_A) \cap \Psi(H_B) = \emptyset$. In particular, $\tau(A) \neq \tau(B)$. Thus τ is a proper colouring of the Kneser graph $K(dc,c)$. As $\chi(K(dc,c)) = dc-2c+2$, we conclude that $t=|S| \geq dc-2c+2$. This completes the proof of Theorem 2.

For a positive integer d, let $p=p(d)$ be the minimum number of vertices of a graph G with girth 6 and $\chi_f(G) \geq 8.1d$. It follows from Theorem 2 that for any integer $q \geq p^2 2^{p-1}$, $f(2(d-1)(4q+2)+2) \leq (4q+2)d$. As $f(n)$ is non-decreasing, for integers n in the interval $[2(d-1)(4q+2)+2, 2(d-1)(4q+6)+2]$, we have $f(n) \leq (4q+6)d$.

Hence for all integers $n \geq 2(4q+2)(d-1)+2$,

$$\frac{f(n)}{n} \leq \frac{(4q+6)d}{2(4q+2)(d-1)+3} = \frac{1}{2} + \frac{4q+4d+1}{2(d-1)(4q+2)+2}.$$

Note that if $d \to \infty$, then $p=p(d)$ goes to infinity, and $q \geq p^3 2^p$ goes to infinity. Therefore

$$\limsup_{n \to \infty} \frac{f(n)}{n} \leq \frac{1}{2}.$$

Theorem 2 improves the result of He and Wigderson [4]. However, He and Wigderson use a modification of Shitov's method, which might be of independent interest.

In the proof of Theorem 2, we actually showed that a tiny subgraph of $K_{dc}^{G[K_q]}$ has chromatic number close to $2dc$. It is not clear if the remaining part of the graph

$K_{dc}^{G[K_q]}$ can be used to show that this graph actually has a much larger chromatic number. We observe that if one can show that the chromatic number of $K_{dc}^{G[K_q]}$ is more than kdc for some positive integer k, then Stahl's conjecture implies that $\limsup_{n\to\infty} \frac{f(n)}{n} \leq \frac{1}{k+1}$.

5 Lower bound for $f(n)$

The breakthrough result of Shitov leads to an improvement of the upper bound for the function $f(n)$. On the other hand, the only known lower bound for $f(n)$ is that $f(n) \geq 4$ for $n \geq 4$. We do not know if $f(n)$ is bounded by a constant or not. What we do know is that if $f(n)$ is bounded by a constant, then the smallest such constant is at most 9.

To prove this result, we need to consider the product of digraphs. For a digraph D, we use $A(D)$ to denote the set of arcs of D. An arc in D is either denoted by an ordered pair (x,y), or by an arrow $x \to y$. Digraphs are allowed to have digons, i.e., a pair of opposite arcs.

Assume D_1, D_2 are digraphs. The product $D_1 \times D_2$ has vertex set $V(D_1) \times V(D_2)$, where $(x,y) \to (x',y')$ is an arc if and only if (x,x') is an arc in D_1 and (y,y') is an arc in D_2. The chromatic number of a digraph D is defined to be $\chi(\underline{D})$, where $\underline{D}$ is the underlying graph of D, i.e., obtained from D by replacing each arc (x,y) with an edge xy. Given a digraph D, let D^{-1} be the digraph obtained from D by reversing the direction of all its arcs. It is easy to see that for any digraphs D_1, D_2,

$$\underline{D_1} \times \underline{D_1} = (\underline{D_1 \times D_2}) \cup (\underline{D_1 \times D_2^{-1}}).$$

Hence

$$\chi(\underline{D_1} \times \underline{D_1}) \leq \chi(D_1 \times D_2) \times \chi(D_1 \times D_2^{-1}).$$

Let

$$\begin{aligned} g(n) &= \min\{\chi(D_1 \times D_2) : \chi(D_1), \chi(D_2) \geq n\}, \\ h(n) &= \min\{\max\{\chi(D_1 \times D_2), \chi(D_1 \times D_2^{-1})\} : \chi(D_1), \chi(D_2) \geq n\}. \end{aligned}$$

Since $E(\underline{D}_1 \times \underline{D}_2) = E(\underline{D_1 \times D_2}) \cup E(\underline{D_1 \times D_2^{-1}})$, we have

$$g(n) \leq h(n) \leq f(n) \leq h(n)^2.$$

The following result was proved by Poljak and Rödl in [9].

Theorem 3. *If $g(n)$ (respectively $h(n)$) is bounded by a constant, then the smallest such constant is at most* 4. *Consequently, if $f(n)$ is bounded by a constant, then the smallest such constant is at most* 16.

Proof. For a graph D, let $\partial(D)$ be the digraph with vertex set $A(D)$, where $(x,y) \to (x',y')$ is an arc of $\partial(D)$ if and only if $y = x'$. In particular, if $(x,y),(y,x)$ is a digon in D, then $(x,y) \to (y,x)$ and $(y,x) \to (x,y)$ is a digon in $\partial(D)$.

Lemma 4. *For any digraph D,*

$$\min\{k : 2^k \geq \chi(D)\} \leq \chi(\partial(D)) \leq \min\{k : \binom{k}{\lceil k/2 \rceil} \geq \chi(D)\}.$$

Proof. If $\phi : V(\partial(D)) \to [k]$ is a proper colouring of $\partial(D)$, then for each vertex v of D, let $\psi(v) = \{\phi(e) : e \in A^+(v)\}$, where $A^+(v)$ is the set of out-arcs at v. Then ψ is a proper colouring of D (with subsets of $[k]$ as colours). Indeed, if $e = (x,y)$ is an arcs of D, then $\phi(e) \in \psi(x) - \psi(y)$. So $\psi(x) \neq \psi(y)$. The number of colours used by ψ is at most the number of subsets of $[k]$, which is 2^k.

If $\psi : V(D) \to \binom{k}{\lceil k/2 \rceil}$ is a proper colouring of D (where the colours are $\lceil k/2 \rceil$-subsets of $[k]$), then for any arc $e = (x,y)$ of D, let $\phi(e)$ be any integer in $\psi(y) - \psi(x)$ (as $\psi(y) \neq \psi(x)$, such an integer exists). Then if $(x,y) \to (y,z)$ is an arc in $\partial(D)$, then $\phi(x,y) \in \psi(y)$ and $\phi(y,z) \notin \psi(y)$. Hence $\phi(x,y) \neq \phi(y,z)$. I.e., ϕ is a proper colouring of $\partial(D)$. This completes the proof of Lemma 4.

It follows easily from the definition that

$$\partial(D_1 \times D_2) = \partial(D_1) \times \partial(D_2),$$
$$\partial(D^{-1}) = (\partial(D))^{-1}.$$

Suppose $g(n)$ is bounded and C is the smallest upper bound. As $g(n)$ is non-decreasing, there is an integer n_0 such that $g(n) = C$ for all $n \geq n_0$. Let $n_1 = 2^{n_0}$, and let D_1, D_2 be digraphs with $\chi(D_1), \chi(D_2) \geq n_1$ and $\chi(D_1 \times D_2) = C$. It follows from Lemma 4 that $\chi(\partial(D_1)), \chi(\partial(D_2)) \geq n_0$ and hence $\chi(\partial(D_1) \times \partial(D_2)) = \chi(\partial(D_1 \times D_2)) \geq C$. By Lemma 4 again, we have

$$C \geq \chi(D_1 \times D_2) > \binom{C-1}{\lceil (C-1)/2 \rceil}.$$

This implies that $C \leq 4$.

The same argument shows that if $h(n)$ is bounded by a constant, then the smallest such constant is at most 4. Since $h(n) \leq f(n) \leq h(n)^2$, if $f(n)$ is bounded by a constant, then the smallest such a constant is at most 16.

Next we show that if $g(n)$ (respectively, $h(n)$) is bounded by a constant, then the smallest such constant cannot be 4. Assume to the contrary that the smallest constant bound for $g(n)$ is 4. Let n_0 be the integer given above, and let $n_1 = 2^{n_0}, n_2 = 2^{n_1}$. Then $g(n_2) = g(n_1) = g(n_0) = 4$. Let D_1, D_2 be two digraphs with $\chi(D_1), \chi(D_2) \geq n_2$ and $\chi(D_1 \times D_2) = 4$. The same argument as above shows that

$$\chi(\partial(\partial(D_1 \times D_2))) = 4.$$

However, we shall show that if $\chi(D) \leq 4$, then $\chi(\partial(\partial(D))) \leq 3$. Let $\vec{K}_4$ be the complete digraph with vertex set $\{1,2,3,4\}$, where (i,j) is an arc for any distinct $i,j \in \{1,2,3,4\}$. If $\chi(D) = 4$, then D admits a homomorphism to $\vec{K}_4$. Hence $\partial(\partial(D))$ admits a homomorphism to $\partial(\partial(\vec{K}_4))$. So it suffices to show that $\partial(\partial(\vec{K}_4)) \leq 3$. In 1990, I was a Ph.D. student at The University of Calgary. After reading the paper by Poljak and Rödl [9], I found a 3-colouring of $\partial(\partial(\vec{K}_4))$ by brute force. I was happy to tell this to my supervisor Professor Norbert Sauer, who then told the result to Duffus. Then I learned from Duffus the following elegant 3-colouring of $\partial(\partial(\vec{K}_4))$, given earlier by Schelp that was not published.

Each vertex of $\partial(\partial(\vec{K}_4))$ is a sequence ijk with $i,j,k \in [k]$, $i \neq j$, $j \neq k$ (but i may equal to k). Let

$$c(ijk) = \begin{cases} j, & \text{if } j \neq 4, \\ s, & \text{if } j = 4 \text{ and } s \in \{1,2,3\} - \{i,k\} \end{cases}$$

Then it is easy to verify that c is a proper 3-colouring of $\partial(\partial(\vec{K}_4))$. This completes the proof that $g(n)$ is either bounded by 3 or goes to infinity. Similarly, $h(n)$ is either bounded by 3 or goes to infinity, and consequently, $f(n)$ is either bounded by 9 or goes to infinity.

Later I learned from Hell that Poljak also obtained this strengthening independently and that was published later (in 1992) [8].

Tardif and Wehlau [15] proved that $f(n)$ is bounded if and only if $g(n)$ is bounded.

The fractional version of Hedetniemi's conjecture was proved in [19]: For any two graphs G and H, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$. Thus if $f(n)$ is bounded by 9, and G and H are n-chromatic graphs with $\chi(G \times H) \leq 9$, then at least one of G and H has fractional chromatic number at most 9.

In [19], I defined the following Poljak-Rödl type function:

$$\psi(n) = \min\{\chi(G \times H) : \chi_f(G), \chi(H) \geq n\}.$$

I proposed a weaker version of Hedetniemi's conjecture, which is equivalent to the statement that $\psi(n) = n$ for all positive integer n. However, Shitov's proof actually refutes this weaker version of Hedetniemi's conjecture, as the graph G used in the proof of Theorem 1 has large fractional chromatic number. The proof of Theorem 2 shows that

$$\limsup_{n \to \infty} \frac{\psi(n)}{n} \leq \frac{1}{2}.$$

On the other hand, it follows from the definition that $f(n) \leq \psi(n)$. A natural question is the following:

Question 1. Is $\psi(n)$ bounded by a constant? If $\psi(n)$ is bounded by a constant, what could be the smallest such constant?

Remark Very recently, I constructed relatively small counterexample to Hedetniemi's conjecture in [21]: There are graphs G and H with $3,403$ and $10,501$ vertices respetively such that $\chi(G), \chi(H) \geq 126$ and $\chi(G \times H) \leq 125$.

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