Observations on disks with tropical Lagrangian boundary

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Abstract In this survey, we look at some expectations for Lagrangian submanifolds which are built as the lifts of tropical curves from the base of an Lagrangian torus fibration. In particular, we perform a first computation showing that holomorphic triangles can appear with boundary on the Lagrangian submanifold. We speculate how these holomorphic triangles can contribute to the count of holomorphic strips in the Lagrangian intersection Floer cohomology between a tropical Lagrangian submanifold and a fiber of the SYZ fibration.

1 Tropical Lagrangians and Holomorphic Disks

Mirror symmetry is a geometric duality between symplectic geometry on (X,ω) and complex geometry on a "mirror space" $(\check{X},J)$ [2]. The spaces $X,\check{X}$ are expected to arise as the total spaces of dual Lagrangian torus fibrations over a common base space Q [9]. The base of a Lagrangian torus fibration always is equipped with an affine structure, and it is predicted that both the symplectic geometry of X and complex geometry of $\check{X}$ degenerate to tropical geometry on Q [4]. In good examples, one uses the affine structure on Q to identify lattices $T_{\mathbb{Z}}Q$ and $T^*_{\mathbb{Z}}Q$ inside the tangent and cotangent bundle respectively. The mirror spaces can be reconstructed from this data as:

$$X := T^*Q/T_{\mathbb{Z}}Q \xrightarrow{\text{val}} Q \xleftarrow{\check{\text{val}}} \check{X} = TQ/T_{\mathbb{Z}}Q$$

which are equipped with their canonical symplectic and almost complex structures. The simplest example of this (which we will focus on) is the example where $Q=\mathbb{R}^n$ and $X=\check{X}=(\mathbb{C}^*)^n$. Recently, the independent works of [7, 8, 5, 6] constructed

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Lagrangian submanifolds $L(V) \subset X$ whose valuation projection $\text{val}(L(V))$ could be made arbitrarily close to a given tropical subvariety $V \subset Q$. These Lagrangians fit in with predictions from mirror symmetry principles. For example, in [6] the number of tropical Lagrangians found was at least as many as the number of curves in the mirror quintic, and in [5] tropical Lagrangian hypersurfaces were shown to be homologically mirror to sheaves supported on divisors.

Parallel to computations in mirror symmetry, tropical geometry has been employed to understand the counts of holomorphic disks with boundary on a given Lagrangian inside of a symplectic manifold. One particularly visible instance of this method of computation is from [10], which used tropical techniques to understand the holomorphic disk count for Lagrangian tori inside of $\mathbb{CP}^2$. These holomorphic disk counts were used to distinguish Hamiltonian isotopy classes of monotone Lagrangian tori. More generally, the count of holomorphic disks with boundary on non-exact Lagrangian submanifolds $L \subset X$ provide a deformation to the homology of L. These deformations are a necessary ingredient in the mirror symmetry prediction, as they provide "corrections" to the identification of symplectic and complex geometric invariants. For example, the correspondence between the moduli space of Lagrangian tori on X and points on the mirror space $\check{X}$ is only expected to hold once these correction terms have been computed [1].

These invariants are frequently difficult to compute, so the promise of reducing them to combinatorial type computations in the setting of tropical geometry is particularly enticing. We exhibit an explicit computation for holomorphic disks for a fixed example, and speculate on what this computation means for the more general problem of computing holomorphic strips contributing to the differential in Lagrangian intersection Floer theory.

2 Disks via sections of Lagrangian fibrations

The first example we consider is the symplectic manifold $X = (\mathbb{C}^*)^2$ with base $Q = \mathbb{R}^2$ and torus fibration given by $\text{val}(z_1, z_2) = (\log|z_1|, \log|z_2|)$. We consider the three tropical curves drawn in fig. 1, and consider their lift to 3 Lagrangian cylinders in X,

$$L_1 = \{(e^r, e^{i\theta}) \mid r \in \mathbb{R}, \theta \in S^1\}$$
$$L_2 = \{(e^{i\theta}, e^r) \mid r \in \mathbb{R}, \theta \in S^1\}$$
$$L_\lambda = \{(e^{r+i\theta}, e^\lambda e^{-r+i\theta}) \mid r \in \mathbb{R}, \theta \in S^1\}$$

where $\lambda \in \mathbb{R}$ is a parameter picked to define the third line. Although these Lagrangians are non-compact, they are conical at infinity and so we may count holomorphic disks and polygons with boundary on the L_i. For topological reasons, the L_i do not individually bound holomorphic disks. However, the collection of all 3 has a chance to bound a holomorphic triangle. We parameterize this triangle with the

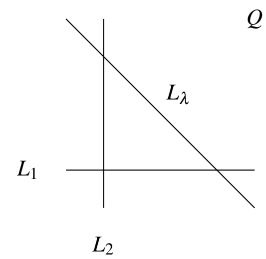


Fig. 1 Projection of three Lagrangians $L_i \subset X$ to the base of the SYZ fibration via the valuation map. These Lagrangians bound a holomorphic section of $\text{val} : X \to Q$ for particular values of λ.

domain $\Delta_\lambda := \{x+iy \mid x,y \geq 0, x+y \leq \lambda\}$. We then consider the holomorphic map:

$$\begin{aligned} u_\lambda : \Delta_\lambda &\to (\mathbb{C}^*)^2 \\ (x+iy) &\mapsto (e^{x+iy}, e^{y-ix}) \end{aligned}$$

One notices that the boundary $u_\lambda(x,0) = (e^x, e^{-ix})$ is contained in L_1 and $u_\lambda(0,y)$ is contained in L_2. However, the remaining boundary $u_\lambda(t,\lambda-t) = \left(e^{t+i(\lambda-t)}, e^{\lambda-t-it}\right)$ will lie in the Lagrangian L_λ if and only if $\lambda \in 2\pi\mathbb{Z}$. This leads to the following strange behaviour: as one modifies the parameter λ, the three Lagrangians L_1, L_2, L_λ periodically bound a holomorphic section over a triangle in the base of $\text{val} : (\mathbb{C}^*)^2 \to \mathbb{R}^2$. These holomorphic triangles are not regular, so it is not unexpected upon taking a generic choice of λ we see no disk. However, for families of Lagrangians parameterized by λ, these holomorphic disks do appear regularly.

This sporadic appearance of Maslov index zero disks also occurs in the descriptions of wall-crossings for Lagrangian tori [1]. In that setting, as one takes a family of Lagrangian tori interpolating between the Chekanov torus and product torus in $\mathbb{C}^2$, a non-regular Maslov index 0 disk with boundary flashes in and out of existence.

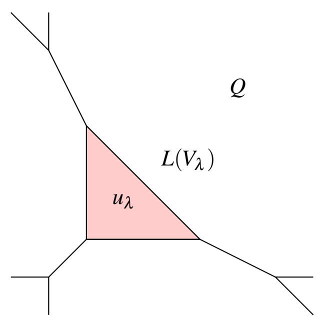
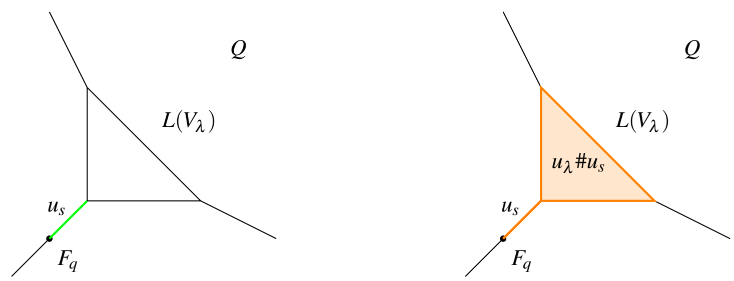


Fig. 2 The valuation projection of a tropical Lagrangian bounding a holomorphic disk. The disk is obtained by taking the holomorphic section of fig. 1 over the triangle, and rounding off the corners. It exists only for certain values of λ determining the tropical Lagrangian submanifold

We can replicate this kind of behaviour by turning our holomorphic triangles into holomorphic disks by performing Lagrangian surgery on the intersections of L_1, L_2, L_λ to obtain a embedded tropical Lagrangian submanifold $L(V_\lambda)$, whose projection to the base Q is drawn in fig. 2. After performing surgery on the Lagrangian, it is expected that holomorphic triangles with boundary on the L_i become holomorphic disks on the Lagrangian $L(V_\lambda)$ [3]. The holomorphic triangles described in the first computation give examples of holomorphic disks with boundary on the tropical Lagrangian $L(V_\lambda)$ when the parameter λ passes through a multiple of 2π. This kind of holomorphic disk is expected to exists: in fact, for some choice of almost complex structure, the wall crossing phenomenon for tropical Lagrangians observed in [5] proves that there is a non-regular disk with boundary on this tropical Lagrangian submanifold corresponding to the wall-crossing phenomenon for the Chekanov and Clifford tori in $\mathbb{CP}^2$.

3 Towards computing Floer Support

The presence of non-regular holomorphic disks plays an important role in homological mirror symmetry, where the flux coordinates parameterizing the space of Lagrangian submanifolds must be corrected by contributions from bubbling of disks. Additionally, the presence of a non-regular holomorphic disk u_λ can honestly modify the behaviour of a regular holomorphic disk u_s with Lagrangian boundary, as u_λ and u_s can be glued along their boundary to produce a new regular holomorphic disk. We now speculate about the Lagrangian intersection Floer theory between the $L(V_\lambda)$ and a fiber of the SYZ fibration, $F_q := \text{val}^{-1}(q)$, as drawn in fig. 3a.



(a) A small holomorphic strip, observed in [5].

(b) Creating a bigger strip by attaching the non-regular disk to the regular strip.

Fig. 3: Examples of holomorphic strips contributing to the differential of $CF^\bullet(L(V_\lambda), F_q)$.

In [5], we gave an example worked out with Diego Matessi regarding an interesting holomorphic strip: $u_s : [0,1] \times \mathbb{R} \to (\mathbb{C}^*)^2$ with boundaries on F_q and V_λ. This holomorphic strip projects under the valuation to the line segment drawn in fig. 3a. u_s is a regular holomorphic strip which persists even as we modify the parameters λ governing the size of the "hole" in the tropical elliptic curve. As a result, for choices of λ which the disk u_λ appears, the strip u_s intersects u_λ, and we conjecture that it is possible to glue together u_s and u_λ to obtain a larger regular holomorphic strip $u_\lambda \# u_s$ contributing to the Floer differential of $CF^\bullet(L(V_\lambda), F_q)$. This conjectured holomorphic strip is drawn in fig. 3b.

The Floer theoretic support, which is the set of Lagrangian branes F_q for which this Floer cohomology does not vanish, gives the equation of an algebraic curve in the mirror $\check{X}$. In order to compute this support, it is necessary to compute all of the holomorphic strips with boundary on F_q and $L(V_\lambda)$. We hope that this combinatorial description of two such holomorphic strips can be extended to produce a combinatorial model for the Lagrangian intersection Floer theory of arbitrary tropical Lagrangian submanifolds, which would significantly improve our understanding of the interplay between homological mirror symmetry and tropical geometry.

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