Aperiodic order meets number theory: Origin and structure of the field

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Abstract Aperiodic order is a relatively young area of mathematics with connections to many other fields, including discrete geometry, harmonic analysis, dynamical systems, algebra, combinatorics and, above all, number theory. In fact, number-theoretic methods and results are present in practically all of these connections. It was one aim of this workshop to review, strengthen and foster these connections.

Aperiodic structures and patterns have revolutionised parts of science, as evidenced by the 2011 Nobel Prize in Chemistry awarded to Dan Shechtman for the discovery of quasicrystals. Beautiful, yet profound, examples in mathematics have captured the attention of many, starting with the famous Penrose tiling from 1974. The deep connection between these topics emerged from the number-theoretic work of Yves Meyer (Abel Prize 2017). During this workshop, an international community of like-minded researchers came together to discuss recent results and develop research collaborations at the interface of aperiodic theory and number theory.

From the very beginnings of research on aperiodic order, intriguing links to number theory were observed; these have become increasingly apparent in recent years. Over the past decade, there has been a tremendous development in various directions, including spectra and transport theory of Schrödinger operators, reversing symmetry groups in dynamical systems and ergodic theory, and topological invariants in symbolic and algebraic dynamics. Additionally, the links between number

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theory, dynamical systems, and theoretical computer science are strengthening, and the lines between them are blurring.

Aperiodic order [36, 5, 6, 24] has several roots in mathematics — predating even the main impetus for the area: the discovery of quasicrystals. These include Harald Bohr's development of the theory of almost periodic functions [13], Robert Berger's proof of the undecidability of the tiling problem [11], and Yves Meyer's work on model sets [35], which was later developed further in [27, 37, 28, 38, 29], as well as early works on tilings and patterns including Roger Penrose's famous fivefold tiling of the plane [40]. From the very start, there have been connections to various areas of mathematics. Number theory features prominently, for instance from the early work of Peter Pleasants, as reviewed in [42]. Arguably the most obvious relation occurs for planar tilings with (non-crystallographic) rotational symmetry, which are closely related to rings of integers in cyclotomic fields.

At the same time, within number theory, the advent of modern (digital) computation has underscored the importance of understanding the relationship between base expansions and algebraic operations. This topic has a rich history, especially in Australian mathematics through the work of Loxton, Mahler and van der Poorten. It focussed mainly on results related to finite automata and their generalisations — structures of importance in aperiodic order.

Recent results and questions at the intersection of aperiodic order and number theory include the following.

- The study of weak model sets [7, 25] was partially motivated by the set of visible points of the integer lattice and the set of kth power-free integers [8, 43], and their connections to Sarnak's programme on the Möbius disjointness conjecture; see [39, 31] for recent developments. Under a rather natural extremality assumption, it is possible to establish pure point spectrum, and that such systems can be seen as natural generalisations of regular model sets.
- The diffraction theory of infinite point sets in Euclidean space with its corresponding inverse problem shows fundamental connections to almost periodicity [32, 48, 47] and Lyapunov exponents [34]. Likewise, there are similar structures in the theory of Schrödinger operators, with surprising applications to spectra on graphs [17].
- Rather than using diffraction, connections with Diophantine approximations can be used to investigate and quantify the nature of order in a cut and project set [21]. Quantities of interest here include the complexity function and the repetitivity function. Another interesting characteristic is the discrepancy, which describes the difference between the expected and actual number of appearances of a given patch in a large region.
- Constant-length substitutions are important objects for both number theory and aperiodic order. A generalisation of these are regular sequences, which are related to finitely generated semigroups of matrices. The growth properties of these sequences are related to questions in both areas, including spectral properties [1] as well as the finiteness conjecture for integer matrices [2, 30]. These results are

connected with the scaling structure of singular continuous measures, as recently analysed for the Thue–Morse measure; see [4] and reference therein.

- Logarithmic Mahler measures occur as the maximal Lyapunov exponents of matrix cocycles for binary constant-length substitutions [3]. In this way, Lehmer's problem for height-one polynomials having minimal Mahler measure becomes equivalent to a natural question from the spectral theory of binary constant-length substitutions. This supports another connection between Mahler measures and dynamics, beyond the well-known appearance of Mahler measures as entropies in algebraic dynamics [44].
- One of the questions that evolved from the early work on Wang tilings [11] and has attracted attention over the years is the question of the minimal set of tiles required to enforce quasiperiodicity. For a long time, it seemed that substitution-based structures were the way to go, until Kari [23] and Culik [16] came up with an ingenious way of assigning rational edge values to Wang tiles in a way that rules out periodicity by arithmetic constraints. This systems continues to attract attention, see for instance [45, 22, 26], but the question whether it opens up a new approach to aperiodic structures remains to be explored. Closely related is the search for planar monotiles of hexagonal shape [41, 46].
- Constant-size substitutions and characteristic-p S-unit equations are also connected to the question of mixing in algebraic dynamics, as described recently in work by Derksen and Masser [18, 19]. Techniques in these articles should shed light on the nature of the symmetry groups of these algebraic dynamical systems, thus providing additional and powerful methods for the characterisation of (extended) symmetry groups in algebraic dynamics, as recognised in [9, 20, 15, 14], as well as the analysis of related systems [12].

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