# Delone sets on spirals

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Motivated by phyllotaxis in botany, the angular development of plants widely found in nature, we give a simple mathematical characterization of Delone sets on spirals.

Let $X$ be a subset of $\mathbb{R}^2$ which is identified with the complex plane $\mathbb{C}$. Denote by $B(x,r)$ the open ball of radius $r$ centered at $x$. We say $X$ is *relatively dense* if there exists $r>0$ such that, for any $x\in\mathbb{C}$, $B(x,r)\cap X\neq\emptyset$ holds, $X$ is *uniformly discrete* if there exists $r>0$ such that, for any $x\in\mathbb{C}$, we have $\operatorname{card}(B(x,r)\cap X)\leq 1$, and finally $X$ is a *Delone set* if it is both relatively dense and uniformly discrete.

Set $\mathbf{e}(z)=e^{2\pi iz}$. Fix an *angle* $\alpha\in[0,1)$ and a strictly increasing function $f$ from $\mathbb{R}_{\geq 0}$ to itself. We wish to characterize when the set

$$X_f=\left\{f(n)\,\mathbf{e}(n\alpha)\mid n\in\mathbb{N}\right\}$$

on a spiral curve $\left\{f(t)\,\mathbf{e}(t\alpha)\mid t\in\mathbb{R}_{\geq 0}\right\}$ forms a Delone set.

Let us collect necessary conditions. Clearly $X_f$ is not relatively dense if the angle $\alpha$ is rational, since $X_f$ is contained in a union of a finite number of lines passing through the origin. An easy discussion leads to the following result.

**Lemma 1.** *If $X_f$ is relatively dense, then $\limsup_{n\to\infty} f(n)/\sqrt{n}<\infty$. If $X_f$ is uniformly discrete, then $\liminf_{n\to\infty} f(n)/\sqrt{n}>0$.*

Hereafter, we assume that $f(n)=\sqrt{n}$ and that $\alpha$ is irrational, and study the set

$$X(\alpha)=\left\{\sqrt{n}\,\mathbf{e}(n\alpha)\mid n\in\mathbb{N}\right\}.$$

In other words, we are interested in the sequence of points on the Fermat spiral that progresses by a constant angle $\alpha$ (see Figure 1).

A real number $\alpha$ is *badly approximable* if there exists a positive constant $C$ so that

$$q|q\alpha-p|\geq C$$

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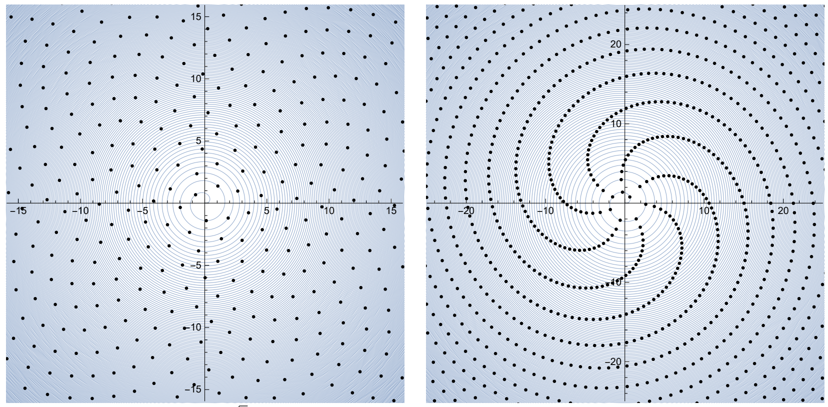

(a) $X(\frac{3-\sqrt{5}}{2})$ (b) $X(\pi - 3)$

**Fig. 1** Constant angular progressions on Fermat spiral

holds for all $(p,q) \in \mathbb{Z} \times \mathbb{N}$. It is well-known that $\alpha$ is badly approximable if and only if the partial quotients of the continued fraction expansion of $\alpha$ are bounded (compare [3, Theorem 23] and [1, Theorem 1.9]). In particular, if $\alpha$ is a real quadratic irrational, then $\alpha$ is badly approximable, due to Lagrange's theorem.

With the help of the three distance theorem on irrational rotation conjectured by Steinhaus and proved by Sós [5, 6] and then Świerczkowski [8], Surányi [7], Halton [2] and Slater [4], we can prove the following result.

**Theorem 1.** *The following four statements are equivalent.*

*a) $X(\alpha)$ is relatively dense,*
*b) $X(\alpha)$ is uniformly discrete,*
*c) $X(\alpha)$ is a Delone set,*
*d) the angle $\alpha$ is badly approximable.*

Searching for a possible higher-dimensional extension is an interesting problem.

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