Extended symmetry groups of multidimensional subshifts with hierarchical structure

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In this contribution, we discuss the automorphism group, i.e., the centralizer of the shift action inside the group of self-homeomorphisms of a subshift, together with the extended symmetry group (the corresponding normalizer) of certain $\mathbb{Z}^d$ subshifts with a hierarchical structure, like bijective substitutive subshifts and the Robinson tiling. This group has been previously studied in the work of Baake, Roberts and Yassawi [1], among others.

Treating these subshifts as geometric objects, we introduce techniques to identify allowed extended symmetries from large-scale structures present in certain special points of the subshift, leading to strong restrictions on the group of extended symmetries. We prove that, in the aforementioned cases, $\mathrm{Sym}(X,\mathbb{Z}^d)$ (and thus $\mathrm{Aut}(X,\mathbb{Z}^d)$) is virtually-$\mathbb{Z}^d$, and we explicitly represent the non-trivial extended symmetries, associated with the quotient $\mathrm{Sym}(X,\mathbb{Z}^d)/\,\mathrm{Aut}(X,\mathbb{Z}^d)$, as a subset of rigid transformations of the coordinate axes. We also show how our techniques carry over to the study of the Robinson tiling, both in its minimal and non-minimal version. We emphasize the geometric nature of these techniques and how they reflect the capability of extended symmetries to capture such properties in a subshift.

Our discussion starts with the computation of the automorphism group for d-dimensional substitutive subshifts coming from bijective rectangular substitutions. By an application of desubstitution and some algebraic manipulations, we generalize Coven's theorem (see [2]) by showing the following.

Theorem 1. *For a non-trivial, primitive, bijective substitution θ on the alphabet $\mathscr{A} = \{0,1\}$, $\mathrm{Aut}(X_\theta,\mathbb{Z}^d)$ is generated by the shifts and the relabeling map (flip map) $\delta(x) := \bar{x}$, where $\bar{x}$ represents the sequence obtained from x by swapping all 1s with 0s and vice versa, and thus is isomorphic to $\mathbb{Z}^d \times (\mathbb{Z}/2\mathbb{Z})$.*

For more general alphabets, the same method of proof yields the following.

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Corollary 1. *Let θ be a non-trivial, primitive, bijective substitution on an alphabet $\mathscr{A}$ with at least two symbols. For any $f \in \mathrm{Aut}(X_\theta, \mathbb{Z}^d)$, there exists a bijection $\tau\colon \mathscr{A} \to \mathscr{A}$ and a value $\boldsymbol{k} \in \mathbb{Z}^d$ such that $f = \sigma_{\boldsymbol{k}} \circ \tau_\infty$. Thus, $\mathrm{Aut}(X_\theta, \mathbb{Z}^d)$ is isomorphic to a subgroup of $\mathbb{Z}^d \times S_{|\mathscr{A}|}$, where S_n is the symmetric group in n elements.*

Next, we divert our attention towards *extended symmetries*, which are a generalization of shift automorphisms, in the sense that they are homeomorphisms $f\colon X \to X$ such that there exists a matrix $A_f \in \mathrm{GL}_d(\mathbb{Z})$ for which the following identity holds,

$$\forall \boldsymbol{n} \in \mathbb{Z}^d : f \circ \sigma_{\boldsymbol{n}} = \sigma_{A_f \boldsymbol{n}} \circ f.$$

The set of all such homeomorphisms is a group, $\mathrm{Sym}(X, \mathbb{Z}^d)$, which is a group extension of $\mathrm{Aut}(X, \mathbb{Z}^d)$ by some subgroup of $\mathrm{GL}_d(\mathbb{Z})$. Extended symmetries satisfy a variant of the Curtis–Hedlund–Lyndon theorem and thus are completely determined by a local mapping $F\colon \mathscr{A}^U \to \mathscr{A}$ (with $U \subset \mathbb{Z}^d$ finite) and the matrix A_f.

We devise a 'fracture method' in which we recognize special pairs of points from a subshift which match only on a half-space, in such a way that the discrepancy in the other half-space is preserved in the images under f by an application of the Curtis–Hedlund–Lyndon theorem. By showing limitations on the possible directions of these fractures, we can compute the extended symmetry group of several subshifts.

It is known that the automorphism group of the Robinson shift X_{Rob} and its minimal subshift M_{Rob} is isomorphic to $\mathbb{Z}^2$ (see e.g. [3]). We show the following.

Proposition 1. *For the Robinson shift, $\mathrm{Sym}(X_{\mathrm{Rob}}, \mathbb{Z}^2) \cong \mathbb{Z}^2 \rtimes D_4$, where D_4 is the dihedral group of order 8. The same holds for M_{Rob}.*

In the case of substitutive subshifts coming from bijective substitutions, the desubstitution technique allows us to apply a variant of the above fracture argument; the bijectiveness imposes a restriction on the possible directions on fracture, which lead to the following result.

Theorem 2. *For a d-dimensional, non-trivial, primitive, bijective substitution θ, the quotient group of all admissible lattice transformations of the subshift X_θ, $\mathrm{Sym}(X_\theta, \mathbb{Z}^d) / \mathrm{Aut}(X_\theta, \mathbb{Z}^d)$, is isomorphic to a subset of the hyperoctahedral group $Q_d \cong (\mathbb{Z}/2\mathbb{Z}) \wr S_d = (\mathbb{Z}/2\mathbb{Z})^d \rtimes S_d$, which is the symmetry group of the d-dimensional cube. Thus, the extended symmetry group $\mathrm{Sym}(X_\theta, \mathbb{Z}^d)$ is virtually-$\mathbb{Z}^d$.*

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