Algebraic invariants for group actions on the Cantor set

María Isabel Cortez

The algebraic invariants[1] associated to the group actions on the Cantor set provide an interesting connection between the fields of dynamical systems and group theory. For instance, Giordano, Putnam and Skau have shown in [29] that the dimension group (see [24] for an introduction about dimension groups) of a minimal $\mathbb{Z}$-action on the Cantor set completely determines its strong orbit equivalence class. Furthermore, the topological full group of such a system, which is known from Juschenko and Monod [38] to be amenable, determines its flip-conjugacy class (see [6] and [30] for more details). On the other hand, the amenability of the topological full groups of minimal $\mathbb{Z}$-actions together with their properties shown in [41] by Matui make them the first known examples of infinite groups which are at the same time amenable, simple and finitely generated. Recently, another algebraic invariant, the group of automorphisms of actions on the Cantor set, has caught the eye of several researchers working in the field [13, 15, 16, 17, 14, 19, 20]. In [5], Boyle, Lind and Rudolph focused their attention on the group of automorphisms of subshifts of finite type, showing that these groups are always countable and residually finite. At the same time, they gave an example of a minimal $\mathbb{Z}$-action on the Cantor set whose group of automorphisms contains $\mathbb{Q}$, which implies that the automorphism group of a minimal action may be a non-residually finite group (recall that the $\mathbb{Z}$-subshifts of finite type are not minimal). This leads to the natural question about the relation between the algebraic properties of the group of automorphisms and the dynamics of the system. Indeed, the residually finite property of the group of automorphisms of the subshifts of finite type is a consequence of the existence of periodic points.

Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile.
e-mail: maria.cortez@usach.cl

[1] By an *algebraic invariant* of the dynamical system (X,T,G) we understand any algebraic structure associated to the system which determines some dynamical properties of (X,T,G) or whose properties depend on the dynamics of (X,T,G).

1 Minimal Cantor systems

By a dynamical system we mean a continuous action $T\colon G\times X\to X$ of a countable group G on a compact metric space X (phase space). We denote this as (X,T,G), and for every $g\in G$, we call $T^g\colon X\to X$ the homeomorphism on X induced by the action of g on X. The dynamical system is *free* or *aperiodic* if $T^g(x)=x$ implies $g=1_G$ (the neutral element in G) for any $x\in X$. The *orbit* of $x\in X$ is the set $O_T(x)=\{T^g(x):x\in G\}$, and we say that the system (X,T,G) is *minimal* if for every $x\in X$ its orbit is dense in X. Minimality is also equivalent to the non-existence of non-trivial sub-dynamical systems of (X,T,G), i.e, the system is minimal if and only if the unique non-empty closed T-invariant set $Y\subseteq X$ is $Y=X$. As a consequence of Zorn's lemma, we get that every dynamical system (X,T,G) has a minimal sub-dynamical system (see for example [1, 3]). It is clear that, if (X,T,G) is aperiodic, the minimal sub-dynamical systems are also aperiodic.

A particular class of dynamical systems are the *Cantor systems*, which are defined as the systems (X,T,G) where X is a Cantor set. An example of a Cantor system is the full G-shift on the finite alphabet Σ. More precisely, given Σ^G, the set of all functions $x\colon G\to\Sigma$, the shift action σ of G on Σ^G is defined as $\sigma^g x(h)=x(g^{-1}h)$, for every $g,h\in G$ and $x\in\Sigma^G$. If we endow Σ with the discrete topology and Σ^G with the product topology, the space Σ^G becomes a Cantor set and every σ^g is a homeomorphism. Thus, (Σ^G,σ,G) is a Cantor dynamical system.

The full G-shift is neither aperiodic nor minimal. However, in [37], Hjorth and Molberg show that for every countable group G there exists an aperiodic Cantor system (X,T,G). Moreover, in [4] and [26], the authors show that this aperiodic Cantor system can be chosen as an aperiodic *G-subshift*, i.e., an aperiodic sub-dynamical system of a full G-shift.

2 Algebraic properties of the topological full group of Toeplitz subshifts

The *full group* of the dynamical system (X,T,G) is the subgroup $[G]$ of the group of homeomorphisms f on X such that for every $x\in X$ there exists $g\in G$ such that $f(x)=T^g(x)$. This is the topological version of the full group introduced by Dye [23] in the context of measure-theoretic dynamical systems. It was shown by Medynets in [43] that the full group of a Cantor aperiodic system is a complete invariant for topological orbit equivalence (see [27, 28, 29, 31] for the notion and results about topological orbit equivalence).

The *topological full group* of the dynamical system (X,T,G) is the subgroup $[[G]]$ of $[G]$ of all the homeomorphisms f on X such that for every $x\in X$ there exist a neighbourhood U of x and $g\in G$ such that $f|_U=T^g$ (see [30, 33] for definitions and results). It is straightforward to check that, when X is a connected space, $[[G]]$ is isomorphic to G. Conversely, when X is a Cantor set, the topological full group depends not only on the group G, but on the dynamics of the system. Indeed, from [43], it is possible to deduce that, for aperiodic Cantor systems, the topological full group is a complete invariant for continuous orbit equivalence (see [9] and [40] for definitions and results about continuous orbit equivalence).

From a group theoretical point of view, Jushenko and Monod have shown in [38] that the topological full group of the minimal Cantor system $(X,T,\mathbb{Z})$ is amenable (see for example [7] for definitions and results about amenability of groups and [34, 35, 41, 42] for more algebraic properties of the topological full group). On the other hand, Elek and Monod exhibited in [25] an example of an aperiodic minimal Cantor system given by a $\mathbb{Z}^2$-action whose topological full group is not amenable. Thus the algebraic properties of the topological full groups of minimal Cantor systems (X,T,G), when the group G is not $\mathbb{Z}$, still remain unclear. In joint work with Medynets and Petite, we are investigating some of these algebraic properties for the class of the Toeplitz G-subshifts.

2.1 Toeplitz G-subshifts

Let Σ be a finite alphabet. An element $x \in \Sigma^G$ is *Toeplitz* if for every $g \in G$ there exists a finite index subgroup Γ of G such that $x(g) = x(\gamma g)$, for every $\gamma \in \Gamma$. A subshift $X \subseteq \Sigma^G$ is a *Toeplitz G-subshift* if there exists a Toeplitz element $x \in X$ such that $X = \overline{O_\sigma(x)}$; see [21] for a survey on Toeplitz $\mathbb{Z}$-subshifts and [8, 11, 12, 39] for results about Toeplitz G-subshifts. It is not difficult to show that the Toeplitz G-subshifts are Cantor minimal systems and that G admits an aperiodic Toeplitz G-subshift if and only if G is residually finite [12]. The aperiodic Toeplitz G-subshifts are characterized as the minimal almost one-to-one symbolic extensions of the G-odometers [12], which correspond to the minimal aperiodic equicontinuous actions of G on the Cantor set [9]. Furthermore, the G-odometers are among the only minimal aperiodic Cantor systems with a topological full group that can be described in an explicit way (see [18] for $\mathbb{Z}$-odometers and [9] for G-odometers when G is residually finite). This description allows to deduce that the topological full group of a G-odometer is amenable if and only if G is amenable (see [9]).

The existence of an almost one-to-one factor map from a Toeplitz G-subshift to a G-odometer makes it possible to define for those systems nice nested sequences of Kakutani–Rohlin partitions (see [22, 36] for definitions and results about Kakutani–Rohlin partitions for $\mathbb{Z}$-actions and [12, 11, 32] for Toeplitz G-subshifts), which provides a useful tool to study the properties of the topological full group of these subshifts in order to find examples of Toeplitz $\mathbb{Z}^2$-subshifts whose topological full groups are not amenable.

We are still working on the following general question: Which are the properties on a Toeplitz G-subshift that ensure that its topological full group is amenable?

3 Algebraic properties of the group of automorphisms of a group action on the Cantor set

Let (X,T,G) be a minimal aperiodic Cantor system. The *normalizer* group of (X,T,G), denoted $\mathrm{Norm}(X,T,G)$, is defined as the subgroup of all the homeomorphisms $h\colon X\to X$ such that there exists an isomorphism $\alpha_h\colon G\to G$ such that $h\circ T^g=T^{\alpha_h(g)}\circ h$, for every $g\in G$. The aperiodicity of the action implies the uniqueness of α_h for any element $h\in\mathrm{Norm}(X,T,G)$. Thus we can define the *automorphism group* of (X,T,G) as

$$\mathrm{Aut}(X,T,G)=\{h\in\mathrm{Norm}(X,T,G):\alpha_h=id\}.$$

It is immediate that $\mathrm{Aut}(X,T,G)$ is a normal subgroup of $\mathrm{Norm}(X,T,G)$. For the case $G=\mathbb{Z}$, the quotient of the normalizer group by the group of automorphisms is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Important progress has been made in the study of the group of automorphisms of minimal $\mathbb{Z}$-subshifts, establishing a connection between the complexity of the subshifts and the algebraic properties of the group of automorphisms; see [2, 15, 16, 17, 19].

In [10], we obtained results concerning the realization of groups as subgroups of the normalizer and the automorphism group of minimal aperiodic actions on the Cantor set as follows.

- Every countable group is the subgroup of the normalizer of some minimal aperiodic action of a countable Abelian free group on the Cantor set.
- Every residually finite group Γ can be realized as the subgroup of the automorphism group of a minimal $\mathbb{Z}$-action on the Cantor set [10, Prop. 7]. A key tool for the proof of this result is the characterization of residually finite groups as those groups G for which every full G-shift has a dense subset of points with finite orbit [7, Thm. 2.7.1].
- For any countable group G, the group of automorphisms of a minimal aperiodic G-action on the Cantor set is a subgroup of the group of automorphisms of a minimal $\mathbb{Z}$-action on the Cantor set.

References

1. Auslander, J.: Minimal Flows and their Extensions. North-Holland, Amsterdam (1988)
2. Baake, M., Roberts, J.A.G., Yassawi, R.: Reversing and extended symmetries of shift spaces. Discr. Cont. Dynam. Syst. **38**, 835–866 (2018)
3. Brin, M., Stuck, G.: Introduction to Dynamical Systems. Cambridge University Press, Cambridge (2002)
4. Aubrun, N., Barbieri, S., Thomassé, S.: Realization of aperiodic subshifts and densities. Groups Geom. Dyn., to appear
5. Boyle, M., Lind, D., Rudolph, D.: The automorphism group of a shift of finite type. Trans. Amer. Math. Soc. **306**, 71–114 (1988)

6. Boyle, M.: Topological Orbit Equivalence and Factor Maps in Symbolic Dynamics. PhD thesis, University of Washington, Seattle (1983)
7. Ceccherini-Silberstein, T., Coornaert, M.: Cellular Automata and Groups. Springer, Berlin (2010)
8. Cortez, M.I.: $\mathbb{Z}^d$ Toeplitz arrays. Discr. Cont. Dynam. Syst. **15**, 859–881 (2006)
9. Cortez, M.I., Medynets, C.: Orbit equivalence rigidity of equicontinuous systems. J. London Math. Soc. **94**, 545–556 (2016)
10. Cortez, M.I., Petite, S.: On the centralizers of aperiodic actions on the Cantor set. Preprint arXiv:1807.04654
11. Cortez, M.I., Petite, S.: Invariant measures and orbit equivalence for generalized Toeplitz subshifts. Groups Geom. Dynam. **8**, 1007–1045 (2014)
12. Cortez, M.I., Petite, S.: G-odometers and their almost one-to-one extensions. J. London Math. Soc. **78**, 1–20 (2008)
13. Coven, E., Quas, A., Yassawi, R.: Computing automorphism groups of shifts using atypical equivalence classes. Discr. Anal. **2016**, 611:1–28 (2016)
14. Cyr, V., Franks, J., Kra, B., Petite, S.: Distortion and the automorphism group of a shift. J. Mod. Dyn. **13**, 147–161 (2018)
15. Cyr, V., Kra, B.: The automorphism group of a shift of linear growth: beyond transitivity. Forum Math. Sigma **3**, e5:1–27 (2015)
16. Cyr, V., Kra, B.: The automorphism group of a shift of subquadratic growth. Proc. Amer. Math. Soc. **144**, 613–621 (2016)
17. Cyr, V., Kra, B.: The automorphism group of a minimal shift of stretched exponential growth. J. Mod. Dyn. **10**, 483–495 (2016)
18. de Cornulier, Y.: Groupes pleins-topologiques [d'après Matui, Juschenko, Monod,...]. Sém. Bourbaki, 65ème anné, no 1064 (2012/13)
19. Donoso, S., Durand, F., Maass, A., Petite, S.: On automorphism groups of low complexity subshifts. Ergodic Th. & Dynam. Syst. **36**, 64–95 (2016)
20. Donoso, S., Durand, F., Maass, A., Petite, S.: On automorphism groups of Toeplitz subshifts. Discr. Anal. **2017**, 11:1–19 (2017)
21. Downarowicz, T.: Survey of odometers and Toeplitz flows. In Kolyada, S., Manin, Y., Ward, T. (eds.), Algebraic and Topological Dynamics, pp. 7–37. AMS, Providence, RI (2005)
22. Durand, F., Host, B., Skau, C.: Substitutional dynamical systems, Bratteli diagrams and dimension groups. Ergodic Th. & Dynam. Syst. **19**, 953–993 (1999)
23. Dye, H.A.: On groups of measure preserving transformations. I. Amer. J. Math. **81**, 119–159 (1959)
24. Effros, E.G.: Dimensions and C*-Algebras. Conference Board of the Mathematical Sciences, Washington, D.C. (1981)
25. Elek, G., Monod, N.: On the topological full group of a minimal Cantor $\mathbb{Z}^2$-system. Proc. Amer. Math. Soc. **141**, 3549–3552 (2013)
26. Gao, S., Jackson, S., Seward, B.: A coloring property for countable groups. Math. Proc. Cambridge Philos. Soc. **147**, 579–592 (2009)
27. Giordano, T., Matui, H., Putnam, I.F., Skau, C.F.: The absorption theorem for affable equivalence relations. Ergodic Th. & Dynam. Syst. **28**, 1509–1531 (2008)
28. Giordano, T., Matui, H., Putnam, I.F., Skau, C.F.: Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems. Invent. Math. **179**, 119–158 (2010)
29. Giordano, T., Putnam, I.F., Skau, C.F.: Topological orbit equivalence and C^*-crossed products. J. Reine Angew. Math. (Crelle) **469**, 51–111 (1995)
30. Giordano, T., Putnam, I.F., Skau, C.F.: Full groups of Cantor minimal systems. Israel J. Math. **111**, 285–320 (1999)
31. Giordano, T., Putnam, I.F., Skau, C.F.: Affable equivalence relations and orbit structure of Cantor dynamical systems. Ergodic Th. & Dynam. Syst. **24**, 441–475 (2004)
32. Gjerde, R., Johansen, O.: Bratteli–Vershik models for Cantor minimal systems: applications to Toeplitz flows. Ergodic Th. & Dynam. Syst. **20**, 1687–1710 (2000)
33. Glasner, E., Weiss, B.: Weak orbit equivalence of Cantor minimal systems. Intern. J. Math. **6**, 559–579 (1995)

34. Grigorchuk, R.I., Medynets, K.S.: On the algebraic properties of topological full groups. Sb. Math. **205**, 843–861 (2014)
35. Grigorchuk, R., Medynets, K.: Presentations of topological full groups by generators and relations. J. Algebra **500**, 46–68 (2018)
36. Herman, R., Putnam, I.F., Skau, C.F.: Ordered Bratteli diagrams, dimension groups and topological dynamics. Intern. J. Math. **3**, 827–864 (1992)
37. Hjorth, G., Molberg, M.: Free continuous actions on zero-dimensional spaces. Topology Appl. **153**, 1116–1131 (2006)
38. Juschenko, K., Monod, N.: Cantor systems, piecewise translations and simple amenable groups. Ann. Math. **178**, 775–787 (2013)
39. Krieger, F.: Sous-décalages de Toeplitz sur les groupes moyennables résiduellement finis. J. London Math. Soc. **75**, 447–462 (2007)
40. Li, X.: Continuous orbit equivalence rigidity. Ergodic Th. & Dynam. Syst. **38**, 1543–1563 (2018)
41. Matui, H.: Some remarks on topological full groups of Cantor minimal systems. Intern. J. Math. **17**, 231–251 (2006)
42. Matui, H.: Some remarks on topological full groups of Cantor minimal systems II. Ergodic Th. & Dynam. Syst., in press
43. Medynets, K.: Reconstruction of orbits of Cantor systems from full groups. Bull. London Math. Soc. **43**, 1104–1110 (2011)