Lyapunov exponents: recent applications of Fürstenberg's theorem in spectral theory

David Damanik

Abstract We discuss the phenomenon of Anderson localization and a new proof of it in one space dimension. This proof is due to V. Bucaj, D. Damanik, J. Fillman, V. Gerbuz, T. VandenBoom, F. Wang, Z. Zhang, and it is centered around the positivity of and large deviation estimates for the Lyapunov exponent — a strategy originally developed in non-random settings by J. Bourgain, M. Goldstein, W. Schlag.

1 The Anderson model

The Anderson model was proposed in 1958 by P. W. Anderson. Its main feature is that randomness can trap quantum states, a phenomenon called Anderson localization. Anderson received the Physics Nobel Prize in 1977 for this work. The model is given by a discrete Schrödinger operator on the d-dimensional standard lattice with potential values given by independent identically distributed random variables.

Concretely, given a probability measure ν on $\mathbb{R}$ whose topological support is compact and contains at least two points, we consider $\Omega = (\operatorname{supp}\nu)^{\mathbb{Z}^d}$ and $\mu = \nu^{\mathbb{Z}^d}$. For every $\omega \in \Omega$ and $n \in \mathbb{Z}^d$, we set $V_\omega(n) = \omega_n$. This defines, for $\omega \in \Omega$, a potential $V_\omega : \mathbb{Z}^d \to \mathbb{R}$, and a Schrödinger operator in $\ell^2(\mathbb{Z}^d)$:

$$[H_\omega \psi](n) = \sum_{|m-n|_1 = 1} \psi(m) + V_\omega(n)\psi(n).$$

The spectrum and the spectral type of H_ω are μ-almost surely independent of ω. The almost sure spectrum Σ is explicitly given by $\Sigma = [-2d, 2d] + \operatorname{supp}(\nu)$.

Anderson localization comes in two standard flavors: spectral localization and dynamical localization. Here, spectral localization refers to the statement that, μ-almost surely, H_ω has pure point spectrum in a suitable energy region $\Sigma_\ell \subseteq \Sigma$ with

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David Damanik
Rice University, Houston, USA. e-mail: damanik@rice.edu

exponentially decaying eigenvectors. Dynamical localization refers to the statement that quantum states remain trapped. Concretely, this means that $e^{-itH_\omega}\chi_{\Sigma_\ell}(H_\omega)\delta_0$ remains mostly in some fixed finite region for all times.

The size and shape of the localized energy region Σ_ℓ depends on the dimension d and the single-site measure ν:

- $d = 1$: It is *known* that $\Sigma_\ell = \Sigma$.
- $d = 2$: It is *conjectured* that $\Sigma_\ell = \Sigma$. Currently, the known result is the same as in the case $d \geq 3$.
- $d \geq 3$: It is *known* that Σ_ℓ contains nontrivial neighborhoods of $\partial\Sigma$. It is *conjectured* that $\Sigma_\ell \neq \Sigma$ if the diameter of $\operatorname{supp}\nu$ is not too large, and that there is a sharp transition from localization to diffusive transport.

We will focus on the case $d = 1$. The first localization proof was given by Carmona, Klein and Martinelli in [4]. A different proof by Shubin, Vakilian and Wolff appears in [6]. Both of these proofs rely on multiscale analysis. A simpler and more direct proof was recently given by Bucaj et al. in [3]. The latter proof will be discussed in what follows.

This proof is centered around the positivity of and large deviation estimates for the Lyapunov exponent. Approaching localization proofs in this way is a strategy due to Bourgain, Goldstein in the case of quasi-periodic potentials [1] and to Bourgain, Schlag in the case of potentials generated by the doubling map [2].

This new proof also suggests how new results can be obtained. For example, Damanik, Fillman and Sukhtaiev implemented this approach in the setting of Anderson models on metric and discrete tree graphs and proved spectral and dynamical localization for these operators [5].

2 Main results

Here is the pair of theorems for the family $\{H_\omega\}_{\omega\in\Omega}$ of random Schrödinger operators, acting in $\ell^2(\mathbb{Z})$ via

$$[H_\omega\psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n),$$

where the potential V_ω is given by independent identically distributed random variables with a common distribution that has a compact support that contains at least two elements.

Theorem 1 (Spectral localization for the 1D Anderson model). *Almost surely, H_ω is spectrally localized, that is, it has pure point spectrum with exponentially decaying eigenfunctions.*

Theorem 2 (Exponential dynamical localization for the 1D Anderson model). *There is a constant $\gamma > 0$ so that for almost every ω and every $\varepsilon > 0$, there is a constant $C = C_{\omega,\gamma,\varepsilon} > 0$ such that, for all $m, n \in \mathbb{Z}$,*

$$\sup_{t\in\mathbb{R}} |\langle \delta_n, e^{-itH_\omega}\delta_m\rangle| \le Ce^{\varepsilon|m|}e^{-\gamma|n-m|}.$$

3 Lyapunov exponents: positivity and large deviation estimates

The difference equation (or generalized eigenvalue equation) for the operator H_ω:

$$u(n+1)+u(n-1)+V_\omega(n)u(n) = Eu(n)$$

admits a two-dimensional solution space, as any two consecutive values of u determine all other values. Fixing $(u(0),u(-1))^\top$ as the point of reference, the linear map taking this vector to $(u(n),u(n-1))^\top$ is given by the so-called transfer matrix $M_n^E(\omega)$. Ergodicity of the standard shift transformation $T:\Omega\to\Omega$, $[T\omega]_n = \omega_{n+1}$ implies that for each E, there are $L(E)\ge 0$ and $\Omega_-^E, \Omega_+^E \subseteq \Omega$ with $\mu(\Omega_-^E) = \mu(\Omega_+^E) = 1$ such that

$$L(E) = \begin{cases} \lim_{n\to\infty} \frac{1}{n}\log\|M_n^E(\omega)\| & \text{for } \omega\in\Omega_+^E, \\ \lim_{n\to-\infty} \frac{1}{|n|}\log\|M_n^E(\omega)\| & \text{for } \omega\in\Omega_-^E. \end{cases}$$

The number $L(E)$ is called the *Lyapunov exponent*.

For E, let ν_E be the push-forward of ν under the map

$$x\mapsto \begin{pmatrix} E-x & -1 \\ 1 & 0 \end{pmatrix}$$

and let G_E be the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ that contains $\operatorname{supp}\nu_E$.

By a result of Fürstenberg, a sufficient condition for $L(E)>0$ is

1. G_E is not compact,
2. G_E is strongly irreducible (which means that there is no finite non-empty invariant set of directions).

A modification of a result of Ishii shows that the condition

3. $\exists A,B\in G_E$ with $\mathrm{tr}(A)\neq 0$, $\mathrm{tr}(B)\neq 0$, $\det(AB-BA)\neq 0$,

implies 1 and 2. This condition is often easy to check.

For the Anderson model on $\mathbb{Z}$, let us verify 1. and 2. for arbitrary $E\in\mathbb{R}$. Since the support of the single-site distribution has cardinality at least two, it follows that ν_E also has at least two points in its support. Thus, G_E contains at least two distinct elements of the form

$$M_x = \begin{bmatrix} x & -1 \\ 1 & 0 \end{bmatrix},$$

say, M_a and M_b with $a\neq b$. Note that

$$A = M_a M_b^{-1} = \begin{bmatrix} 1 & a-b \\ 0 & 1 \end{bmatrix} \in G_E.$$

Taking powers of the matrix A, we see that G_{v_E} is not compact, verifying 1.

Now, consider $V_1 := \mathrm{span}(\mathbf{e}_1)$, the projection of $\mathbf{e}_1 := (1,0)^\top$ to $\mathbb{RP}^1$. Then, one has $AV_1 = V_1$ and, for every $V \in \mathbb{RP}^1$, $A^n V$ converges to V_1. Thus, if there is a nonempty finite invariant set of directions $\mathscr{F} \subseteq \mathbb{RP}^1$, one must have $\mathscr{F} = \{V_1\}$. However, we also have

$$A' = M_a^{-1} M_b = \begin{bmatrix} 1 & 0 \\ a-b & 1 \end{bmatrix} \in G_E$$

and $A'V_1 \neq V_1$. This establishes 2.

So far, we have seen that the transfer matrices are almost surely asymptotically exponentially large. That is,

$$\lim_{|n| \to \infty} \frac{1}{|n|} \log \|M_n^E(\omega)\| = L(E) > 0$$

for every $E \in \mathbb{R}$ and μ-almost every ω.

It is natural to ask what can be said about the size of $\log \|M_n^E(\omega)\|$ before n is taken to infinity. Large deviation estimates address this issue.

Theorem 3 (Uniform LDT for the Lyapunov exponent). *For any $\varepsilon > 0$, there exist $C = C(\varepsilon) > 0$, $\eta = \eta(\varepsilon) > 0$ such that, for all $n \in \mathbb{Z}_+$ and all $E \in \Sigma$,*

$$\mu \left\{ \omega \in \Omega : \left| \frac{1}{n} \log \left\| M_n^E(\omega) \right\| - L(E) \right| \geq \varepsilon \right\} \leq C e^{-\eta n}.$$

4 Transfer matrices, Dirichlet determinants, and Green's functions

There are well known connections between the transfer matrices discussed above, the determinant of the restriction $H_{\omega,N}$ of H_ω to the finite interval $[0, N-1] \cap \mathbb{Z}$ with Dirichlet boundary conditions, and the Green function of $H_{\omega,N}$, defined by

$$G^E_{\omega,N}(m,n) = \langle \delta_m, (H_{\omega,N} - E)^{-1} \delta_n \rangle$$

for $0 \leq m, n \leq N-1$.

The following formula connects the transfer matrix and the determinants:

$$M_N^E(\omega) = \begin{pmatrix} \det(E - H_{\omega,N}) & -\det(E - H_{T\omega,N-1}) \\ \det(E - H_{\omega,N-1}) & -\det(E - H_{T\omega,N-2}) \end{pmatrix}, \quad N \geq 2.$$

This formula shows that if the transfer matrix has exponentially large norm, then at least one of the determinants must be exponentially large.

The following formula connects the determinants and the Green function:

$$G^E_{\omega,N}(m,n) = \frac{\det[H_{\omega,m} - E]\det[H_{T^{n+1}\omega,N-n-1} - E]}{\det[H_{\omega,N} - E]}.$$

If one has exponentially large determinants, this formula shows that the Green function must have exponential off-diagonal decay.

Finally, the following lemma connects the Green function and the solutions.

Lemma 1. *If u is a solution of the difference equation $H_\omega u = Eu$ and $E \notin \sigma(H_{\omega,N})$, then*

$$u(n) = -G^E_{\omega,N}(n,0)u(-1) - G^E_{\omega,N}(n,N-1)u(N)$$

for $0 \le n \le N-1$.

With this lemma and the results discussed earlier one can readily deduce for almost all ω's that a polynomially bounded solution must in fact decay exponentially, thus establishing spectral localization. A second look at the semi-uniform localization properties of the eigenvectors then allows one to establish dynamical localization as well.

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