Extended symmetries of Markov subgroups

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A symmetry of a tesselation is an isometry of the plane, or space, preserving the tesselation. What symmetry groups can one get? This is a classical problem in geometry, leading to the wallpaper groups of the plane or crystallographic groups in higher dimensions. For dynamical systems with a $\mathbb{Z}^d$-action on X, the symmetries are the homeomorphisms that commute with the action. This is the centralizer of $\mathbb{Z}^d$ in $Homeo(X)$. The extended symmetries are given by the normalizer. Baake, Roberts and Yassawi [1] showed that the centralizer can be non-trivial for well-known systems such as the Thue–Morse shift or the Ledrappier shift. The latter is a standard example of a Markov subgroup and the topic of this talk is the extended symmetry group of arbitrary Markov subgroups in $\mathbb{Z}^2$ shifts.

A 2D Markov subgroup is described by a polynomial with two indeterminates $p(X,Y)$ with coefficients in $\mathbb{F}_2$. A fundamental result by Quas and Trow [3] gives precise conditions such that the symmetry group is 'algebraic': this occurs if $p(X,Y)$ has no collinear factors. Using results from algebraic geometry, one can deduce from this that the extended symmetry group is a finitely generated Abelian group if such a $p(X,Y)$ is squarefree. It is infinitely generated if it is not squarefree. This leaves the case of a polynomial with collinear factors. A prime example of this is $p(X,Y) = (1+X)(1+Y)$. It turns out that the elements of its extended symmetry group correspond to automorphisms of the Bernoulli shift $\{0,1\}^{\mathbb{Z}}$ that commute with the flip (the involution that flips 0 and 1). Which automorphisms have this property? This is not an easy question, and I left this as a homework exercise during the talk.

It seems likely that the extended symmetry group is non-amenable if $p(X,Y)$ has collinear factors. For some cases, other than $(1+X)(1+Y)$, this is not so dfficult to prove. The full result is topic of ongoing research with Dan Rust and Reem Yassawi [2]. I would like to thank the participants of the workshop for stimulating discussions and am looking forward to their solutions of the homework exercise.

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References

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