Renormalisation for inflation tilings I: General theory

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Inflation tilings are generated by iterating an inflation procedure ρ, which first expands a (partial) tiling linearly by a factor λ, and then divides each expanded tile (called supertile) according to a fixed rule into a set of original tiles. The relative positions of the tiles of type i within a supertile of type j are encoded in a set T_{ij} (we assume here finitely many tile types, up to translations). The associated inflation matrix M_ρ, which we assume to be primitive, has entries $\mathrm{card}(T_{ij})$ and a leading eigenvalue λ^d. The information contained in T_{ij} is also encoded in the matrix-valued function

$$B_{ij}(k) = \sum_{t \in T_{ij}} \mathrm{e}^{2\pi \mathrm{i} t \cdot k},$$

which is known as the Fourier matrix of the inflation. Note that the n^{th} power of ρ has the Fourier matrix $B^{(n)}(k) = B(k)B(\lambda k) \cdots B(\lambda^{n-1}k)$.

Suppose now an inflation tiling is decorated with point measures on the control points of its tiles, with weights which may depend on the tile type. The diffraction spectrum of the resulting measure is then given by the Fourier transform of its pair correlation measure. This is again a measure, which can be decomposed into pure-point (pp), absolutely continuous (ac), and a singular continuous (sc) parts. Here, we are here mainly interested in the presence or absence of an ac part.

As first observed in [2], and further elaborated in [3], the self-similarity of inflation tilings results in exact renormalisation equations, which the pair correlation measure of the tiling must satisfy. These in turn lead to exact scaling relations for the diffraction measure, which must hold for each spectral component separately. For instance, the Radon–Nikodym density $v(k)$ of the ac part of the Fourier amplitude (a vector with one component per tile type) must satisfy the relation

$$v(\lambda k) = \lambda^{d/2} B^{-1}(k)\, v(k),$$

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provided $B(k)$ is invertible for almost all k. As $v(k)$ must be translation bounded, ac spectrum can exist only if the minimal Lyapunov exponent governing the asymptotic growth of $v(k)$,

$$\chi_{\min}(k) = \log \lambda^{d/2} + \liminf_{n\to\infty} \frac{1}{n} \log \big\| B(k)B(\lambda k)\cdots B(\lambda^{n-1}k) \big\|_{\mathrm{F}}^{-1},$$

vanishes for almost all k. Setting $\chi_{\min}(k) = \log \lambda^{d/2} - \chi^{B}(k)$, we need to investigate the behaviour of $\chi^{B}(k)$. Taking into account that the Frobenius norm is submultiplicative, we get the estimate

$$\chi^{B}(k) = \limsup_{n\to\infty} \frac{1}{n} \log \|B^{(n)}(k)\|_{\mathrm{F}} \leqslant \frac{1}{N} \mathbb{M}\big(\log \|B^{(N)}(k)\|_{\mathrm{F}}\big)$$

for any fixed N, where $\mathbb{M}(f)$ is the mean of the quasiperiodic function f. Moreover, to compute the mean of the quasiperiodic function $\log \|B^{(n)}(k)\|_{\mathrm{F}}$, or rather $\log \|B^{(n)}(k)\|_{\mathrm{F}}^{2}$, we can lift it to a section through a periodic function, and compute the mean as an integral over the unit cell,

$$\frac{1}{N} \mathbb{M}\big(\log \|B^{(N)}(.)\|_{\mathrm{F}}^{2}\big) = \frac{1}{N} \int_{\mathbb{T}^{D}} \log\Big(\sum_{i,j} \big|P_{ij}^{(N)}(\tilde{k})\big|^{2}\Big)\, \mathrm{d}\tilde{k},$$

where the $P_{ij}^{(N)}$ are trigonometric polynomials. In this way, for each N, an upper bound for $\chi^{B}(k)$ is obtained, which is readily computable for many examples. If that upper bound implies that $\chi^{B}(k) < c \cdot \log \lambda^{d/2}$ for some $c < 1$, the presence of ac spectrum can be ruled out.

This criterion has successfully been applied to many examples, among them several with non-Pisot inflation factors. These are known to have no non-trivial pp part in the spectrum, but the nature of the continous part has long remained unclear. Using our approach, it could be shown that the binary non-Pisot tiling [1], the (non-FLC) Frank–Robinson tiling [4], and the well-known Godrèche–Lançon–Billard tiling [5], all have singular diffraction spectrum. The same conclusion is obtained for several other examples with mixed pp and continuous spectrum. In fact, except for the few examples known to have an ac part in the spectrum, such as the Rudin–Shapiro tiling, in all examples studied the upper bound on $\chi^{B}(k)$ quickly drops below the threshold $\log \lambda^{d/2}$, showing that the diffraction spectrum is singular.

References

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