Problems in number theory related to aperiodic order

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This talk concerns properties of dilated floor functions $f_\alpha(x) = [\alpha x]$, where α takes a fixed real value. Such functions perform quantization of the linear function αx at length scale $\frac{1}{\alpha}$. For $\alpha > 1$, the set of values $[\alpha n]$ for positive integer n is called the Beatty sequence associated to α. It is an aperiodic sequence if α is irrational, and when extended to all integers it is a one-dimensional cut and project set; compare [1, Ex. 9.8]. This talk studies the composed functions $f_\alpha \circ f_\beta(x) = [\alpha[\beta x]]$. The set of parameter values (α, β) where the two floor functions commute were characterized in joint work [2] with Takumi Murayama and D. Harry Richman (2016). The solution set consists of three straight lines through the origin $(0,0)$ plus a countable set of 'exceptional' rational solutions $(\frac{1}{m}, \frac{1}{n})$ for positive integer m, n.

Ongoing joint work [3] with D. Harry Richman (2019) determines the set S of all values (α, β) that satisfy $[\alpha[\beta x]] \geq [\beta[\alpha x]]$ for all real x. When α, β have opposite signs then $(\alpha, \beta) \in S$ if and only if $\alpha < 0$ and $\beta > 0$. For positive α, β, the solution set is a countable collection of half-lines and rectangular hyperbolas, passing through $(0,0)$, plus the vertical lines $\alpha = \frac{1}{m}, \beta > 0$ for positive integer m. The hyperbola solutions are associated to disjoint Beatty sequences when both α, β are irrational, but also include extra solutions with rational values. For negative α, β, the solution set consists of countably infinite families of lines and rectangular hyperbolas passing through $(0,0)$, plus vertical finite line segments at every rational $\alpha = -\frac{m}{n}$ (with $-\frac{1}{m} \leq \beta < 0$), plus a countable set of 'sporadic rational solutions'.

The existence of the sporadic rational solutions relates to the Diophantine Frobenius problem in two variables. The classification establishes that the set S is closed. It establishes various internal symmetries of the set S given by linear and bilinear changes of variables, for positive α, β (resp., negative α, β). The classification implies a pre-partial ordering on nonzero α where one says $\alpha \prec \beta$ if $(\alpha, \beta) \in S$. Namely, if $(\alpha, \beta) \in S$ and $(\beta, \gamma) \in S$ with $\alpha\beta\gamma \neq 0$, then $(\alpha, \gamma) \in S$.

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References

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