Automatic sequences are orthogonal to aperiodic multiplicative functions

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In 2010, P. Sarnak [7] formulated the following conjecture: For each zero entropy topological dynamical system (X,T), we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N} f(T^n x)\mu(n)=0 \tag{1}$$

for all $f\in C(X)$ and $x\in X$. Sarnak's conjecture has been proved in many classes of zero entropy systems [1], including so-called automatic sequences (C. Müllner, [6]), that is when $X=X_\theta\subset A^{\mathbb{Z}}$ and $T=S$ (shift) is determined by a primitive substitution $\theta : A\to A^\lambda$ of constant length λ. One can ask, however, whether Eq. (1) holds when we replace μ by other arithmetic functions. Especially, we are interested in the class of multiplicative functions. My talk is to present the main strategies to prove the following result:

Theorem 1 (M. Lemańczyk, C. Müllner, 2018). *Each automatic sequence is orthogonal to an arbitrary bounded, aperiodic and multiplicative function, i.e. for each primitive substitution $\theta : A\to A^\lambda$, all $f\in C(X_\theta)$ and $x\in X$, we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N} f(S^n x)u(n)=0$$

for each $u:\mathbb{N}\to\mathbb{C}$ as above.

Our main tool is the so-called DKBSZ criterion [3] which says that every bounded sequence (a_n) of complex numbers is orthogonal to *all* bounded multiplicative functions if

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N} a_{pn}\overline{a}_{qn}=0 \tag{2}$$

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for each pair of sufficiently large different primes p,q. Then, in the dynamical context, we consider $a_n = f(S^n x)$, and Eq. (2) leads us to study

$$\frac{1}{N}\sum_{n\leq N} f(S^{pn}x)\overline{f(S^{qn}x)} = \int_X f\otimes\overline{f}\,d\left(\frac{1}{N}\sum_{n\leq N}\delta_{(S^{pn}x,S^{qn}x)}\right),$$

which, by passing to a convergent subsequence of the empiric measures, yields the following: $\frac{1}{N_k}\sum_{n\leq N_k}\delta_{(S^{pn}x,S^{qn}x)} \to \rho$ implies that the limit in Eq. (2) is equal to $\int_{X\times X} f\otimes\overline{f}\,d\rho$ and that ρ is a joining of S^p and S^q (remembering that primitivity implies that (X_θ,S^p) and (X_θ,S^q) are uniquely ergodic, say, ν denotes the unique S-invariant measure). Note that we cannot prove the theorem above for *all* bounded multiplicative functions, as periodic sequences are always automatic and can also be multiplicative.

In fact, more than that is true. By taking $u(1) = u(2) = 1$, $u(2n) = u(n)$ and $u(2n+1) = (-1)^n$ we obtain an automatic sequence which is not periodic but represents a completely multiplicative function. To explain this phenomenon and the use of the DKBSZ criterion, we should remember that we do not expect the limit joinings ρ to be product measure $\nu\otimes\nu$. This is impossible as (X_θ,S) has the odometer (H_λ,R) as its factor. If we consider the odometer H_λ with its unique invariant measure (Haar measure) ν_{H_λ} the measure-theoretic systems $(H_\lambda,\nu_{H_\lambda},R^p)$ and $(H_\lambda,\nu_{H_\lambda},R^q)$ are isomorphic! Whence the only ergodic joinings between them are graphs of relevant isomorphisms. Hence the simplest possible (ergodic) joinings between S^p and S^q are relative products over graph joinings. However, if these are the *only* ergodic joinings, each pair $(x,x)\in X_\theta\times X_\theta$ is generic for such a relative product and the DKBSZ criterion works for all continuous functions $f\in C(X_\theta)$ provided that $f\perp L^2(H_\lambda,\nu_{H_\lambda})$. Therefore, we have two tasks:

- to show that in the case of primitive substitutions we have the minimal number of possible ergodic joinings between S^p and S^q and
- to describe the structure of continuous functions; more precisely, to show that there are continuous functions orthogonal to the L^2-space of the underlying odometer, in fact (surprisingly) that the conditional expectation of each continuous function with respect to the odometer factor remains continuous.

The first task is done by using some results from the 1980s of Host and Parreau [2] (and also of Lemańczyk and Mentzen [4]) on the measure-theoretic centralizer of substitutions of constant length by showing non-isomorphism of different prime powers and using Mentzen's theorem on factors (of substitutions) to conclude relative disjointness. The second task is fulfilled by developing a (new) theory of substitution joinings which culminates in showing that each substitution has a representation in which it is relatively bijective over its synchronizing part.

References

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