# Similarity isometries of shifted lattices and point packings

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## 1 Some preliminaries

A *lattice* $\Gamma$ (of rank and dimension $d$) is a discrete subset of $\mathbb{R}^d$ that is the $\mathbb{Z}$-span of $d$ linearly independent vectors $v_1,\ldots,v_d \in \mathbb{R}^d$ over $\mathbb{R}$. The set $\{v_1,\ldots,v_d\}$ is called a basis for $\Gamma$, and $\Gamma = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_d$. As a group, $\Gamma$ is isomorphic to the free Abelian group of rank $d$. Alternatively, a lattice $\Gamma$ may be defined as a discrete co-compact subgroup of $\mathbb{R}^d$.

On the other hand, a (*crystallographic*) *point packing* $\Lambda$ is a non-empty point set of $\mathbb{R}^d$ such that there exists a lattice $\Gamma$ in $\mathbb{R}^d$ and a finite point set $F$ such that

$$\Lambda = \Gamma + F = \{\ell + f \mid \ell \in \Gamma \text{ and } f \in F\}.$$

That is, a point packing is the union of a lattice $\Gamma$ and a finite number of translated copies of $\Gamma$. We refer to $\Gamma$ as a generating lattice for $\Lambda$ and the shifted lattice $x+\Gamma$, where $x \in F$, as a component of $\Lambda$. Observe that a point packing need not be a lattice.

Point packings have appeared in the literature under different names and different contexts. For instance, they appeared as non-lattice periodic packings in relation to the sphere packing problem in [3]. Dolbilin et al. referred to point packings in [4] as ideal or perfect crystals, and gave minimal sufficient geometric conditions on a discrete subset of $\mathbb{R}^d$ to be an ideal crystal. The term multilattice has also been used to pertain to a point packing, and arithmetic classification of multilattices have been studied in [9, 6].

Point packings serve as a standard model for 'ideal crystals', that is, crystals having multiple atoms per primitive unit cell. Examples of point packings include the honeycomb lattice, diamond lattice (crystal structure of diamond, tin, silicon, and germanium), and hexagonal closed packing (crystal structure of quartz).

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As point sets, point packings are Meyer sets (relatively dense sets $\Lambda$ such that $\Lambda - \Lambda$ is uniformly discrete) [2]. Recall that a periodic point set is a discrete set $\Lambda \subset \mathbb{R}^d$ for which $\mathrm{per}(\Lambda) := \{t \in \mathbb{R}^d \mid t + \Lambda = \Lambda\}$ is non-trivial. If $\mathrm{per}(\Lambda)$ forms a lattice in $\mathbb{R}^d$, then we say that $\Lambda$ is crystallographic. Point packings are exactly the locally finite point sets that are crystallographic [2].

## 2 Symmetry groups of point packings

The symmetry group of a point packing is a crystallographic group [4]. In particular, denote an isometry of $\mathbb{R}^d$ by $(v,R)$, where $(v,R)x = v + Rx$, with $v \in \mathbb{R}^d$ and $R \in \mathrm{O}(d,\mathbb{R})$. The crystallographic restriction for point packings [2] states that if $\Lambda$ is a point packing with $\mathrm{per}(\Lambda) = \Gamma$, then $R$ is a symmetry of the lattice $\Gamma$ whenever $(v,R)$ is a symmetry of $\Lambda$. Here, we discuss several additional results on the symmetry group of a given point packing $\Lambda$.

Suppose $\Lambda = \Gamma + \{x_0 = 0, x_1, \ldots, x_{m-1}\}$. It is easy to see that $\mathrm{per}(\Gamma) \subseteq \mathrm{per}(\Lambda)$. The reverse inclusion does not always hold. It has been shown in [8] that there exists a (maximal) generating lattice $\Gamma'$ of $\Lambda$ that contains $\Gamma$ such that $\mathrm{per}(\Lambda) = \mathrm{per}(\Gamma')$. In fact, if $S := \{x_i \mid (x_i, \mathrm{id})\Lambda = \Lambda\}$, then $\Gamma' = \Gamma + S$ is a lattice that generates $\Lambda$ with $\mathrm{per}(\Lambda) = \mathrm{per}(\Gamma')$; see [5].

Hence, without loss of generality, we may assume that $\Lambda$ is a point packing with $\mathrm{per}(\Lambda) = \mathrm{per}(\Gamma)$. Then, the isometry $(v,R)$ is a symmetry of $\Lambda$ if and only if for each $i \in \{0, \ldots, m-1\}$, there exists $j \in \{0, \ldots, m-1\}$ such that $(v,R)(x_i + \Gamma) = x_j + \Gamma$; see [5]. In words, the symmetries of $\Lambda \subseteq \mathbb{R}^d$ are precisely the isometries of $\mathbb{R}^d$ that induce a permutation of the components of $\Lambda$.

## 3 Similarity isometries of shifted lattices and point packings

Two lattices $\Gamma$ and $\Gamma'$ of $\mathbb{R}^d$ are said to be commensurate if the intersection of $\Gamma$ and $\Gamma'$ is a sublattice (of finite index) of both $\Gamma$ and $\Gamma'$. A linear isometry $R$ of $\mathbb{R}^d$ is called a (*linear*) *similarity isometry* of $\Gamma$ whenever $\Gamma$ and $\alpha R\Gamma$ are commensurate for some $\alpha \in \mathbb{R}^+$. Equivalently, $R$ is a similarity isometry of $\Gamma$ whenever $\beta R\Gamma$ is a sublattice of $\Gamma$ for some $\beta \in \mathbb{R}^+$. The lattice $\beta R\Gamma$ is referred to as a similar sublattice of $\Gamma$. Given a similarity isometry $R$ of $\Gamma$, the set $\mathrm{scal}_\Gamma(R)$ of scaling factors of $R$ is defined to be the set of all real numbers $\alpha$ for which $\Gamma$ and $\alpha R\Gamma$ are commensurate. Meanwhile, the set $\mathrm{Scal}_\Gamma(R)$ is comprised of all real numbers $\beta$ for which $\beta R\Gamma \subseteq \Gamma$. The set of similarity isometries of $\Gamma$ forms a group which we denote by $\mathrm{OS}(\Gamma)$. We use the notation $\mathrm{SOS}(\Gamma)$ for the group of similarity rotations of $\Gamma$. Various studies have examined the existence of similar sublattices as well as the properties of similarity isometries for particular lattices. We now give some initial results on similarity isometries of a point packing [1]. To this end, we first consider affine similarity isometries of lattices which allow us to study similarity isometries of shifted lattices. This line of attack is analogous to the one used to investigate coincidence isometries of point packings [7].

Let $(v,R)$ be an isometry of $\mathbb{R}^d$ and $\Gamma$ be a lattice in $\mathbb{R}^d$. Then, there exists $\alpha \in \mathbb{R}^+$ such that $\Gamma$ and $(v,\alpha R)\Gamma$ are commensurate if and only if $\Gamma$ and $\alpha R\Gamma$ are commensurate and $v \in \Gamma + \alpha R\Gamma$. On the other hand, there exists $\beta \in \mathbb{R}^+$ such that $(v,\beta R)\Gamma \subseteq \Gamma$ if and only if $\beta R\Gamma \subseteq \Gamma$ and $v \in \Gamma$. Observe that we obtain inequivalent statements when we extend the equivalent definitions of linear similarity isometries of lattices to affine similarity isometries of lattices. The same phenomenon occurs when we look at similarity isometries of shifted lattices.

Let $\mathrm{OS}_1(x+\Gamma)$ be the set of linear isometries $R$ such that $x+\Gamma$ is commensurate with $\alpha R(x+\Gamma)$ for some $\alpha \in \mathbb{R}^+$, and $\mathrm{OS}_2(x+\Gamma)$ be the set of linear isometries $R$ such that $\beta R(x+\Gamma) \subseteq x+\Gamma$ for some $\beta \in \mathbb{R}^+$. We obtain that

$$\mathrm{OS}_1(x+\Gamma) = \{R \in \mathrm{OS}(\Gamma) \mid \alpha Rx - x \in \Gamma + \alpha R\Gamma \text{ for some } \alpha \in \mathrm{scal}_\Gamma(R)\}, \text{ and}$$
$$\mathrm{OS}_2(x+\Gamma) = \{R \in \mathrm{OS}(\Gamma) \mid \beta Rx - x \in \Gamma \text{ for some } \beta \in \mathrm{Scal}_\Gamma(R)\}.$$

This implies that $\mathrm{OS}_2(x+\Gamma) \subseteq \mathrm{OS}_1(x+\Gamma)$. The reverse inclusion does not always hold. Nonetheless, a sufficient condition for $R \in \mathrm{OS}_1(x+\Gamma)$ to be in $\mathrm{OS}_2(x+\Gamma)$ is if there exists $M \in \mathbb{N}$ such that $Mkx - x \in \Gamma$, where $k = [\alpha R\Gamma : \Gamma \cap \alpha R\Gamma]$ with $\alpha \in \mathrm{scal}_\Gamma(R)$.

To illustrate, suppose $\Gamma$ is the square lattice $\mathbb{Z}[i]$. Then $\mathrm{OS}_2(\sqrt{2}+\Gamma) = \varnothing$ while $\mathrm{OS}_1(\sqrt{2}+\Gamma) = \langle T_r \rangle$, where $T_r$ is the reflection along the real axis. On the other hand,

$$\mathrm{SOS}_1(\tfrac{1}{1-i}+\Gamma) = \mathrm{SOS}_2(\tfrac{1}{1-i}+\Gamma) \subset \mathrm{SOS}(\Gamma).$$

From the above discussion, given a point packing $\Lambda = \Gamma + \{x_0 = 0, x_1, \ldots, x_{m-1}\}$, we will say that $R$ is a similarity isometry of $\Lambda$ if there exists $\beta \in \mathbb{R}^+$ such that $\beta R\Lambda = \bigcup_{i=0}^{m-1} \beta R(x_i + \Gamma) \subset \Lambda$.

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