Renormalisation for inflation tilings II: Connections to number theory

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In the study of spectral properties of a d-dimensional aperiodic tiling which arises from an inflation rule ρ on a finite set of prototiles, one recovers a system of renormalisation relations for measures which make up the diffraction measure $\widehat{\gamma}$; see [2, 3] for general notions and [5, 4, 6] for the rigorous treatment of certain classes. One can show that each component of the Lebesgue decomposition of $\widehat{\gamma}$ satisfies these relations independently. In particular, the Radon–Nikodym density $h(k)$ representing the absolutely continuous component $\widehat{\gamma}_{\mathsf{ac}}$ exhibits a certain scaling behaviour, which is encoded in the Fourier matrix $B(k)$. Here, $k \in \mathbb{R}^d$ and the dimension of $B(k)$ is given by the number of prototiles.

When $d = 1$, the exponential growth of $h(k)$ along orbits of the dilation map $k \mapsto \lambda k$ is determined by the Lyapunov exponent $\chi^B(k)$ of the matrix cocycle induced by $B(k)$, where λ is the inflation multiplier of ρ. If, for a set of real parameters k of full measure, this exponent is bounded from above by $\log\sqrt{\lambda} - \varepsilon$ for some $\varepsilon > 0$, the diffraction $\widehat{\gamma}$ is singular, i.e., $\widehat{\gamma}_{\mathsf{ac}} = 0$.

What is described below is based on joint work with Michael Baake, Michael Coons, Franz Gähler, and Uwe Grimm. In general, one can view $B(k)$ as a section of a function $\widetilde{B}$ on $\mathbb{T}^r$, where r is the algebraic degree of λ. This allows one to obtain a sequence of upper bounds for $\chi^B(k)$ via the mean of $\log|\widetilde{B}(.)|$, which are normalised logarithmic Mahler measures of multivariate polynomials.

Proposition 1 ([7]). *Let ρ be a one-dimensional primitive inflation with inflation multiplier λ of algebraic degree r. Assuming that $B(k)$ is invertible for some $k \in \mathbb{R}$, there exists a sequence of multivariate polynomials $P_N \in \mathbb{Z}[x_1, \ldots, x_r]$ such that, for each $N \in \mathbb{N}$,*

$$\chi^B(k) \leqslant \frac{1}{N}\mathfrak{m}(P_N)$$

for a.e. $k \in \mathbb{R}$, where $\mathfrak{m}(P)$ is the logarithmic Mahler measure of P. □

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Whenever λ is an integer or a Pisot number, the Lyapunov exponent $\chi^B(k)$ exists as a limit and is constant for a.e. $k \in \mathbb{R}$. In some cases, this a.e. value is given as a logarithmic Mahler measure. Finding suitable bounds for $\chi^B(k)$ then reduces to bounding logarithmic Mahler measures of certain polynomials.

A Borwein polynomial is a polynomial whose coefficients lie in $\{-1,0,1\}$. For an inflation derived from a binary substitution of constant length, one can show that $\chi^B(k) = \mathfrak{m}(P)$, for a.e. $k \in \mathbb{R}$, where P is a Borwein polynomial. Indeed, for this class of inflations, $\mathfrak{m}(P)$ is always strictly bounded from above by $\log \sqrt{\lambda}$, thus implying the singularity of $\widehat{\gamma}$ [6]. In fact, the correspondence also goes the other way as follows.

Proposition 2 ([1]). *Let P be a Borwein polynomial. Then, there exists at least one binary constant-length substitution ρ such that*

$$\mathfrak{m}(P) = \chi^B(k)$$

for a.e. $k \in \mathbb{R}$. □

A famous problem regarding logarithmic Mahler measures of polynomials in $\mathbb{Z}[x]$ is Lehmer's problem, which asks whether there is a universal non-zero constant c that serves as a lower bound for all non-zero logarithmic Mahler measures $\mathfrak{m}(P)$, with $P \in \mathbb{Z}[x]$. This question has been answered affirmatively for certain subclasses, but the general case remains open.

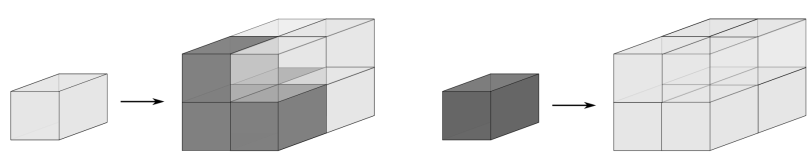
In view of Lehmer's problem, Borwein polynomials form an important class due to a result by Pathiaux which states that a polynomial $P \in \mathbb{Z}[x]$ with $\mathfrak{m}(P) < \log(2)$ must divide a Borwein polynomial, i.e., $Q = PR$ for some Borwein Q.

From his previous calculations, Boyd noted that the other divisor R can be chosen to have relatively small degree with respect to P and so that $\mathfrak{m}(R) = 0$. So far, there is still no proof of the existence of such a mollifier polynomial R; see [1] and references therein for details. If such an R always exists, the correspondence given in Proposition 2 gives the following dynamical version of Lehmer's problem.

Conjecture 1 ([1]). For all binary substitutions ρ of constant length, whose associated Lyapunov exponent $\chi^B(k)$ is a.e. non-zero, there exists a non-zero constant c such that $\chi^B(k) \geqslant c$ for a.e. $k \in \mathbb{R}$.

One should note that Proposition 2 extends to higher dimensions, where one can always realise a logarithmic Mahler measure of a multivariate Borwein polynomial as a Lyapunov exponent of a binary block inflation.

As an example, the logarithmic Mahler measure of $1+x+y+z$ is realised as the Lyapunov exponent associated to the following binary block substitution in $\mathbb{R}^3$.



It is interesting to note that $\mathfrak{m}(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$, where $\zeta(s)$ is Riemann's zeta function.

References

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