The Penrose and the Taylor–Socolar tilings, and first steps to beyond

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The Penrose hexagonal tilings form a family of aperiodic tilings comprised of arrowed double-hexagon tiles based on the standard periodic tiling of the plane by equilateral triangles [7, 8]. Each Penrose tile consists of a hexagon whose edges are arrowed, and within it a smaller hexagon of $1/3$ the area with orthogonal orientation whose edges are also arrowed. The matching of the arrowing forces aperiodicity, and remarkably, in spite of the rich symmetry of the usual hexagonal tilings, none of the Penrose hexagonal tilings has any non-trivial symmetry at all.

As observed in [5], each Penrose hexagonal tiling can be described algebraically by a pair of inverse sequences based on the nesting of equilateral triangles. This description is almost always (in a measure theoretical sense) unambiguous, and the exceptions (singular cases) are seen algebraically to be directly due to the symmetries of the underlying triangular lattice. In fact, from this perspective, one can see the Penrose tilings as being a symmetry-breaking construction.

This talk, which is based on joint work with Jeong-Yup Lee, is concerned with these symmetries—how they appear both geometrically and algebraically, and how they are broken by the Penrose tilings themselves. We also explore the relationship of these singularities to the singular Taylor–Socolar tilings. The Penrose tilings have long been known to be closely connected with another aperiodic tiling based on the usual hexagonal tiling of the plane, this one due to Joan Taylor and Joshua Socolar [10, 9]. Still, the connection between the two tilings is subtle [1]; see also [2, Sec. 6.4]. Here we show the connection directly, seeing it as the outcome of a $3:1$ mapping of the inverse sequences which pass from the Penrose tilings to Taylor–Socolar tilings. Under this mapping, each Penrose tiling gives rise to a unique Taylor–Socolar tiling, although only one third of its tiles actually appear directly from the Penrose hexagons (as the smaller interior hexagons of the Penrose hexagons). In the reverse direction, each Taylor–Socolar tiling gives rise to three different Penrose tilings.

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A key feature of both tilings is the underlying geometry of what is called a Coxeter Euclidean kaleidoscope [3]—the infinite configuration of hyperplanes that define the reflections of a Euclidean Coxeter group. It seems to us that putting these two remarkable tilings in the setting of kaleidoscopes might lead to a deeper understanding of how they actually arise. We conclude the talk with a brief introduction to the full classification of all the Euclidean kaleidoscopes that range through the famous A,B,C,D,E,F,G series, of which the Penrose and Taylor–Socolar tilings belong to the Euclidean G_2 kaleidoscope. In spite of the complications of higher dimensions, the geometry of these kaleidoscopes is quite articulately described, and in particular there are good descriptions of both their Voronoi and Delaunay cells [6]. We make the suggestion that each kaleidoscope may give rise to families of aperiodic tilings in ways similar to those from which the Penrose and Taylor–Socolar tilings can be derived. We give a short initial foray into this by creating a new square substitution tiling which arises by the same sort of ideas from the Euclidean B_2 kaleidoscope.

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