Scaling properties of the Thue–Morse measure: A summary

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This is an extended abstract of the paper 'Scaling properties of the Thue–Morse measure' by Baake, Gohlke, Kesseböhmer and Schindler [1].

The Thue–Morse diffraction measure for the balanced-weight case is given by the infinite Riesz product

$$\mu_{\mathrm{TM}} = \prod_{\ell=0}^{\infty} \bigl(1 - \cos(2\pi 2^{\ell} k)\bigr), \tag{1}$$

with convergence in the vague topology, see [2, Sec. 10.1] and references therein. As such, μ_{TM} is a translation-bounded, positive measure on $\mathbb{R}$ that is purely singular continuous and 1-periodic. Clearly, $\mu_{\mathrm{TM}} = \nu * \delta_{\mathbb{Z}}$, with $\nu = \mu_{\mathrm{TM}}|_{[0,1)}$ being a probability measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the latter represented by $[0,1)$ with addition modulo 1. In this case, ν is the weak limit of Radon–Nikodym densities of finite products as the right-hand side of (1).

If we denote by $B(x,r)$ the ball around x with radius r (either with respect to the Euclidean or the subshift metric), one way to quantify how concentrated the measure ν is at a given point $x \in \mathbb{T}$ is to determine its *local dimension*, given by

$$\dim_{\nu}(x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log(r)},$$

provided that the limit exists. Due to their highly irregular structure, we cannot hope to pin down the level sets of $\dim_{\nu}$ explicitly. However, the corresponding *Hausdorff dimension*,

$$f(\alpha) = \dim_{\mathrm{H}}\{x \in \mathbb{T} : \dim_{\nu}(x) = \alpha\},$$

yields a properly behaved function of α. The analysis of the *dimension spectrum* $f(\alpha)$ is one of the open questions considered in [7]. This problem turns out to be

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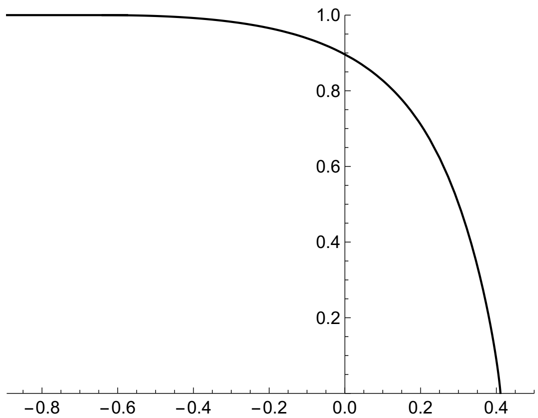

Fig. 1 The graph of the Birkhoff spectrum b from Eq. (5).

intimately related to pointwise scaling properties of the approximants in Eq. (1). More precisely, we consider

$$\beta(x) := \lim_{n\to\infty} \frac{1}{n\log(2)} \log \prod_{\ell=0}^{n-1} \left(1-\cos(2^{\ell+1}\pi x)\right),$$

for all $x \in \mathbb{T}$ for which the limit exists. The limit is known for Lebesgue-a.e. $x \in \mathbb{T}$, in which case it equals -1, and for some particular examples of non-typical points; see [4, 3].

There is a natural way to interpret β in terms of the Birkhoff average of some function $\psi\colon \mathbb{T} \longrightarrow [-\infty, \log(2)]$,

$$\psi(x) = \log\left(1-\cos(2\pi x)\right), \quad \beta(x) = \lim_{n\to\infty} \frac{\psi_n(x)}{n\log(2)}, \tag{2}$$

where $\psi_n(x) = \sum_{\ell=0}^{n-1} \psi(2^\ell x)$. With this, we are interested in the *Birkhoff spectrum* $b(\alpha) = \dim_{\mathrm{H}} \mathscr{B}(\alpha)$ with

$$\mathscr{B}(\alpha) = \left\{x \in \mathbb{T} : \lim_{n\to\infty} \frac{\psi_n(x)}{n} = \alpha\right\} = \left\{x \in \mathbb{T} : \beta(x) = \frac{\alpha}{\log(2)}\right\}. \tag{3}$$

It is one of the strengths of the thermodynamic formalism to connect such locally defined functions to the Legendre transform of a globally defined quantity. An adequate choice for the latter in our situation is the *topological pressure* of the function $t\,\psi$, $t \in \mathbb{R}$, defined by

$$p(t) := \mathscr{P}(t\,\psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{J\in I_n} \sup_{x\in J} \exp\left(t\,\psi_n(x)\right), \tag{4}$$

where, for each $n \in \mathbb{N}$, I_n forms a partition of $[0,1]$ into intervals of length 2^{-n}.

Indeed, the relation between $f(\alpha)$ and $b(\alpha)$ given in [1, Thm. 1.1] is analogous to known results for Hölder continuous potentials [6, Cor. 1]: If p^* denotes the Legendre transform of p, one obtains

$$b(\alpha) = \max\left\{\frac{-p^*(\alpha)}{\log(2)}, 0\right\} \text{ and } f(\alpha) = b\big(\log(2)(1-\alpha)\big), \tag{5}$$

with the graph of $b(\alpha)$ given in Figure 1, confirming some numerical and scaling-based results of [5].

References

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