Weak model sets

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1 Square-free integers

In this talk, which is based on joint work with M. Baake and C. Huck, we will review the properties of weak model sets of extremal density. These results have been proved independently in [3] and [6], and we recommend these for more details.

Let us start by recalling the example of the set S of square-free integers,

$$S := \{n \in \mathbb{Z} : \forall p \in \mathbb{P}, p^2 \nmid n\},$$

where $\mathbb{P}$ denotes the set of primes.

Theorem 1 ([4]). *The autocorrelation measure γ of S, with respect to the natural van Hove sequence $\big(A_m = [-m,m]\big)_{m\in\mathbb{N}}$, exists. The correponding diffraction measure, $\widehat{\gamma}$, is a pure point measure.*

Note that, with respect to other van Hove sequences, S can have mixed diffraction spectrum. We will see below why the choice of the natural van Hove sequence is important, and leads to a connection to a cut and project scheme (CPS).

We assume below that the reader is familiar with the cut and project formalism and with regular model sets. For a review of this, we recommend the monograph [1] for $G = \mathbb{R}^d$, and [8] for general G.

To describe S, consider the following CPS,

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\;\pi_G\;} & \mathbb{R} \times H & \xrightarrow{\;\pi_H\;} & H := \prod_{p\in\mathbb{P}} \mathbb{Z}/p^2\mathbb{Z} \\
\cup & & \cup & & \cup \text{ dense} \\
L & \xleftarrow{\;1-1\;} & \mathscr{L} := \{(n, \tau(n)) : n \in \mathbb{Z}\} & \longrightarrow & L^{\star}
\end{array}$$

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where $\tau\colon\ \mathbb{Z} \longrightarrow H$ is defined by $\tau(n) = \big(n \bmod p^2\big)_{p\in\mathbb{P}}$. Consider the set

$$W := \prod_{p\in\mathbb{P}} \big((\mathbb{Z}/p^2\mathbb{Z}) \setminus \{0\}\big),$$

which acts as a window for the above CPS.

Theorem 2 ([4, 2]). *For the square-free integers, the following properties hold.*

1. *The window W is compact, with $W = \partial W$ and $S = \curlywedge(W)$.*
2. *The natural autocorrelation and diffraction measures of S are given by*

$$\gamma = \omega_{1_W * \widetilde{1_W}}\,, \quad \widehat{\gamma} = \omega_{|\check{1_W}|^2}\,. \tag{1}$$

While (1) is the usual formula of the diffraction of regular model sets, the window W has no interior and a boundary of positive measure. Thus, the standard proofs for the diffraction of regular model sets do not apply.

2 Weak model sets of maximal density

Definition 1. Let $(G,H,\mathscr{L})$ be a CPS. If $W \subseteq H$ is compact, we say that $\curlywedge(W)$ is a *weak model set*. We say that the weak model set $\curlywedge(W)$ has *maximal density* with respect to $\mathscr{A} = (A_n)_{n\in\mathbb{N}}$ if

$$\mathrm{dens}_{\mathscr{A}}(\curlywedge(W)) := \lim_n \frac{\mathrm{card}(\curlywedge(W)\cap A_n)}{\theta_G(A_n)} = \mathrm{dens}(\mathscr{L})\,\theta_H(W),$$

where θ_G and θ_H are the Haar measures of the groups G and H, respectively.

Note that the right-hand side is always an upper bound for the left-hand side. Regular model sets have maximal density. Square-free integers, as well as visible lattice points, have maximal density with respect to the natural van Hove sequence. The next result shows that generic positions of compact windows define weak model sets of maximal density.

Proposition 1 ([7]). *Let $(G,H,\mathscr{L})$ be a CPS, let $W \subseteq H$ be compact, and let $\mathscr{A}$ be a tempered van Hove sequence. Then, for generic $(x,y)+\mathscr{L} \in (G\times H)/\mathscr{L}$, the weak model set $-x+\curlywedge(y+W)$ has maximal density with respect to $\mathscr{A}$.*

For weak model sets of maximal density, we have the following result; see [3, 6] for details.

Theorem 3 ([3, 6]). *Let $(G,H,\mathscr{L})$ be a CPS, and $\curlywedge(W)$ a weak model set of maximal density with respect to $\mathscr{A} = (A_n)_{n\in\mathbb{N}}$. Then, the following properties hold.*

1. *With respect to $\mathscr{A}$, the set $\curlywedge(W)$ has autocorrelation and diffraction*

$$\gamma = \mathrm{dens}(\mathscr{L})\,\omega_{1_W * \widetilde{1_W}} \quad \textit{and} \quad \widehat{\gamma} = (\mathrm{dens}(\mathscr{L}))^2\, \omega_{|\check{1_W}|^2}\,.$$

2. For each $\chi \in \widehat{G}$, the Fourier–Bohr coefficient a_χ exists, with

$$a_\chi := \lim_n \frac{1}{\theta_G(A_n)} \sum_{x \in \curlywedge(W) \cap A_n} \overline{\chi(x)} = \mathrm{dens}(\mathscr{L}) \int_W \overline{\chi^\star(t)}\, \mathrm{d}t.$$

3. There exists an ergodic measure ν for the dynamical system $\big(\mathbb{X}(\curlywedge(W)), G\big)$ such that $\curlywedge(W)$ is generic for ν.

The measure ν can be identified as the unique invariant measure with maximal density for generic configurations.

References

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