Doubly sparse measures on locally compact Abelian groups

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In this work, joint with Michael Baake and Nicolae Strungaru [3], we are interested in *doubly sparse* measures on a locally compact Abelian group (LCAG) G. By a doubly sparse measure, we mean a Fourier-transformable Radon measure μ such that both $\mathrm{supp}(\mu)$ and $\mathrm{supp}(\widehat{\mu})$ are locally finite point sets in G and $\widehat{G}$, respectively. In particular, both μ and $\widehat{\mu}$ must then be pure point measures.

This work has its origins in the study of crystals and quasicrystals: a physical structure, represented by a point measure in $\mathbb{R}^d$, is considered to have long range order when its diffraction is also a pure point measure. In the simplest case, we have a periodic structure represented by the Dirac comb of a lattice, in which case its diffraction is also periodic: for a general lattice $\Gamma \subseteq \mathbb{R}^d$, we have from the Poisson summation formula (PSF) that $\widehat{\delta_\Gamma} = \mathrm{dens}(\Gamma) \cdot \delta_{\Gamma^*}$ and hence the diffraction

$$\widehat{\gamma_\Gamma} = \mathrm{dens}(\Gamma)^2 \cdot \delta_{\Gamma^*}\,;$$

see [1] for general background.

A model set is a point set gained from a cut and project scheme (CPS) by projecting onto a group, G, from a lattice in a higher-dimensional superspace, $G \times H$, via a sufficiently nice, relatively compact window in H. Such sets form natural mathematical models of quasicrystals, being non-periodic with pure point diffraction. We apply some recent results of Strungaru [7], who characterised measures that may be written as model combs in a CPS, and Richard and Strungaru [6], who proved the PSF for measures supported on a model set, using the PSF of the underlying lattice.

Meyer sets possess a strong form of finite local complexity and may always be constructed as relatively dense subsets of model sets. In fact, a Fourier-transformable measure μ supported inside a Meyer set with pure point Fourier transform $\widehat{\mu}$ can be written as a model comb in a CPS, with its coefficients determined by a continuous function of compact support on the internal space, H. Using this, we show that, un-

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der relatively mild conditions of sparseness on the support of $\widehat{\mu}$, both μ and $\widehat{\mu}$ (and hence the diffraction of μ) are supported on finitely many translates of a lattice, and thus have a periodic structure.

If a measure μ has uniformly discrete support and is positive definite, with pure point Fourier transform $\widehat{\mu}$, then $\widehat{\mu}$ may again be written as a model comb in a CPS, however with coefficients determined by a continuous function vanishing at infinity on the internal space, H. In this case, with similarly mild conditions of sparseness on the support of $\widehat{\mu}$, we show that μ is the limit of a sequence of measures with periodic structure.

Our results can be seen as a generalisation of many of those of Lev and Olevskii [4, 5] from the Euclidean to the general LCAG setting. To conclude, we consider some consequences of our results for measures supported on $\mathbb{R}^d$. In particular, we show the following. If a measure μ, supported inside a model set in a fully Euclidean CPS, is such that the support of the pure point part of $\widehat{\mu}$, that is, $\mathrm{supp}(\widehat{\mu}_{\mathsf{pp}})$, is locally finite then, in fact, $\widehat{\mu}_{\mathsf{pp}} = 0$; see [3] for details.

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