The mean-median map

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Consider a finite multiset $\xi = [x_1, \ldots, x_n]$ of real numbers. The *arithmetic mean* $\langle \xi \rangle$ and the *median* $\mathscr{M}(\xi)$ of ξ are defined, respectively, as

$$\langle \xi \rangle = \frac{1}{|\xi|} \sum_{x \in \xi} x \qquad \text{and} \qquad \mathscr{M}(\xi) = \begin{cases} x_{j_{\frac{n+1}{2}}} & n \text{ odd}, \\ \frac{1}{2}\left(x_{j_{\frac{n}{2}}} + x_{j_{\frac{n}{2}+1}} \right) & n \text{ even}, \end{cases}$$

where $x_{j_1} \leqslant x_{j_2} \leqslant \cdots \leqslant x_{j_n}$, for some permutation $k \mapsto j_k$ of indices.

We enlarge ξ by adjoining to it a new real number x_{n+1} determined by the requirement that the arithmetic mean of the enlarged multiset be equal to the median of the original multiset:

$$x_{n+1} = (n+1)\mathscr{M}(\xi) - n\langle \xi \rangle.$$

This rule is known as the *mean-median map* (MMM), which was introduced in [5], and subsequently studied in [2, 1, 3].

Since the MMM commutes with affine transformations [5], the simplest non-trivial case — three distinct initial numbers — may be studied in full generality by considering the initial multiset $[0, x, 1]$, with $x \in [\frac{1}{2}, \frac{2}{3}]$, exploiting symmetries [2]. Here one finds already substantial difficulties, which are synthesised in the following conjectures.

Conjecture 1 (Strong terminating conjecture [5]). The MMM sequence of any initial multiset is eventually constant.

For the system $[0, x, 1]$, we let the *transit time* $\tau(x)$ be the time at which the MMM sequence becomes constant (letting $\tau(x) = \infty$ if this does not happen). If the MMM sequence $(x_n)_{n=1}^{\infty}$ converges at x — with finite or infinite transit time — we have

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a real function $x \mapsto m(x)$, called the *limit function*, which gives the limit of this sequence. This function has an intricate, distinctive structure.

Conjecture 2 (Continuity conjecture [2]). The function $x \mapsto m(x)$ is continuous.

In [2], both conjectures were proved to hold in a neighbourhood of $x = \frac{1}{2}$, where m turns out to be affine. Using a computer-assisted proof, this result was then substantially extended in [1], where the limit function was constructed in small neighbourhoods of all rational numbers with denominator at most 18 lying in the interval $\left[\frac{1}{2}, \frac{2}{3}\right]$. The authors also identified 17 rational numbers at which m is non-differentiable.

In this joint work with Jonathan Hoseana [4], motivated by the above investigations, we study the mean-median map as a dynamical system on the space of finite multisets $[Y_1(x), \ldots, Y_n(x)]$ of piecewise-affine continuous functions with rational coefficients, the MMM map being defined pointwise. We study the limit function in the vicinity of its local minima. The latter occur at a distinctive family of rational points, the so-called *X-points*, which are transversal intersections of the functions Y_k. We prove the existence of local symmetries (homologies) around X-points, which result in affine functional equations for the limit function. We establish the general form of the limit function near an X-point, and show that the X-points form a hierarchical structure, whereby each X-point typically generates an *auxiliary sequence* of like points; such sequences form the scaffolding of the intricate structure of local minima of the limit function.

We then show that there is a one-parameter family of dynamical systems over $\mathbb{Q}$ — the *reduced system* — which, after suitable scaling, represent the dynamics near any X-point with given transit time. This simplification results from the fact that the reduced dynamics is largely unaffected by the earlier history of the X-point.

By exploiting the dynamics of the reduced system, we have established the strong terminating conjecture for $[0, x, 1]$ in neighbourhoods of 2791 rational numbers in the interval $\left[\frac{1}{2}, \frac{2}{3}\right]$, thereby extending the results of [1] by two orders of magnitude. This large data collection makes it clear that the domains over which the limit function is regular do not account for the whole Lebesgue measure, suggesting the existence of a drastically different, yet unknown, dynamical behaviour.

For a quantitative assessment of this phenomenon, we have computed a lower bound for the total variation of the limit function, sampled over a set of some 202,000 Farey points. Our data suggest the following conjecture.

Conjecture 3. The Hausdorff dimension of the graph of the limit function of the system $[0, x, 1]$ is greater than 1.

References

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