Similar sublattices and submodules

Peter Zeiner

In this talk, we want to give an overview of similar sublattices and submodules and the recent progress in this area. A milestone are some general existence results for similar sublattices of rational lattices [6] by Conway, Rains and Sloane. Since then, detailed explicit results have been achieved for a large collection of lattices and $\mathbb{Z}$-modules in dimensions $d \leq 4$, including planar modules related to cyclotomic integers [1], root lattices such as the A_4-lattice [2] and related modules in 4 dimensions, where one makes use of certain quaternion algebras [3], but also for less symmetric lattices in the plane [4]. Another important result is the establishment of a close connection between similar submodules and coincidence site modules (CSMs) [7, 8, 10].

A similar sublattice (SSL) of a lattice Γ is a sublattice of full rank that is similar to Γ; see [6, 5]. By a $\mathbb{Z}$-module, we mean a $\mathbb{Z}$-module which is (properly) embedded in $\mathbb{R}^d$, that is, a module $M \subset \mathbb{R}^d$ such that there is a $\mathbb{Z}$-basis $\{b_1, \ldots, b_n\}$ of M whose $\mathbb{R}$-span is $\mathbb{R}^d$; see [5]. Likewise, a similar submodule (SSM) of M is a submodule of full rank that is similar to M. In particular, every similar submodule is of the form αRM, where $\alpha \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $R \in \mathrm{O}(d, \mathbb{R})$. The key objects are the group of similarity isometries

$$\mathrm{OS}(M) = \{R \in \mathrm{O}(d, \mathbb{R}) \mid \exists \alpha \in \mathbb{R}^+ \text{ such that } \alpha RM \subseteq M\}$$

and the sets of scaling factors

$$\begin{aligned} \mathrm{Scal}_M(R) &:= \{\alpha \in \mathbb{R} \mid \alpha RM \subseteq M\} \quad \text{and} \\ \mathrm{scal}_M(R) &:= \{\alpha \in \mathbb{R} \mid \alpha RM \text{ is commesurate to } M\}, \end{aligned}$$

which are non-trivial if and only if R is a similarity isometry.

Peter Zeiner
Xiamen University Malaysia, Jalan Sunsuria, Bandar Sunsuria, 43900 Sepang, Selangor, Malaysia.
e-mail: pzeiner@xmu.edu.my

If E is the identity operation, then we have $\mathrm{Scal}_\Gamma(E) = \mathbb{Z}$ for a lattice Γ and $\mathrm{scal}_\Gamma(E) \cup \{0\} = \mathbb{Q}$ is the corresponding field of fractions. In general, the set of 'trivial' scaling factors $\mathrm{Scal}_M(E)$ is an order in a real number field, whose rank satisfies certain restrictions [10, 5]. The set $\mathrm{scal}_M(E) \cup \{0\}$ is again the corresponding field of fractions.

The family of these sets, $\{\mathrm{scal}_M(R) : R \in \mathrm{OS}(M)\}$, has a natural group structure, which allows one to define a homomorphism [7, 8, 10, 5]

$$\begin{aligned} \phi : \mathrm{OS}(M) &\to \mathbb{R}/(\mathrm{scal}_M(E)), \\ R &\mapsto \mathrm{scal}_M(R). \end{aligned}$$

The kernel of this homomorphism is $\mathrm{OC}(M)$, the so-called group of coincidence isometries of M. The latter is defined as the group of all $R \in \mathrm{O}(d,\mathbb{R})$ such that M and RM are commensurate [10, 5]. This establishes a connection between similar submodules and coincidence site modules. In particular, $\mathrm{OS}(M)/\mathrm{OC}(M)$ is an Abelian group. If $M = \Gamma$ is a lattice, all elements of $\mathrm{OS}(M)/\mathrm{OC}(M)$ have a finite order which is a divisor of d. For instance, if M is the square lattice, then $\mathrm{OS}(M)/\mathrm{OC}(M)$ is an infinite 2-group [7]. In case of modules in general, this is not true any more, and $\mathrm{OS}(M)/\mathrm{OC}(M)$ may have factors isomorphic to $\mathbb{Z}$ [5].

Finding $\mathrm{OS}(M)$ and $\mathrm{Scal}_M(R)$ for all $R \in \mathrm{OS}(M)$ are typically the first steps if one wants to count the SSMs of a given modules. If $b(n)$ denotes the number of SSMs of a given index n, then $b(n)$ is a supermultiplicative arithmetic function, that is, $b(mn) \geq b(m)b(n)$ whenever m and n are coprime. If the modules under consideration are related to number fields of class number one, this counting function is typically multiplicative and it makes sense to consider generating functions of Dirichlet series type

$$\Phi(s) = \sum_{n\in\mathbb{N}} \frac{b(n)}{n^s}.$$

For many cases, these generating functions have been calculated explicitly, among others for certain planar modules of N-fold symmetry [1], and some root lattices and related modules up to dimension 4, see e.g. [3, 2]. These generating functions can be used to determine the asymptotic behaviour of the number of SSMs via Delange's theorem [9]. In particular, one can calculate the asymptotic behaviour of the summatory function $\sum_{n\leq x} b(n)$ by using some information on the poles of $\Phi(s)$. For explicit calculations on the examples mentioned above (and many more), see [1, 3, 2, 5, 4] and references therein.

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