Short time existence for higher order curvature flows with and without boundary conditions

Yuhan Wu

Abstract We prove short time existence for higher order curvature flows of plane curves with and without generalised Neumann boundary condition.

1 Introduction

We first introduce the ideal curve flow of plane curves with Neumann boundary condition. This ideal curve flow is a sixth order curvature flow which is the steepest descent gradient flow of the energy functional in L^2. The Neumann boundary condition here is that there are two parallel lines which has a distance between them, the two end points of the curve we study are on these two lines respectively and the ideal curves are orthogonal to the boundaries. We then give the definition of generalized $(2m+4)$th-order curvature flows of plane curves with Neumann boundary condition. Secondly, we introduce the closed curve diffusion flow of plane curves with constrained length.

1.1 The curvature flows of open curves with Neumann boundary condition

Let η_1, η_2 denote two parallel vertical lines in $\mathbb{R}^2$, with distance between them. The immersed curves $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^2$ satisfying Neumann boundary condition.

$$\gamma(-1) \in \eta_1(\mathbb{R}), \gamma(1) \in \eta_2(\mathbb{R}).$$

Yuhan Wu
University of Wollongong, Australia, e-mail: yw120@uowmail.edu.au

Denote $\tau = \gamma_s$ is the unit tangent vector field and ν the unit normal vector along γ. The Neumann condition is equivalent to $\langle \nu(\pm 1, t), \nu_{\eta_{1,2}} \rangle = 0$, here $\nu_{\eta_{1,2}}$ is the unit normal vector field to $\eta_{1,2}$. See Figure 1.1.

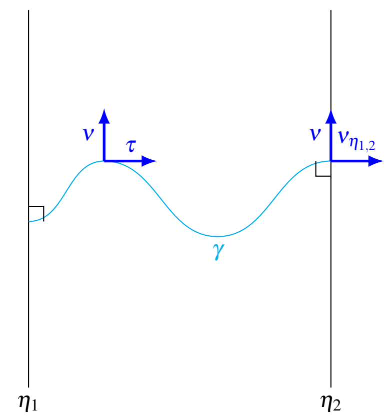

Figure 1.1

1.1.1 A sixth order flow of plane curves with Neumann boundary condition

We consider the energy functional

$$E(\gamma) = \frac{1}{2} \int_\gamma k_s^2 ds,$$

where k is the scalar curvature, ds the arclength element and k_s is the derivative of curvature with respect to arclength s. The corresponding gradient flow has normal speed given by F, that is

$$\partial_t \gamma = F \nu.$$

Under the evolution of the functional $E(\gamma)$ a straightforward calculation yields

$$\frac{d}{dt} \frac{1}{2} \int_\gamma k_s^2 ds = - \int_\gamma F \cdot \left(k_{s^4} + k^2 k_{ss} - \frac{1}{2} k k_s^2 \right) ds,$$

where $k_{s^4} = k_{ssss}$. For the flow to be the steepest descent gradient flow of $E(\gamma)$ in L^2, we require

$$F = k_{s^4} + k^2 k_{ss} - \frac{1}{2} k k_s^2. \tag{1}$$

Let γ be a smooth curve satisfying $F = 0$, that is a stationary solution to the L^2-gradient flow of E. We call such curves *ideal*.

We define sixth order curvature flow with Neumann boundary condition as follows, see more details in [3].

Definition 1. [3] Let $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^2$ be a family of smooth immersed curves. γ is said to move under sixth order curvature flow (1) with homogeneous Neumann boundary condition, if

$$\begin{cases} \frac{\partial}{\partial t}\gamma(s,t) = F\nu, & \forall (s,t) \in [-1,1] \times [0,T) \\ \gamma(\cdot,0) = \gamma_0, & \\ \langle \nu, \nu_{\eta_{1,2}} \rangle = k_s = k_{s^3} = 0, \; \forall (s,t) \in \eta_{1,2}(\mathbb{R}) \times [0,T) & \end{cases} \tag{2}$$

where $F = k_{s^4} + k_{ss}k^2 - \frac{1}{2}k_s^2 k$ denotes the normal speed of the curves, ν and $\nu_{\eta_{1,2}}$ are the unit normal fields to γ and $\eta_{1,2}$ respectively.

Here we give the long-time existence result as:

Theorem 1. *[3] Let γ_0 be a smooth embedded regular curve. Let $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^n$ be a solution to (2). If the initial curve γ_0 satisfies $\omega = 0$ and*

$$\|\kappa_s\|_2^2 \leq \frac{\pi^3}{7L_0^3},$$

here L_0 is the length of γ_0 and $\kappa = k(\cdot,0)$ is the curvature, then the flow exists for all time $T = \infty$ and $\gamma(\cdot,t)$ converges exponentially to a horizontal line segment γ_∞ in the C^∞ topology.

We use w to denote the winding number, defined here as

$$w := \frac{1}{2\pi}\int_\gamma k ds.$$

For closed curves, $w \in \mathbb{Z}$, in our setting, the winding number must be a multiple of $\frac{1}{2}$. For example in Figure 1.1.1,

$$\gamma_1 \text{ has } w[\gamma_1] = \tfrac{1}{2\pi}\textstyle\int_{\gamma_1} k ds = 1;$$
$$\gamma_2 \text{ has } w[\gamma_2] = \tfrac{1}{2\pi}\textstyle\int_{\gamma_2} k ds = \tfrac{1}{2}.$$

Lemma 1. *The hypothesis of theorem 1 implies that $\omega[\gamma] = \omega[\gamma_0] = 0$.*

1.1.2 Higher order flows of plane curves with Neumann boundary condition

We generalise the sixth order case where we considered the L^2-gradient flow for the energy

$$\frac{1}{2}\int_\gamma k_s^2 ds.$$

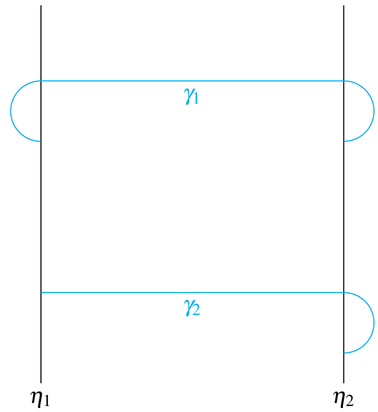

Figure 1.1.1

Our work is also the arbitrary even order generalisation of [11], where the fourth order curve diffusion and elastic flow of curves between parallel lines are investigated.

We consider the L^2-gradient flow for the energy

$$E(\gamma) = \frac{1}{2}\int_\gamma k_{s^m}^2 ds$$

with suitable associated generalised Neumann boundary conditions, here $m \in \mathbb{N} \cup \{0\}$.

Under a normal variation of the energy, straightforward calculations yield the normal flow speed

$$F = (-1)^{m+1} k_{s^{2m+2}} - \sum_{j=1}^{m} (-1)^j k k_{s^{m+j}} k_{s^{m-j}} - \frac{1}{2} k k_{s^m}^2. \tag{3}$$

And we set the Neumann boundary condition as:

$$< \nu, \nu_{\eta_{1,2}} > (\pm 1, t) = k_s(\pm 1, t) = \ldots = k_{s^{2m-1}}(\pm 1, t) = k_{s^{2m+1}}(\pm 1, t) = 0.$$

We define (2m+4)th order curvature flow with Neumann boundary condition in Definition 2, see more details in [5].

Definition 2. [5] Let $\gamma : [-1,1] \times [0,T] \to \mathbb{R}^2$ be a family of smooth immersed curves. γ is said to move under (2m+4)th-order curvature flow (3) with homogeneous Neumann boundary condition, if

$$\begin{cases} \frac{\partial}{\partial t}\gamma(s,t) = -F\nu, & \forall (s,t) \in [-1,1] \times [0,T) \\ \gamma(\cdot, 0) = \gamma_0, & \\ < \nu, \nu_{\eta_{1,2}} > = k_s = \ldots = k_{s^{2m-1}} = k_{s^{2m+1}} = 0, & \forall (s,t) \in \eta_{1,2}(\mathbb{R}) \times [0,T) \end{cases} \tag{4}$$

where $F=(-1)^{m+1}k_{s^{2m+2}}+\sum_{j=1}^{m}(-1)^{j+1}kk_{s^{m+j}}k_{s^{m-j}}-\frac{1}{2}kk_{s^m}^2$ denotes normal speed of the curves, $m\in\mathbb{N}\cup\{0\}$, ν and $\nu_{\eta_{1,2}}$ are the unit normal fields to $\gamma(\pm 1)$ and $\eta_{1,2}$ respectively.

We are also interested in one-parameter families of curves $\gamma(\cdot,t)$ satisfying the polyharmonic curvature flow

$$\frac{\partial}{\partial t}\gamma(s,t)=(-1)^{m+1}k_{s^{2m+2}}\nu, \tag{5}$$

here general $m\in\mathbb{N}\cup\{0\}$. Above ν is the smooth choice of unit normal such that the above flow is parabolic in the generalised sense.

Lemma 2. *While a solution to the flow* (5) *with generalised Neumann boundary conditions exists, we have*

$$\frac{d}{dt}L(t)=-\int_\gamma k_{s^{m+1}}^2\,ds,$$

where $L(t)$ denotes the length of the curve.

In view of this lemma and the separation of the supporting parallel lines $\eta_{1,2}$, the length $L(t)$ of the evolving curve $\gamma(\cdot,t)$ remains bounded above and below under the flow (5).

1.2 The length-constrained curve diffusion flow of closed curves

We consider one-parameter families of immersed closed curves $\gamma:\mathbb{S}^1\times[0,T)\to\mathbb{R}^2$. The energy functional

$$L(\gamma)=\int_\gamma|\gamma_u|\,du.$$

The curve diffusion flow is the steepest descent gradient flow for length in H^{-1}. We define the constrained curve diffusion flow here, see more details about this flow in [4].

Definition 3. [4] Let $\gamma:\mathbb{S}^1\times[0,T)\to\mathbb{R}^2$ be a $C^{4,\alpha}$-regular immersed curve. The length constrained curve diffusion flow

$$\begin{cases}\partial_t\gamma=-(k_{ss}-h(t))\nu,\ \forall(s,t)\in\mathbb{S}^1\times[0,T)\\ \gamma|_{t=0}=\gamma_0,\end{cases} \tag{6}$$

where ν denotes a unit normal vector field on γ.

To preserve length of the evolving curve $\gamma(\cdot,t)$, we take

$$h(t) = -\frac{\int k_s^2 ds}{2\pi w}.$$

Length-constrained curve diffusion flow fixes length and increases area. Regular curve diffusion flow fixes area and reduces length. We can say that the length-constrained curve diffusion flow is ”dual” to curve diffusion flow.

The following theorem is the long time existence result for the length-constrained curve diffusion flow.

Theorem 2. *[4] Suppose γ_0: $\mathbb{S}^1 \to \mathbb{R}^2$ is a regular smooth immersed closed curve with $A[\gamma_0] > 0$ and $w[\gamma_0] = 1$. Then there exists a constant $K^* > 0$ such that if*

$$K_{osc}[\gamma_0] < K^*, I[\gamma_0] < \frac{4\pi^2}{4\pi^2 - K^*},$$

then the length-constrained curve diffusion flow γ with initial data γ_0 exists for all time and converges exponentially to a round circle with radius $\frac{L_0}{2\pi}$.

In Theorem 2, $A[\gamma]$ denotes the area, $K_{osc}[\gamma]$ is the oscillation and $I[\gamma]$ is the isoperimetric of the flow. From our calculation, we know that $K^* \simeq \frac{1}{9}$.

In the setting for this flow, the winding number must be an integer and is always 1 under the assumption in Theorem 2. For example in Figure 1.2,

γ_1 has $w[\gamma_1] = \frac{1}{2\pi}\int_{\gamma_1} kds = 0$;
γ_2 has $w[\gamma_2] = \frac{1}{2\pi}\int_{\gamma_2} kds = 1$;
γ_3 has $w[\gamma_3] = \frac{1}{2\pi}\int_{\gamma_3} kds = 2$.

Lemma 3. *The hypothesis of theorem 2 implies that $\omega[\gamma] = \omega[\gamma_0] = 1$.*

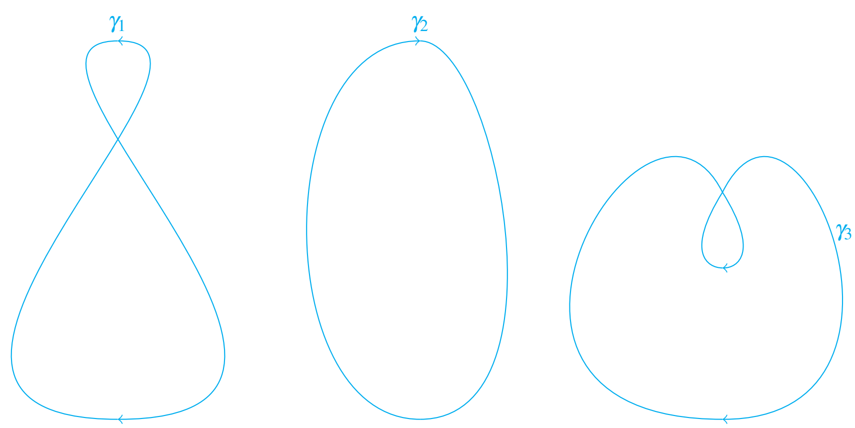

Figure 1.2

2 Short time existence for higher order curvature flows with Neumann boundary condition

Here we state the way to prove the short time existence for higher order curvature flows of plane curves with generalised Neumann boundary condition. The first step is to convert the weakly parabolic system (2) together with boundary conditions to a corresponding nonlinear scalar parabolic equation. This involves fixing a graphical parametrisation over a reference curve. The reference curve here is a straight line segment. The conversion process using generalised Gaussian coordinates in the case with boundary conditions is described for example in Section 2 of [7] (The case of higher codimension is covered in [9]). The second step is for the scalar parabolic equation with boundary conditions, we consider the corresponding linearized equation, for which existence of a unique (smooth) solution is well-known. By using the solution existence of the linearized problem together with the general result on the nonlinear evolutionary boundary value problems (for example, Theorem 4.4 in [6]) to see that the scalar graph equation has a unique solution at least for a short time. We then prove scalar graph equation is equivalent to the flow system (2), thus a solution to (2) exists for a short time. The solution to (2) is necessarily not unique due to the possibility of choosing different parametrisations, however the image curve is unique. This method also works for the generalised case (4).

2.1 A sixth order flow of plane curves with boundary condition

Let $l([-1,1])$ be a straight line segment which is perpendicular to boundaries η_1,η_2. Define the flux lines $\Phi=\Phi(u,\cdot)$ to $l([-1,1])$ are perpendicular to $l([-1,1])$ and tangential to η_1,η_2. Define a neighbourhood $\mathscr{U}\subset\mathbb{R}^2$ of $l([-1,1])$, $\mathscr{U}_\varepsilon := \{\Phi(u,x) : u\in[-1,1], |x|<\varepsilon\}$. In $\mathscr{U}$, let $\rho(p_0)$ denote tangential coordinate of p_0 on $l([-1,1])$, we can define a smooth normal vector field ξ with the following properties:

$$\langle\xi,\rho\rangle|_{l([-1,1])}=0, \quad \xi|_{\eta_{1,2}\cap\mathscr{U}_\varepsilon}\in T\eta_{1,2}, \quad \|\xi\|=1,$$

where $\eta_{1,2}\cap\mathscr{U}_\varepsilon=\{p\in\mathbb{R}^2 : p=\Phi(u,x), u\in\eta_{1,2}, x\in(-\varepsilon,\varepsilon)\}$.

Hence for any given point $p=\Phi(u,x)$, we can define $x(p)$ is the length of the flux line through p between p and intersection point $p_0=\Phi(u,0)$ on $l([-1,1])$. We define $M=\{p\in\mathbb{R}^2 : p=\Phi\left(u,w(u,t)\right), u\in[-1,1]\}$, here $w(u,t):[-1,1]\times[0,\sigma]\to\mathbb{R}$ and $\sigma\in[0,T)$.

We transform the problem (2) to a scalar initial-boundary-value graph problem as follows,

$$\begin{cases} \frac{\partial w}{\partial t}(u,t)=f(u,t), & \forall(u,t)\in[-1,1]\times[0,\sigma] \\ w(\cdot,0)=w_0, & \\ w_u=w_{u^3}=w_{u^5}=0, \ \forall(u,t)\in\eta_{1,2}\times[0,\sigma] & \end{cases} \tag{7}$$

where $f(u,t) = v^{-6} w_{u^6} + g(w_u, w_{uu}, w_{u^3}, w_{u^4}, w_{u^5})$, $v(u,t) = |\gamma_u|$ and g is a function depending only on $w_u, w_{uu}, w_{u^3}, w_{u^4}, w_{u^5}$.

Next let $\bar{\gamma}(\phi(u,t),t) = (\phi(u,t), w(\phi(u,t),t))$ and define $\phi : [-1,1] \times [0,\sigma] \to [-1,1]$ by the following system of ordinary differential equation:

$$\begin{cases} \frac{d}{dt}\phi(u,t) = -(\bar{\gamma}_u)^{-1} \cdot \left(\frac{\partial}{\partial t}\bar{\gamma}\right)^T (\phi(u,t),t) \\ \phi(u,0) = u, \end{cases}$$

where $\alpha^T := \alpha - \langle \alpha, \bar{v} \rangle \cdot \bar{v}$ denotes the tangential component of a vector α, $\bar{v}(\phi(u,t),t) = v^{-1} \cdot \left(-w_\phi, 1\right)$ denotes normal vector field, $w_\phi\left(\phi(u,t),t\right) = \frac{\partial w}{\partial \phi}\left(\phi(u,t),t\right)$.

At least for a short time, ϕ is a diffeomorphism on $[-1,1]$, it's equivalent to that ϕ is tangential to the boundaries $\eta_{1,2}$, i.e. $u \in \eta_{1,2} \Longrightarrow \phi(u,t) \in \eta_{1,2}, \forall t \in [0,\sigma]$.

We prove the original problem (2) and the scalar graph problem (7) are equivalent under tangential diffeomorphism. See [8] for the proof.

Lemma 4. *The boundary conditions in* (7) *satisfy the compatibility condition,* $\forall (u,t) \in \eta_{1,2} \times [0,\sigma]$*, we have*

$$\partial_t^j w_u\Big|_{t=0} = \partial_t^j w_{u^3}\Big|_{t=0} = \partial_t^j w_{u^5}\Big|_{t=0} = 0, \quad j = 0,1,2,...,n.$$

Lemma 5. *The boundary conditions in* (7) *satisfy the normal boundary conditions.*

For the definition of the normal boundary condition, see [6].

Now we do the linearization at any $a \in \{w : [-1,1] \times [0,\sigma] \to \mathbb{R}\}$ for nonlinear problem (7). The linear problem of (7) can be written as:

$$\begin{cases} \frac{\partial w}{\partial t}(u,t) = f_a(a) w(u,t),\ \forall (u,t) \in [-1,1] \times [0,\sigma] \\ w(\cdot,0) = w_0, \\ w_u = w_{u^3} = w_{u^5} = 0, \qquad \forall (u,t) \in \eta_{1,2} \times [0,\sigma] \end{cases} \tag{8}$$

where $f_a(a)w(u,t) = -v^{-6}(a) \cdot w_{u^6} + g_n w_{u^n}$, g_n are depends only on $a, a_u, ..., a_{u^{7-n}}$ and are all smooth in space and time, $n = 1,2,...,5$.

Proposition 1. *There is always a unique solution for the linear problem (8).*

The proof for Proposition 1 refers to classical results on linear parabolic boundary value problem (for example [2], Ch IV, 6.4).

Lemma 4, Lemma 5 and Proposition 1 allow us to use the result in [6], then our graph boundary value problem (7) has a unique solution. As (7) is equivalent to (2) under tangential diffeomorphism, thus we get the short time existence for (2):

Theorem 3. *There exists a smooth solution* $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^2$ *of the flow system (2), unique up to parametrisation. This solution is in the class* $C^{6,1,\alpha}\left([-1,1] \times [0,T)\right)$ *(with arbitrary* $0 < \alpha < 1$*).*

2.2 Higher order flows of plane curves with boundary conditions

The way to proof the short time existence for the $(2m+4)$th-order curvature flows of plane curves with generalised Neumann boundary condition (4) is similar to the method used for the sixth order curvature flow problem (2).

Firstly, we transform the given problem into an equivalent initial-boundary-value problem for a scalar function and using standard results of the parabolic theory. The scalar initial-boundary-value problem:

$$\begin{cases} \frac{\partial w}{\partial t}(u,t) = f(u,t), & \forall (u,t) \in [-1,1] \times [0,\sigma] \\ w(\cdot,0) = w_0, & \\ w_u = w_{u^3} = w_{u^5} = \ldots = w_{u^{2m+3}} = 0, & \forall (u,t) \in \eta_{1,2} \times [0,\sigma] \end{cases} \tag{9}$$

here $w : [-1,1] \times [0,\sigma] \to \mathbb{R}$.

Secondly, we prove that scalar nonlinear initial-boundary-value problem (9) has a unique solution for a short time. Next we show that equations (9) and (4) are equivalent, see the sixth order case for the proof. Thus, flow problem (4) has a unique solution for finite time up to reparameterization.

Theorem 4. *There exists a smooth solution $\gamma : [-1,1] \times [0,T) \to \mathbb{R}^2$of the system (4), unique up to parametrisation. This solution is in the class $C^{2m+4,1,\alpha}([-1,1] \times [0,T))$ (with arbitrary $0 < \alpha < 1$).*

Directly, we can get the short time existence for flow (5) satisfying Neumann boundary condition and $k_s = \ldots = k_{s^{2m-1}} = k_{s^{2m+1}} = 0$ at the boundary and with smooth initial curve $\gamma(\cdot,0) = \gamma_0$ compatible with the boundary conditions, the solution is also unique up to parametrisation.

3 Short time existence for flow of closed planar curves without boundary

The framework of short time existence for flow of closed planar curves without boundary is that we first write the length-constrained curve diffusion flow as a graph over the initial curve for unknown function of time, we have the scalar nonlinear parabolic problem. Secondly, we prove there is a unique solution for the graph problem and the length-constrained curve diffusion flow is invariant under tangential diffeomorphisms. Then these is a unique solution for length-constrained flow with the unknown time function. Thirdly, we use the Schauder fixed point theorem to prove the unique solution exists for our original problem with specific $h(t)$.

The constrained curve diffusion flow (6) is introduced in Definition 3. Firstly we write $\gamma : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ as a graph for unknown function of time $\bar{h}(t)$ over the initial curve γ_0, using $\nu(u,t)$, $\tau(u,t)$ to denote the tangential and normal vector fields of the curve $\gamma(u,t)$ respectively, then $\langle \nu, \tau \rangle(u,t) = 0$, $\nu(u,t) = rot_{\pi/2}\tau(u,t)$. Let $f : \mathbb{R} \times [0,T) \to \mathbb{R}$, $\nu_0(u) = \nu(u,0)$ write

$$\gamma(u,t) = \gamma_0(u) + f(u,t)\nu_0(u).$$

We write the scalar nonlinear parabolic problem as:

$$\begin{cases} (\partial_t f)(u,t) = Q(f),\ \forall (u,t) \in \mathbb{S}^1 \times [0,T) \\ f(\cdot,0) = 0, \end{cases} \tag{10}$$

where

$$Q(f) = -\frac{V_0^2(1-k_0 f)^2}{V^6} \cdot f_{u^4} + b\left(\bar{h}, f, f_u, f_{uu}, f_{u^3}\right),$$

here $V = |\gamma_u(u,t)|, V_0 = |\gamma_u(u,0)|$ and b is a function depending only on $\bar{h}(t), f, f_u, f_{uu}, f_{u^3}$. As $f \in C^{4,1,\alpha}\left(\mathbb{S}^1 \times [0,T)\right)$ which means that f is $C^{4,\alpha}$ in space and $C^{1,\alpha}$ in time, then $b\left(f, f_u, f_{uu}, f_{u^3}\right)$ is bounded and continuous in sapce and time.

Secondly, we linearize $Q(f)$ at $f_0 = f(\cdot,0) = 0$, then our linearized scalar graph problem at $f_0 = 0$ is

$$\begin{cases} (\partial_t f)(u,t) = -V_0^{-4} \cdot f_{u^4} + g_l \cdot f_{u^l},\ \forall (u,t) \in \mathbb{S}^1 \times [0,T) \\ f(\cdot,0) = 0. \end{cases}$$

As $f \in C^{4,1,\alpha}\left(\mathbb{S}^1 \times [0,T)\right)$, thus the leading coefficient $-V_0^{-4}$ and $g_l, l = 0,1,2,3$ are continuous at u,t and uniformly bounded. We also can see that the leading coefficient satisfies Legendre-Hadamard condition (See [1], there exists a positive constant $\lambda \in \mathbb{R}$ such that the leading coefficient satisfies $|-V_0^{-4}| \geq \lambda$). Thus we can refer to Main Theorem 5 in [1] and proof that there is a unique solution for the nonlinear scalar graph problem (10) when $\bar{h}(t)$ is an unknown function of time.

Proposition 2. *There exists a positive time $T > 0$ such that the problem* (10) *has a unique solution $f \in C^{4,1,\alpha}(\mathbb{S}^1 \times [0,T))$.*

Lemma 6. *Length-constrained curve diffusion flow is invariant under tangential diffeomorphisms.*

For the proof of Lemma 6, we refer to Lemma 2.11 in [10].

Before giving the fixed point argument, we calculate $\frac{d^2}{dt^2}h(0) \leq c(\gamma_0)$ first, we do the second derivative of $h(t) = -\frac{\int_\gamma k_s^2 ds}{2\pi w}$ with respect to time, the highest order term is $\int_\gamma k_{s^5}^2 ds$. So $\frac{d^2}{dt^2}h(0)$ is bounded if $\gamma_0 \in C^{7,\alpha}(\mathbb{S}^1)$.

Theorem 5. *(Schauder fixed point theorem) Let I be a compact, convex subset of a Banach space B and let J be a continuous map of I into itself. Then J has a fixed point.*

We get the short time existence for the flow problem (6) by applying Theorem 5 together with $\frac{d^2}{dt^2}h(0)$ is bounded. (For the proof see [10].)

Theorem 6. *Let $\gamma_0 : \mathbb{S}^1 \to \mathbb{R}^2$ be a $C^{7,\alpha}$-regular immersed curve. Then there exists a maximal $T \in (0,\infty]$ such that the constrained curve diffusion flow system* (6) *is uniquely solvable with γ of degree $C^{4,1,\alpha}\left(\mathbb{S}^1 \times [0,T)\right)$.*

References

1. C. Baker, The mean curvature flow of submanifolds of high codimension, PhD thesis, ANU (2010).
2. J.L. Lions and E. Magenes, Non-homogeneous boundary value problems and application, vol. I–II, Springer, Berlin (1972).
3. J. McCoy, G. Wheeler and Y. Wu, A sixth order curvature flow of plane curves with boundary conditions, Tohuko Mathematical Journal, (2019).
4. J. McCoy, G. Wheeler and Y. Wu, Evolution of closed curves by length-constrained curve diffusion, Proc. Amer. Math. Soc. Vol. 147 (2019), pp. 3493-3506.
5. J. McCoy, G. Wheeler and Y. Wu, Higher order curvature flows of plane curves with generalised Neumann boundary conditions, arXiv:2001.06140.
6. M. Poppenberg, Nash-moser techniques for nonlinear boundary-value problems, Electronic Journal of Differential Equations, Vol. 2003(2003), No. 54, pp. 1-33.
7. A.Stahl, Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition, Calculus of Variations and Partial Differential Equations, Vol. 4(1996), pp. 385-407.
8. A.Stahl, Uber den mittleren Krümmungsfluss mit Neumannrandwerten auf glatten Hyperflächen, PhD thesis, Fachbereich Mathematik, Eberhard-Karls-Universität, Tüebingen, Germany, (1994).
9. A. Spener, Short time existence for the elastic flow of clamped curves, Mathematische Nachrichten, Vol. 291 (2018), pp. 2115-2116.
10. G. Wheeler, Fourth order geometric evolution equations, PhD thesis, UOW (2009).
11. G. Wheeler and V. Wheeler, Curve diffusion and straightening flows on parallel lines, arXiv:1703.10711.