Hankel transforms and weak dispersion

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Abstract This survey is concerned with a general strategy, based on Hankel transforms and special functions decompositions, to prove weak dispersive estimates for a class of PDE's. Inspired by [2], we show how to adapt the method to some scaling critical dispersive models, as the Dirac-Coulomb equation and the fractional Schrödinger and Dirac equation in Aharonov-Bohm field.

1 Introduction

Let $a \in \mathbb{R}$ and let us consider the Hamiltonians

$$H_0 := -\Delta, \qquad H_a := -\Delta + \frac{a}{|x|^2},$$

on $L^2(\mathbb{R}^n)$, with $n \geq 2$. It is well known that, under the condition

$$a \geq -\frac{(n-2)^2}{4}, \tag{1}$$

the Hamiltonian H_a can be realized as the Friedrichs' extension of the symmetric semi-bounded operator $-\Delta + a/|x|^2$, acting on the natural domain induced by the quadratic form

$$q[u] := \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx + a \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2} \, dx.$$

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In particular, by the Spectral Theorem we can define the Schrödinger flow e^{itH_a} on the domain of H_a, for any a satisfying (1). For $a \neq 0$, in dimension $n \geq 3$, we can consider H_a as a critical linear perturbation of H_0, due to the Hardy's inequality

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^2}\,dx \leq \frac{4}{(n-2)^2}\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx \qquad (n \geq 3).$$

In addition, the Schrödinger equation

$$\partial_t u(t,\cdot) = -iH_a u(t,\cdot) \tag{2}$$

is invariant under the scaling

$$u_\lambda(t,x) := u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right).$$

In the recent years, a new interest has been devoted to the study of the dispersive properties of flows as e^{itH_a}, once it was realized that the somehow vintage business of special functions and Hankel transforms could play a role in the analysis of critical and scaling invariant models (see e.g. [2, 3, 6, 7, 10, 11, 12, 14, 17, 20]). In this topic, we will point our attention on dispersive models which usually arise in Quantum Mechanics and always enjoy the above mentioned property of criticality. The inspiration, and main motivation of the project, comes form the papers [2, 3], of which we now briefly review the main results. Let us first recall the spherical harmonics decomposition of $L^2(\mathbb{R}^2)$, which is a peculiar feature of the 2D-space. Given the complete orthonormal set $\{\phi_m\}_{m\in\mathbb{Z}}$ on $L^2(\mathbb{S}^1)$, with $\phi_m = \phi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi}}$, $\theta \in [0,2\pi)$, one has the canonical isomorphism

$$L^2(\mathbb{R}^2) \cong \bigoplus_{m\in\mathbb{Z}} L^2(\mathbb{R}_+, rdr) \otimes [\phi_m] \tag{3}$$

where we are denoting with $[\phi_m]$ the one dimensional space spanned by ϕ_m and with $\|f\|^2_{L^2_{rdr}} = \int_0^\infty |f(r)|^2 rdr$. We denote by $L^2_{\geq d(\mathbb{R}^2)}$, the subspace of L^2 consisting of all functions that are orthogonal to all spherical harmonics of degree less than d. In [2], the authors managed to prove the following family of estimates

$$\| |x|^{-1/2-2\alpha}(H_a^{1/4-\alpha}) e^{itH_a} f\|_{L^2_t L^2_x} \leq C\|f\|_{L^2(\mathbb{R}^n)} \tag{4}$$

where we are denoting with

$$H_a = -\Delta + \frac{a}{|x|^2}$$

for $n \geq 2$, $\alpha \in (0, \frac{1}{4} + \frac{1}{2}\mu_d)$, with $\mu_d = \sqrt{(\lambda(n)+d)^2 + a}$, $d \geq 0$, $\lambda(n) = \frac{n-2}{2}$, and $f \in L^2_{\geq d(\mathbb{R}^n)}$. The strategy developed in [2] can be roughly summarized in the following steps.

1. Use *spherical harmonics decomposition* to reduce the equation to a radial problem;

2. Use *Hankel transform* to "diagonalize" the reduced problem and to define fractional powers of the operator $-\Delta+\frac{a}{|x|^2}$;
3. Prove the smoothing estimate on a fixed spherical space using Hankel transform properties and the explicit integral representation of the fractional powers;
4. Sum back: use triangle inequality and L^2-orthogonality of spherical harmonics to obtain the desired estimate for the original dynamics. To conclude, it will be crucial to show that the constant obtained in step (3) is a bounded function of the spherical parameter.

In [2, 3], as an appliction of of (4), the authors proved that the usual Strichartz estimates hold for the flow e^{itH_a}, when a satisfies (1). More precisely,

$$\left\|e^{itH_a}f\right\|_{L_t^pL_x^q}\leq C\|f\|_{L^2}, \tag{5}$$

for some $C>0$ independent on f, provided (1) and

$$\frac{2}{p}=\frac{n}{2}-\frac{n}{q}, \qquad p\geq 2, \qquad (p,q,n)\neq(2,+\infty,2).$$

At that time, it was a striking result, since it was completely unclear whether Strichartz estimates would have been true for critical perturbations of the free Hamiltonian. Moreover, it is known that the inverse-square potential represents a threshold, among homogeneous perturbations, for the validity of Strichartz estimates (see [13, 16]). In addition, it is now known that the usual time decay estimate

$$\sup_{x\in\mathbb{R}^n}\left|e^{itH_a}f(x)\right|\leq C|t|^{-\frac{n}{2}}\|f\|_1$$

fails, in general, as soon as $a<0$ (see [10, 11, 12]), which possibly gives strength to the averaging property of Strichartz estimates. To complete the state of the art, for the critical value $a=-(n-2)^2/4$, in the recent papers [21, 25] the authors proved the validity of Strichartz estimates for the Schrödinger and wave equations, provided the admissible couple is not endpoint.

To prove (5) by (4) is a quite simple application of a $TT^\star$ argument, mixing free Strichartz estimates and (4) when $\alpha=\frac{1}{4}$, $d=0$, which is

$$\||x|^{-1}e^{itH_a}f\|_{L_t^2L_x^2}\leq C\|f\|_{L^2(\mathbb{R}^n)}. \tag{6}$$

It is important to notice that $\alpha=\frac{1}{4}$ is in the range of estimate (4), for $d=0$, thanks to (1), which implies $\mu_d>0$ (see [2, 3] for details).

The aim of this survey is to describe under which extent we can hope to generalize estimates (4) to other dispersive models and which is the quantitative role played by the number a, interpreted as the bottom of the spectrum of the angular component of H_a. In the following, we will restrict our attention on fractional Schrödinger equations in Aharonov-Bohm fields and Dirac equations, both in Coulomb and Aharonov-Bohm fields.

2 Fractional Schrödinger in Aharonov-Bohm field

An interesting 2D-example of scaling critical, first order perturbation of the Laplace operator is given by the so called *Aharonov-Bohm field*: such a field is given by

$$A_B:\mathbb{R}^2\setminus\{(0,0)\}\to\mathbb{R}^2,\qquad A_B(x)=\left(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\right),\qquad x=(x_1,x_2) \tag{1}$$

so that we can define for each $\alpha\in\mathbb{R}$ the Hamiltonian

$$H_\alpha=\left(-i\nabla+\alpha\left(-\frac{x_2}{|x|^2},\frac{x_1}{|x|^2}\right)\right)^2 \tag{2}$$

which is self-adjoint. By the Spectral Theorem we can thus define fractional powers of the hamiltonian H_α, and thus we can in particular consider for $a>0$ the following Cauchy problems

$$\begin{cases}\partial_t u = iH_\alpha^{a/2}u\\ u(0,\cdot)=f(\cdot)\in L^2(\mathbb{R}^2)\end{cases} \tag{3}$$

which we will refer to as *fractional Schrödinger equation with Aharonov-Bohm field*. We note that the cases $a=1$ and $a=2$ correspond, respectively, to the Schrödinger and wave flows.

The main result in this case is given by the following Theorem, that is proved in [6].

Theorem 1 ([6]). *Let $a>0$, $\alpha\in\mathbb{R}$ and*

$$0<\varepsilon<\frac{1}{4}+\frac{1}{2}dist(\alpha,\mathbb{Z}).$$

Then for every $f\in L^2$ the following estimate holds

$$\||x|^{-\frac{1}{2}-2\varepsilon}H^{\frac{a-1}{4}-\varepsilon}e^{itH^{a/2}}f\|_{L^2_tL^2_x}\le C\|f\|_{L^2} \tag{4}$$

with a constant C depending on α and ε.

In addition, in the endpoint case $\varepsilon=0$ the following local estimate holds

$$\sup_{R>0}R^{-1/2}\|e^{itH^{a/2}}f\|_{L^2_tL^2_{|x|<R}}\le C\|H^{\frac{1-a}{4}}f\|_{L^2_x}. \tag{5}$$

Remark 1. It is woth noticing that (4) fails for $\alpha\in\mathbb{Z}$ and $\varepsilon=\frac{1}{4}$. Indeed, the dimension $d=2$ is critical with respect to estimate (4), with $\varepsilon=\frac{1}{4}$, due to the fact that the weight $|x|^{-1}$ is too singular at the origin. Nevertheless, the presence of the field A_B, as it is well known, improves the angular ellipticity of H, if $\alpha\notin\mathbb{Z}$, and this usually permits to obtain better estimates than in the free case, as (4) shows. Roughly speaking, the higher the spherical frequency is, the better is the dispersive phenomenon we are measuring. The improvement arises since the introduction of the external potential is cutting the 0-frequency from the spectrum of the spherical operator.

Remark 2. Notice that estimates above can be extended, by following the argument in [8], to deal with the Klein-Gordon flow $e^{it\sqrt{H^a+1}}$. Also, as an immediate corollary of the result above, it is possible to prove weighted Strichartz estimates for the dynamics (3) (simply by interpolating estimate (4) with the 2D Sobolev inequality)

2.1 The massless Dirac-Coulomb equation

The Cauchy problem for the 3D massless Dirac equation with a Coulomb potential reads as

$$\begin{cases} i\partial_t u + \mathscr{D}u + \dfrac{\nu}{|x|}u = 0, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}^4 \\ u(0,x) = f(x) \end{cases} \tag{6}$$

where we recall that the massless Dirac operator $\mathscr{D}$ is defined in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7}$$

as

$$\mathscr{D} = -i\sum_{k=1}^{3} \alpha_k \partial_k = -i(\alpha \cdot \nabla)$$

where the 4×4 Dirac matrices are given by

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3. \tag{8}$$

The charge ν is assumed to be in the interval $(-1,1)$, as the operator $\mathscr{D}_\nu = \mathscr{D} + \frac{\nu}{|x|}$ needs to be self-adjoint (see [9]).

The adaptation of the machinery to this setting is a bit more tricky, and requires to deal with some additional technical difficulties, which are mainly due to the rich algebraic structure of the Dirac operator, that are essentially the following:

- The Dirac operator does not preserve radiality, meaning that the standard spherical harmonics decomposition does not represent a "good" setting, which is instead given by the so called *partial wave decomposition*, that we now briefly introduce. First of all, we use spherical coordinates to write

  $$L^2(\mathbb{R}^3, \mathbb{C}^4) \cong L^2((0,\infty), r^2 dr) \otimes L^2(S^2, \mathbb{C}^4)$$

  with S^2 being the unit sphere. Then, we have the orthogonal decomposition on S^2:

$$L^2(S^2,\mathbb{C}^4) \cong \bigoplus_{k\in\mathbb{Z}\setminus\{0\}} \bigoplus_{m=-|k|+1}^{|k|} h_{m,k}$$

where the spaces $h_{m,k} := \mathbb{C}\Phi^+_{m,k} + \mathbb{C}\Phi^-_{m,k}$ with

$$\Phi^+_{m,k} = \begin{pmatrix} \phi^+_{m,k} \\ 0 \end{pmatrix}, \qquad \Phi^+_{m,k} = \begin{pmatrix} 0 \\ \phi^-_{m,k} \end{pmatrix}$$

and the functions $\phi^\pm_{m,k}$ can be explicitly written in terms of standard spherical harmonics as

$$\phi^\pm_{m,k} = \frac{1}{\sqrt{|2k\pm 1|}} \begin{pmatrix} \sqrt{|k\mp(m-1)|}Y^{m-1}_{|k|-H(\mp k)} \\ \mp\mathrm{sgn}(k)\sqrt{|k\pm m)|}Y^{m}_{|k|-H(\mp k)} \end{pmatrix}$$

and H is the Heaviside function. The action of the Dirac-Coulomb operator leaves invariant these subspaces and this decomposition it is represented by the radial matrix

$$\mathscr{D}_{\nu,k} = \begin{pmatrix} \frac{\nu}{r} & -\frac{d}{dr} + \frac{1+k}{r} \\ \frac{d}{dr} - \frac{1-k}{r} & \frac{\nu}{r} \end{pmatrix}. \tag{9}$$

We mention the fact that a similar decomposition holds in any dimension $n \geq 2$. The standard reference for this and related problem is the book [24]

Remark 3. We have to stress the fact that the decomposition introduced in [24] (see in particular Subsection 4.6.5) is slightly different, as it also relies on the isomorphism $u \to r\tilde{u}$. This has the effect of rather "simplifying" the expression of the action of the radial Dirac operator given in (9) and, of course, affects the presence of a weight in the radial scalar product. This same approach is used also e.g. in [4]. Also, we stress the fact that our index m is shifted by $1/2$ with respect to the one in [24], in order to avoid half integers.

- The construction of the analogous of the Hankel transform is a more delicate problem: roughly speaking, we will need to define a two-dimensional operator which projects onto the positive and negative part of the continuous spectrum of the Dirac Coulomb operator (we recall that these generalized eigenfunctions are explicit and well known, see e.g. [19]). If we thus define, for a fixed admissible couple m,k, the "relativistic Hankel transform" to be

$$\mathscr{H}^\pm_{m,k}\Phi(r) = \langle \Psi^{\pm E}_{m,k}(r), \Phi(r)\rangle_{L^2(r^2dr)}$$

  we obtain the properties we need, simply relying on the self-adjointess of the operator $\mathscr{D}_{\nu,k}$ with respect to the $L^2(r^2dr)$ scalar product

$$\mathscr{H}^\pm_{m,k}\mathscr{D}_{\nu,k} = \pm E\mathscr{H}^\pm_{m,k}. \tag{10}$$

  With this, we mean that the transform $\mathscr{H}^\pm_{m,k}$ "diagonalizes" the equation.

- By relying on property (10) we can define fractional powers of the restricted Dirac-Coulomb operator. As a last (and technical) step then we will have to show that these fractional powers admit an integral kernel which can be explicitly written by solving suitably weighted interaction integrals of generalized eigenstates; this will allow to prove the estimate on a fixed partial wave subspace with a suitable constant depending on the spherical parameters which, again, will need to be bounded in order to allow the application of triangle inequality to sum back in the partial wave decomposition.

The following result is proved in [7] (we mention the fact that a similar result holds in 2D as well).

Theorem 2 ([7]). *Let K be a positive integer, and set*

$$h_{\geq K} = \bigoplus_{|k|\geq K} \bigoplus_{m=-|k|+1}^{|k|} h_{m,k}.$$

Then for any

$$1/2 < \varepsilon < \sqrt{K^2 - \nu^2} + 1/2$$

and any $f \in L^2((0,\infty), r^2 dr) \otimes h_{\geq K}$ *there exists a constant* $C = C(\nu, \varepsilon, K)$ *such that the following estimate holds*

$$\| |x|^{-\varepsilon} |\mathscr{D}_\nu|^{1/2-\varepsilon} e^{it(\mathscr{D}+\frac{\nu}{|x|})} f \|_{L^2_t L^2_x} \leq C \|f\|_{L^2_x}. \tag{11}$$

Remark 4. We need to point out a typo in formula (2.7) in [7]: in the definition of the space $\mathscr{H}^3_{\geq \bar{k}_3}$ the sum in j is in the range $j \geq |\bar{k}_3| - 1/2$. Also, we stress the fact that we are here providing a rather different (and somehow simplified) representation of the partial wave subspaces, and therefore of the spaces $h_{\geq K}$, with respect to [7]: in particular, we are here neglecting the sum in $j \in \frac{1}{2}\mathbb{N}$, that is "englobed" in the one in k (which is an integer) and, as mentioned, we have "shifted" the index m by $1/2$.

Remark 5. The analogous of estimate (5) seems to be more complicated to be proved in this contest. This is ultimately due to the much more complicated structure of the Hankel transform, which involves confluent hypergeometric functions: indeed, the key step is represented by the proof of a bound, uniform in R and l, of the form

$$\frac{1}{R} \int_0^R \chi_l(r)^2 r^{n-1} dr < C$$

for the radial components of the generalized eigenstates. This is well known in the case of Bessel functions (see [23]) but seems to be more complicated to be obtained in the Dirac-Coulomb case.

Remark 6. The restriction to the massless case is crucial in our result, as our strategy deeply relies on the scaling invariant structure of the equation, therefore leaving open the question whether the same result (or at least similar) holds in presence of a mass. The application of the strategy presented in [8] to pass from the wave to the Klein-Gordon equation is not indeed completely straightforward, as here we are not simply "shifting" but we are "opening a gap" as, we recall, the spectrum of the free Dirac operator is unbounded both from above and below. Still, it seems to be possible to make things work in this contest, and this will be the object of future investigations. In any case, it is well known (see e.g. [19]) that the massive Dirac-Coulomb operator has eigevalues in the gap: therefore, in order to be able to obtain any kind of dispersive estimates, it will be necessary to project out of the point spectrum.

2.2 The massless Dirac equation in Aharonov-Bohm field

The two results above can be somehow merged to deal with the massless Dirac equation in Aharonov-Bohm field; in this case, the Hamiltonian reads as $\mathscr{D}_A = i\big(\sigma_1(\partial_x + A^1) + \sigma_2(\partial_y + A^2)\big)$, where the σ matrices are defined as in (7) and the magnetic field $A(x)$ is given by (1). In this case we claim that the following result holds (for simplicity we restrict to the case $\alpha \in (0,1)$ without losing in generality).

Theorem 3. *[5] Let $\alpha \in (0,1)$ and $A(x)$ given by* (1)*. Then for any*

$$1/2 < \varepsilon < 1 + |l + \alpha|. \tag{12}$$

and any $f \in L^2((0,\infty), rdr) \otimes \mathscr{H}_{\geq \bar{l}}$ there exists a constant $C = c(\alpha, \varepsilon, l)$ such that the following estimate holds

$$\left\| \Omega^{-\varepsilon} \mathscr{D}_A^{1/2-\varepsilon} e^{it\mathscr{D}_A} f \right\|_{L_t^2 L_x^2} \leq c \|f\|_{L_x^2}. \tag{13}$$

In addition, in the endpoint case $\gamma = 1/2$ the following estimate holds

$$\sup_{R>0} R^{-1/2} \|e^{it\mathscr{D}_A} f\|_{L_t^2 L^2_{|x|\leq R}} \lesssim \|f\|_{L_x^2}, \tag{14}$$

Remark 7. Notice that the range (12) is better than the one in the free case, as soon as $\alpha \notin \mathbb{Z}$. This fact can be interpreted as a *diamagnetic* behavior of this model, which is quite surprising for Dirac-type equations. Indeed, an original conjecture about universal paramagnetism in [18] was later disproved in [1]. The above model seems to go in the same direction of the example in [1], with the difference that it is critically singular at the origin.

Remark 8. It is important to notice that the local smoothing estimates obtained for the three models above do not allow to recover, following the perturbation arguments in [2, 3], the full set of Strichartz estimates for the corresponding flows. To make the argument work we would indeed need, for al the three models, an estimate of the form

$$\||x|^{-1/2} u\|_{L_t^2 L_x^2} \leq C \|f\|_{L_x^2}$$

with u being a solution of any of the models above with initial condition f. But this, as it is seen, is exactly at the endpoint (and just outside) of our admissible ranges; in fact, we seem to have no hope to prove such an estimate even imposing further restrictions (say, radial initial data) as it does fail even for the free wave equation (see e.g. [22]). The loss of (fractional) derivatives in our local smoothing seems on the other hand to suggest that some "weak" (namely, with loss of derivatives) Strichartz estimates should hold in this setting.

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