A priori bounds for the kinetic DNLS

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Abstract In this note, we consider the kinetic derivative nonlinear Schrödinger equation (KDNLS), which arises as a model of propagation of a plasma taking the effect of the resonant interaction between the wave modulation and the ions into account. In contrast to the standard derivative NLS equation, KDNLS does not conserve the mass and the energy. Nevertheless, the dissipative structure of KDNLS enables us to show an a priori bound in the energy space and a lower bound of the L^2 norm for its solution, as we see in this note. Combined with the local well-posedness result, which we plan to show in a forthcoming paper, these bounds will give a global existence result in the energy space for small initial data.

1 Introduction

We consider the kinetic derivative NLS equation (KDNLS):

$$\partial_t u = \partial_x \Big[iu_x + \alpha |u|^2 u + \beta \mathscr{H}(|u|^2)u \Big], \qquad \alpha, \beta \in \mathbf{R}, \quad \beta < 0, \tag{1}$$

where the spatial domain is either $\mathbf{R}$ or $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. We write $\mathscr{F}$ to denote the Fourier transform and use the notation: $u_x := \partial_x u$, $\mathscr{H} := \mathscr{F}^{-1}\big[-i\,\mathrm{sgn}(\xi)\big]\mathscr{F}$, $D := (-\partial_x^2)^{1/2} = \mathscr{F}^{-1}|\xi|\mathscr{F} = \partial_x \mathscr{H}$. The negative constant β represents the ratio of plasma pressure to magnetic pressure, which can be positive, negative or zero according to each physical situation. Equation (1) takes the resonant interaction be-

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tween the wave modulation and the ions into account, while it is ignored in the derivative NLS, i.e., in the case of $\beta = 0$. The word "kinetic" implies that the collective motion of ions in a plasma is modeled by the Vlasov equation and not by the fluid equation. If Maxwell's equations and the Euler equations are taken as a model system, then we have DNLS, i.e., (1) with $\beta = 0$. If Maxwell's equations and the Vlasov equation are taken as a model system, then we have KDNLS (1) (see Dysthe and Pécseli [1] and Mjølhus and Wyller [3, 4].

Due to the presence of the Hilbert transform, the mass and the energy corresponding to the standard DNLS ($\beta = 0$ in (1)) are not conserved under the flow when $\beta < 0$. However, the nonlinear term $\beta\partial_x(\mathscr{H}(|u|^2)u)$ has dissipative structure when $\beta < 0$. The aim of this note is to derive an a priori bound in the energy space H^1 by using this structure. The main result reads as follows:

Theorem 1. *Let u be a smooth solution to* (1) *on* $[0,T]\times Z$*, where Z is either* $\mathbf{R}$ *or* $\mathbf{T}$. *Then, it holds that*

$$\|u(t)\|_{L^2}^2 + |\beta|\int_0^t \|D^{1/2}(|u(\tau)|^2)\|_{L^2}^2\,d\tau \;=\; \|u(0)\|_{L^2}^2, \qquad t\in[0,T].$$

Moreover, there exist $C_*, C > 0$ depending only on α,β (and bounded when $\beta \to 0$) such that if $\|u(0)\|_{L^2} \le C_*^{-1}$, then

$$\|u(t)\|_{H^1}^2 + \frac{|\beta|}{4}\int_0^t \|D^{1/2}\partial_x(|u(\tau)|^2)\|_{L^2}^2\,d\tau \;\le\; 4\|u(0)\|_{H^1}^2 e^{C\|u(0)\|_{L^2}^2},$$
$$\|u(t)\|_{L^2}^2 \;\ge\; \|u(0)\|_{L^2}^2 \exp\Big[-C\|u(0)\|_{H^1} e^{C\|u(0)\|_{L^2}^2}|\beta|^{1/2}t^{1/2}\Big], \qquad t\in[0,T].$$

The proof of the theorem is based on the differential equalities (see Corollary 2) for the mass $\|u(t)\|_{L^2}^2$ and the energy functional

$$E[u] := \int \Big\{|u_x|^2 - \frac{3}{2}\Big(\alpha|u|^2 + \beta\mathscr{H}(|u|^2)\Big)\operatorname{Im}(\bar{u}u_x) + \frac{1}{2}\alpha^2|u|^6\Big\}\,dx.$$

In a forthcoming paper [2], we will consider the case $Z = \mathbf{T}$ and construct local-in-time solutions to the associated Cauchy problem for small initial data in Sobolev space $H^s(\mathbf{T})$ with $s > 1/2$. More precisely, we will prove the following result:

Theorem 2. *We assume $\alpha = 0$ and $\beta < 0$. Let $s \ge s_0 > 1/2$, then there exist $\eta = \eta(s_0,s) > 0$ and $T > 0$ such that for any $u_0 \in H^s(\mathbf{T})$ with $\|u_0\|_{H^{s_0}} \le |\beta|^{1/2}\eta$, the Cauchy problem of* (1) *with $u|_{t=0} = u_0$ has a unique solution $u \in C([0,T];H^s(\mathbf{T}))$ on $(0,T)\times\mathbf{T}$, which belongs to certain auxiliary spaces. Furthermore, the map $u_0 \mapsto u$ is continuous.*

The H^1 a priori bound in Theorem 1 and a standard approximation argument then show:

Corollary 1. *Let $\alpha = 0$ and $\beta < 0$. There exists $\eta > 0$ such that if $u_0 \in H^1(\mathbf{T})$ satisfies $\|u_0\|_{H^1} \le |\beta|^{1/2}\eta$, then the solution $u \in C([0,T];H^1(\mathbf{T}))$ of* (1) *on $(0,T)\times\mathbf{T}$ with $u|_{t=0} = u_0$ constructed in Theorem 2 can be extended to a global-in-time H^1 solution which is bounded and continuous in t.*

2 Energy conservation

In this section, we give a proof of Theorem 1. The next two lemmas follow from a direct calculation, so we omit their proofs.

Lemma 1. *Let u be a smooth solution to* (1). *Then, it holds that*

$$\partial_t(|u|^2) = \partial_x\Big[-2\,\mathrm{Im}(\overline{u}u_x) + \frac{3}{2}\alpha|u|^4 + \beta|u|^2\mathscr{H}(|u|^2)\Big] + \beta|u|^2 D(|u|^2),$$

$$\partial_t\,\mathrm{Im}(\overline{u}u_x) = \partial_x\Big[\frac{1}{2}\partial_x^2(|u|^2) - 2|u_x|^2 + \Big(\alpha|u|^2+\beta\mathscr{H}(|u|^2)\Big)\,\mathrm{Im}(\overline{u}u_x)\Big] + 2\,\mathrm{Im}(\overline{u}u_x)\,\partial_x\Big(\alpha|u|^2+\beta\mathscr{H}(|u|^2)\Big).$$

Lemma 2. *Let u be a smooth solution to* (1). *Then, it holds that*

$$\partial_t\int|u_x|^2\,dx \;=\; -3\int\big(\alpha|u|^2+\beta\mathscr{H}(|u|^2)\big)\partial_x(|u_x|^2)\,dx + \beta\big\|D^{1/2}\partial_x(|u|^2)\big\|_{L^2}^2,$$

$$\partial_t\int|u|^2\,\mathrm{Im}(\overline{u}u_x)\,dx = -2\int|u|^2\partial_x(|u_x|^2)\,dx + \int\Big[2\alpha\partial_x(|u|^4) + 4\beta|u|^2D[|u|^2]\Big]\,\mathrm{Im}(\overline{u}u_x)\,dx,$$

$$\begin{aligned}\partial_t\int\mathscr{H}(|u|^2)\,\mathrm{Im}(\overline{u}u_x)\,dx &= -2\int\mathscr{H}(|u|^2)\partial_x(|u_x|^2)\,dx - 2\big\|D^{1/2}\,\mathrm{Im}(\overline{u}u_x)\big\|_{L^2}^2 + \frac{1}{2}\big\|D^{1/2}\partial_x(|u|^2)\big\|_{L^2}^2\\ &\quad+\alpha\int\Big\{\frac{3}{2}D(|u|^4) - |u|^2D(|u|^2) + 2\mathscr{H}(|u|^2)\partial_x(|u|^2)\Big\}\,\mathrm{Im}(\overline{u}u_x)\,dx\\ &\quad+\beta\int\Big\{D[|u|^2\mathscr{H}(|u|^2)] + \mathscr{H}\big(|u|^2D(|u|^2)\big) + \mathscr{H}(|u|^2)D(|u|^2)\Big\}\,\mathrm{Im}(\overline{u}u_x)\,dx,\end{aligned}$$

and that

$$\partial_t\int|u|^6\,dx \;=\; 6\int\partial_x(|u|^4)\,\mathrm{Im}(\overline{u}u_x)\,dx + 2\beta\int|u|^6D(|u|^2)\,dx.$$

¿From these lemmas, we immediately obtain the following:

Corollary 2. *Let u be a smooth solution to* (1). *Then, we have*

$$\partial_t\int|u|^2\,dt \;=\; \beta\big\|D^{1/2}(|u|^2)\big\|_{L^2}^2$$

and

$$\begin{aligned}\partial_t E[u] &= \frac{1}{4}\beta\big\|D^{1/2}\partial_x(|u|^2)\big\|_{L^2}^2 + 3\beta\big\|D^{1/2}\operatorname{Im}(\bar{u}u_x)\big\|_{L^2}^2 + \alpha^2\beta\int |u|^6 D(|u|^2)\,dx\\ &\quad - \frac{3}{2}\alpha\beta\int\Big\{\frac{3}{2}D(|u|^4) + 2\partial_x\big[|u|^2\mathscr{H}(|u|^2)\big] + |u|^2 D(|u|^2)\Big\}\operatorname{Im}(\bar{u}u_x)\,dx\\ &\quad - \frac{3}{2}\beta^2\int\Big\{D[|u|^2\mathscr{H}(|u|^2)] + \frac{1}{2}\partial_x\big[\big(\mathscr{H}(|u|^2)\big)^2\big] + \mathscr{H}\big[|u|^2 D(|u|^2)\big]\Big\}\\ &\qquad\qquad \times \operatorname{Im}(\bar{u}u_x)\,dx.\end{aligned}$$

We prepare some more lemmas:

Lemma 3. *There exists $C_0 > 0$ depending only on α, β (and bounded as $\beta \to 0$) such that*

$$\|u\|_{L^2} \le C_0^{-1} \qquad \Longrightarrow \qquad 2^{-1}\|u\|_{H^1}^2 \le E[u] + \|u\|_{L^2}^2 \le 2\|u\|_{H^1}^2.$$

Proof. This follows from the Gagliardo-Nirenberg inequality:

$$\|u\|_{L^6(Z)}^6 \lesssim \begin{cases} \|u\|_{L^2}^4\|u_x\|_{L^2}^2 & (Z = \mathbf{R}),\\ \|u\|_{L^2}^6 + \|u\|_{L^2}^4\|u_x\|_{L^2}^2 & (Z = \mathbf{T}). \end{cases} \qquad \square$$

Lemma 4. *The following estimates hold:*

$$\|D^{1/2}(|u|^2)\|_{L^2}^2 \lesssim \|u\|_{L^2}^2\|D^{1/2}\partial(|u|^2)\|_{L^2}, \tag{2}$$

$$\|\mathscr{F}^{-1}\big[|\mathscr{F}(|u|^2)|\big]\|_{L^\infty}\|D^{1/2}(|u|^2)\|_{L^2} \lesssim \|u\|_{L^2}^2\|D^{1/2}\partial(|u|^2)\|_{L^2}, \tag{3}$$

$$\|u\|_{L^\infty}\|\mathscr{F}^{-1}\big[|\mathscr{F}D^{1/2}(|u|^2)|\big]\|_{L^\infty} \lesssim \|u\|_{L^2}\|D^{1/2}\partial(|u|^2)\|_{L^2}. \tag{4}$$

Proof. We first derive (2). In the non-periodic case, by interpolation we have

$$\begin{aligned}\||u|^2\|_{L^2} &\lesssim \||u|^2\|_{L^\infty}^{1/2}\||u|^2\|_{L^1}^{1/2} \lesssim \|\partial(|u|^2)\|_{L^2}^{1/4}\||u|^2\|_{L^2}^{1/4}\|u\|_{L^2}\\ &\lesssim \|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/6}\||u|^2\|_{L^2}^{1/3}\|u\|_{L^2},\end{aligned}$$

which implies

$$\||u|^2\|_{L^2} \lesssim \|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/4}\|u\|_{L^2}^{3/2}.$$

Using this, we have

$$\|D^{1/2}(|u|^2)\|_{L^2}^2 \lesssim \|D^{1/2}\partial(|u|^2)\|_{L^2}^{2/3}\||u|^2\|_{L^2}^{4/3} \lesssim \|D^{1/2}\partial(|u|^2)\|_{L^2}\|u\|_{L^2}^2.$$

The above argument also works in the periodic case if $|u|^2$ is replaced with $|u|^2 - \frac{1}{2\pi}\int_0^{2\pi}|u|^2\,dx$. Since we estimate $D^{1/2}(|u|^2)$, the same result (2) holds in the periodic case.

In what follows, $\chi = 0$ if $Z = \mathbf{R}$ and $\chi = 1$ if $Z = \mathbf{T}$. By interpolation inequalities, we have

$$\begin{aligned}\|u\|_{L^\infty}^2 = \||u|^2\|_{L^\infty} &\lesssim \chi\||u|^2\|_{L^2} + \|\partial(|u|^2)\|_{L^2}^{1/2}\||u|^2\|_{L^2}^{1/2}\\ &\lesssim \chi\|u\|_{L^\infty}\|u\|_{L^2} + \|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/3}\||u|^2\|_{L^2}^{2/3}\\ &\lesssim \chi\|u\|_{L^\infty}\|u\|_{L^2} + \|u\|_{L^\infty}^{2/3}\|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/3}\|u\|_{L^2}^{2/3},\end{aligned}$$

which implies that

$$\|u\|_{L^\infty}^2 \;\lesssim\; \chi\|u\|_{L^2}^2 + \|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/2}\|u\|_{L^2}. \tag{5}$$

The above argument also shows that

$$\|\mathscr{F}^{-1}\left[|\mathscr{F}(|u|^2)|\right]\|_{L^\infty} \;\lesssim\; \chi\|u\|_{L^2}^2 + \|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/2}\|u\|_{L^2}. \tag{6}$$

If $Z = \mathbf{R}$, (3) is obtained from (2) and (6). If $Z = \mathbf{T}$, it suffices to combine (2), (6) with the trivial estimate $\|D^{1/2}(|u|^2)\|_{L^2} \le \|D^{1/2}\partial(|u|^2)\|_{L^2}$.

The same argument for (3) but using (5) instead of (6) shows

$$\|u\|_{L^\infty}^2\|D^{1/2}(|u|^2)\|_{L^2} \;\lesssim\; \|u\|_{L^2}^2\|D^{1/2}\partial(|u|^2)\|_{L^2}.$$

Using this and interpolation, we see

$$\begin{aligned}\|u\|_{L^\infty}\|\mathscr{F}^{-1}\left[|\mathscr{F}D^{1/2}(|u|^2)|\right]\|_{L^\infty} &\lesssim \|u\|_{L^\infty}\|D^{1/2}(|u|^2)\|_{L^2}^{1/2}\|D^{1/2}\partial(|u|^2)\|_{L^2}^{1/2}\\ &\lesssim \|u\|_{L^2}\|D^{1/2}\partial(|u|^2)\|_{L^2},\end{aligned}$$

which shows (4). □

Proof (Proof of Theorem 1). The L^2 equality follows from the first equality in Corollary 2, so we focus on the H^1 a priori estimate and the exponential L^2 lower bound.

¿From (3), the integrals

$$\int D(|u|^4)\,\mathrm{Im}(\overline{u}u_x)\,dx, \qquad \int \partial_x\left[|u|^2\mathscr{H}(|u|^2)\right]\mathrm{Im}(\overline{u}u_x)\,dx,$$
$$\int D[|u|^2\mathscr{H}(|u|^2)]\,\mathrm{Im}(\overline{u}u_x)\,dx, \qquad \int \partial_x\left[\left(\mathscr{H}(|u|^2)\right)^2\right]\mathrm{Im}(\overline{u}u_x)\,dx$$

are bounded by

$$\begin{aligned}&\|\mathscr{F}^{-1}\left[|\mathscr{F}(|u|^2)|\right]\|_{L^\infty}\|D^{1/2}(|u|^2)\|_{L^2}\|D^{1/2}\,\mathrm{Im}(\overline{u}u_x)\|_{L^2}\\ &\lesssim \|u\|_{L^2}^2\|D^{1/2}\partial(|u|^2)\|_{L^2}\|D^{1/2}\,\mathrm{Im}(\overline{u}u_x)\|_{L^2}\\ &\lesssim \|u\|_{L^2}^2\|D^{1/2}\partial(|u|^2)\|_{L^2}^2 + \|u\|_{L^2}^2\|D^{1/2}\,\mathrm{Im}(\overline{u}u_x)\|_{L^2}^2.\end{aligned}$$

To estimate the integrals

$$\int |u|^2 D(|u|^2)\,\mathrm{Im}(\overline{u}u_x)\,dx, \qquad \int \mathscr{H}\left[|u|^2 D(|u|^2)\right]\mathrm{Im}(\overline{u}u_x)\,dx,$$

we denote the frequency variables for $|u|^2$, $D(|u|^2)$, and $\mathrm{Im}(\overline{u}u_x)$ by k_1, k_2, and k_3, respectively. Note that $k_1+k_2+k_3=0$. If $|k_3| \gtrsim |k_2|$, we can move half a derivative onto $\mathrm{Im}(\overline{u}u_x)$ and argue as before. If $|k_1| \sim |k_2| \gg |k_3|$, we apply (4) to estimate these integrals as

$$\begin{aligned}
&\|\mathscr{F}^{-1}\big[|\mathscr{F}D^{1/2}(|u|^2)|\big]\|_{L^\infty}\|D^{1/2}(|u|^2)\|_{L^2}\|u\|_{L^\infty}\|u_x\|_{L^2} \\
&\quad\lesssim \|u\|_{L^2}\|D^{1/2}\partial(|u|^2)\|_{L^2}\|D^{1/2}(|u|^2)\|_{L^2}\|u_x\|_{L^2} \\
&\quad\lesssim \|u\|_{L^2}^2\|D^{1/2}\partial(|u|^2)\|_{L^2}^2 + \|D^{1/2}(|u|^2)\|_{L^2}^2\|u_x\|_{L^2}^2.
\end{aligned}$$

Finally, using (3) we have

$$\begin{aligned}
\Big|\int |u|^6 D(|u|^2)\,dx\Big| &\lesssim \|\mathscr{F}^{-1}\big[|\mathscr{F}(|u|^2)|\big]\|_{L^\infty}^2\|D^{1/2}(|u|^2)\|_{L^2}^2 \\
&\lesssim \|u\|_{L^2}^4\|D^{1/2}\partial(|u|^2)\|_{L^2}^2.
\end{aligned}$$

Combining these estimates and Corollary 2, we verify that

$$\begin{aligned}
\partial_t E[u(t)] \le &-\frac{|\beta|}{4}\big\|D^{1/2}\partial_x(|u|^2)\big\|_{L^2}^2 - 3|\beta|\big\|D^{1/2}\,\mathrm{Im}(\overline{u}u_x)\big\|_{L^2}^2 \\
&+C(|\alpha|+|\beta|)|\beta|\|D^{1/2}(|u|^2)\|_{L^2}^2\|u_x\|_{L^2}^2 \\
&+C(\alpha^2+|\alpha|+|\beta|)\big(\|u\|_{L^2}^2+\|u\|_{L^2}^4\big) \\
&\quad\times|\beta|\Big(\big\|D^{1/2}\partial_x(|u|^2)\big\|_{L^2}^2+\big\|D^{1/2}\,\mathrm{Im}(\overline{u}u_x)\big\|_{L^2}^2\Big).
\end{aligned}$$

Hence, there exist $C_1, C_2 > 0$ depending only on α, β (bounded as $\beta \to 0$) such that if $\|u\|_{L^2} \le C_1^{-1}$, then

$$\partial_t E[u] \;\le\; -\frac{|\beta|}{8}\big\|D^{1/2}\partial_x(|u|^2)\big\|_{L^2}^2 + C_2|\beta|\|D^{1/2}(|u|^2)\|_{L^2}^2\|u_x\|_{L^2}^2$$

This inequality and Lemma 3, together with the L^2 equality, imply that if $\|u(0)\|_{L^2} \le \min\{C_0^{-1}, C_1^{-1}\}$,

$$\begin{aligned}
\partial_t\Big(E[u(t)]+\|u(t)\|_{L^2}^2\Big) \le &-\frac{|\beta|}{8}\big\|D^{1/2}\partial_x(|u(t)|^2)\big\|_{L^2}^2 \\
&+2C_2|\beta|\|D^{1/2}(|u(t)|^2)\|_{L^2}^2\Big(E[u(t)]+\|u(t)\|_{L^2}^2\Big)
\end{aligned}$$

By the Gronwall inequality, we obtain the desired H^1 a priori bound:

$$
\begin{aligned}
&\|u(t)\|_{H^1}^2 + \frac{|\beta|}{4}\int_0^t \left\|D^{1/2}\partial_x(|u(\tau)|^2)\right\|_{L^2}^2 d\tau \\
&\le 2\Big(E[u(t)] + \|u(t)\|_{L^2}^2 + \frac{|\beta|}{8}\int_0^t \left\|D^{1/2}\partial_x(|u(\tau)|^2)\right\|_{L^2}^2 d\tau\Big) \\
&\le 2\Big(E[u(0)] + \|u(0)\|_{L^2}^2\Big)\exp\Big[2C_2|\beta|\int_0^t \|D^{1/2}(|u(\tau)|^2)\|_{L^2}^2 d\tau\Big] \\
&\le 4\|u(0)\|_{H^1}^2 \exp\Big[2C_2\|u(0)\|_{L^2}^2\Big].
\end{aligned}
$$

For the lower bound of $\|u(t)\|_{L^2}^2$, we first note that $u(0) = 0$ implies $u(t) \equiv 0$ by the L^2 equality. We thus assume $u(0) \neq 0$, and consider the differential inequality for $\|u(t)\|_{L^2}^{-2}$. By Corollary 2 and the estimate (2), we see

$$
\begin{aligned}
\partial_t\|u(t)\|_{L^2}^{-2} &= |\beta|\|u(t)\|_{L^2}^{-4}\left\|D^{1/2}(|u(t)|^2)\right\|_{L^2}^2 \\
&\le C|\beta|\|u(t)\|_{L^2}^{-2}\left\|D^{1/2}\partial_x(|u(t)|^2)\right\|_{L^2},
\end{aligned}
$$

as long as $\|u(t)\|_{L^2} > 0$. Applying the H^1 a priori estimate shown above, we have

$$
\begin{aligned}
\|u(t)\|_{L^2}^{-2} &\le \|u(0)\|_{L^2}^{-2}\exp\Big[C|\beta|\int_0^t \left\|D^{1/2}\partial_x(|u(\tau)|^2)\right\|_{L^2} d\tau\Big] \\
&\le \|u(0)\|_{L^2}^{-2}\exp\Big[C|\beta|^{1/2}t^{1/2}\Big(|\beta|\int_0^t \left\|D^{1/2}\partial_x(|u(\tau)|^2)\right\|_{L^2}^2 d\tau\Big)^{1/2}\Big] \\
&\le \|u(0)\|_{L^2}^{-2}\exp\Big[4C\|u(0)\|_{H^1}e^{C_2\|u(0)\|_{L^2}^2}|\beta|^{1/2}t^{1/2}\Big].
\end{aligned}
$$

This completes the proof of Theorem 1. □

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