MOLES WITH HATS AND THE ERDÖS-HAJNAL CONJECTURE

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WHAT IS THE EFFECT OF EXCLUDING AN INDUCED SUBGRAPH?

STRUCTURE? NO

χ-BOUNDEDNESS? NO

CONJECTURE (ERDŐS - HAJNAL) 1985

$\forall H.\ \exists \varepsilon(H)$ S.T. EVERY H-FREE n-VERTEX GRAPH HAS A CLIQUE OR A STABLE SET OF SIZE $n^{\varepsilon(H)}$.

IN A RANDOM GRAPH EXPECTED CLIQUE/STABLE SET SIZE IS $\log n$

G HAS THE EH-PROP (ε)
IF $\max(\alpha(G), \omega(G)) \geq |V(G)|^{\varepsilon}$.

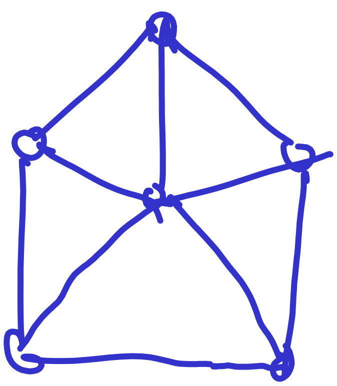
H IS FRIENDLY IF $\exists \varepsilon$
S.T. EVERY $G \in \mathrm{Forb}(H)$
HAS THE EH-PROP (ε)

ERDÖS-HAJNAL CONJ:
EVERY H IS FRIENDLY.

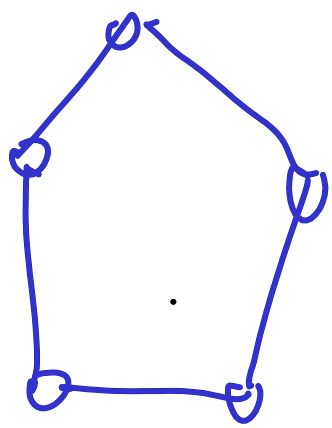
H IS AN INDUCED SUBGRAPH OF G IF

$V(H) \subseteq V(G)$

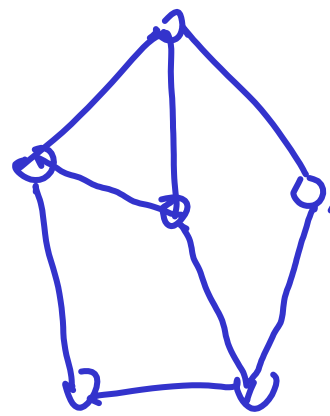
$uv \in E(H)$ iff $u, v \in V(H)$ & $uv \in E(G)$



G



ISG OF G



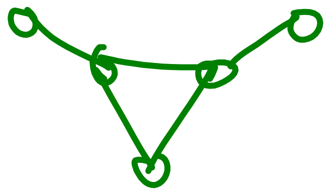
NOT ISG OF G

G IS H-FREE IF NO ISG OF G IS ISOMORPHIC TO H

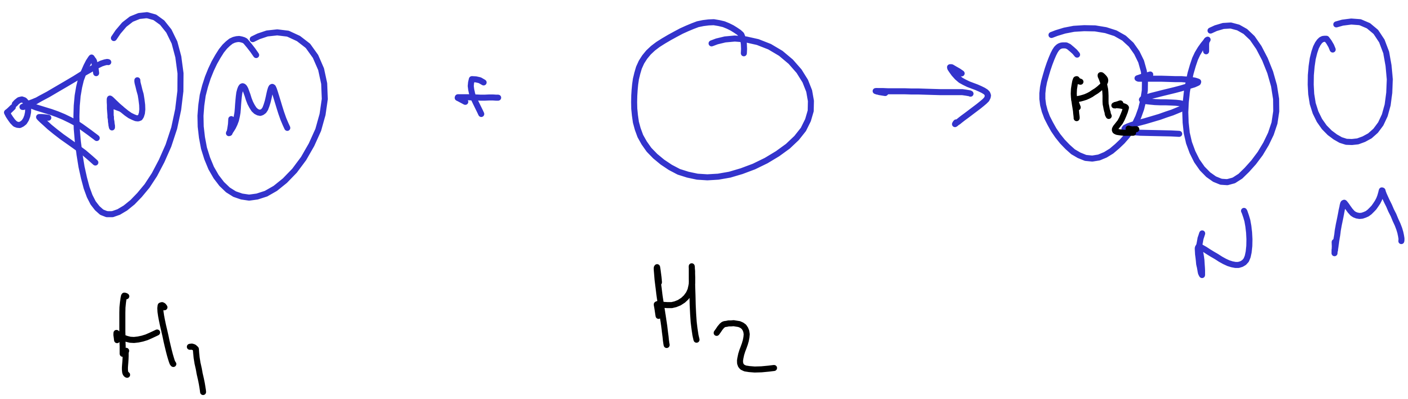
$G \in \mathrm{FORB}(H)$

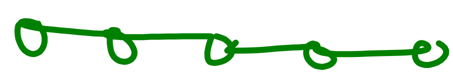
KNOWN FRIENDLY GRAPHS:

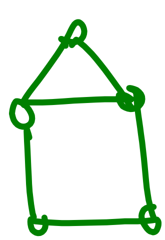
H WITH $|V(H)| \leq 4$

$H =$  (C., SAFRA) 2005

SUBSTITUTION (ALON, PACH, SOLYMOSI) 2001

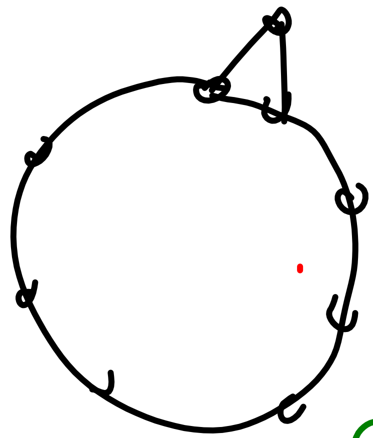


OPEN: $H = C_5$, $H =$ 

$H =$ 

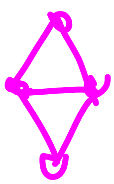
A HOLE WITH A HAT

(A CAP)



C_k

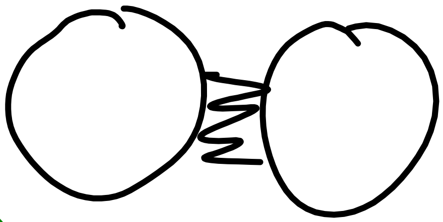
$k \geq 4$

( IS NOT A HOLE WITH A HAT)

MAIN THEOREM:

$\exists\, \varepsilon$ S.T. EVERY HOLE-WITH-A-HAT-FREE GRAPH HAS THE ERDŐS-HAJNAL PROPERTY (ε).

G HAS THE STRONG EH PROPERTY (ε) IF G HAS



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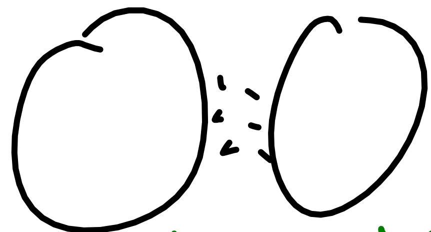


$\varepsilon|V(H)|$ $\varepsilon|V(H)|$ $\varepsilon|V(H)|$ $\varepsilon|V(H)|$

THM LET c, ε BE SUCH THAT $c^{\varepsilon} \leq \frac{1}{2}$.

IF

- ALL INDUCED SUBGRAPHS OF G HAVE THE EH-PROP (ε), AND
- G HAS THE STRONG EH-PROP (c)

THEN

- G HAS THE EH-PROP (ε)

STRONG EH-PROP "$\Rightarrow$" EH-PROP

THEOREM (C., SCOTT, SEYMOUR, SPIRKL)

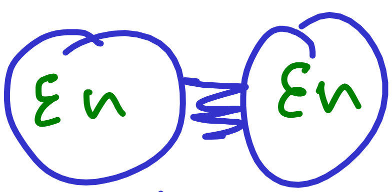
$\forall$ PAIR OF FORESTS (T_1, T_2)

$\exists\ \varepsilon(T_1, T_2)$ S.T. IF

G IS T_1-FREE, AND

G^c IS T_2-FREE

THEN



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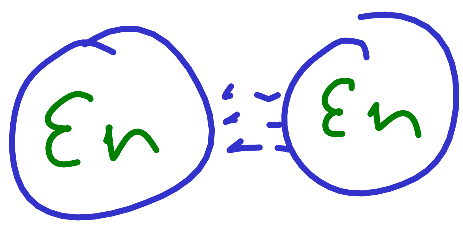


- CONJECTURED BY LIEBENAU-PILIPCZUK
- BEST POSSIBLE

α - NARROWNESS

DEF $f : V(G) \to [0,1]$ IS GOOD

IF $\sum_{v \in V(P)} f(v) \le 1$

FOR EVERY PERFECT INDUCED SUBGRAPH P OF G.

DEF G IS α- NARROW IF

$\sum_{v \in V(G)} f^{\alpha}(v) \le 1$ FOR EVERY GOOD FUNC f

THM (C., SAFRA)

G α-NARROW $\Rightarrow$

G HAS EH- PROP $\left(\frac{1}{2\alpha}\right)$

THM (FOX)

EVERY I.S.G. OF G HAS EH- PROP (ε)

$\Rightarrow$ G IS $3/\varepsilon$ -NARROW

THM (C., SAFRA)
IF G IS OBTAINED BY SUBSTITUTION FROM α-NARROW GRAPHS, THEN G IS α-NARROW

OUR RESULT:

$\exists\, \alpha$ S.T. EVERY GRAPH WITH NO HOLE WITH A HAT IS α-NARROW

PROOF STRATEGY

EITHER STRONG EH-PROP
(THE α-NARROW VERSION)

OR A LARGE PORTION OF THE GRAPH IS SIMPLER
(OBTAINED BY SUBSTITUTION FROM α'-NARROW GRAPHS)

PROOF: FIX α SMALL ENOUGH

G IS NOT α-NARROW; MINIMAL

$\exists\ f\ \ V(G) \to [0,1]$ S.T.

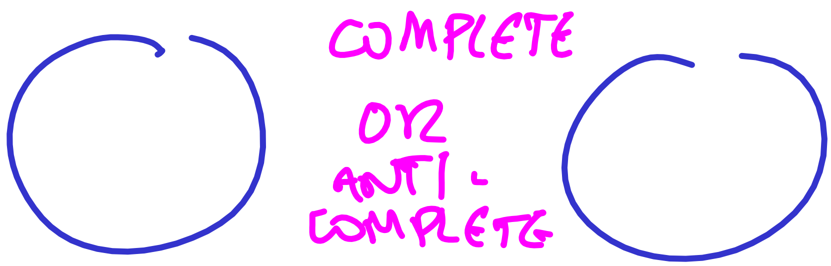
$$\sum_{v \in P} f(v) \le 1 \quad \forall \text{ PERFECT } P.$$

BUT $$\sum_{v \in V(G)} f^{\alpha}(v) > 1$$

DEFINE: $\boxed{w(v) = f^{\alpha}(v)}$

OBSERVATION

G DOES NOT HAVE WEIGHTED STRONG EH-PROP (δ).



$\sum f^{\alpha}(v) > c' \qquad \sum f^{\alpha}(v) > c'$

RÖDL'S THEOREM

$\forall H \; \forall \varepsilon \; \exists c$ S.T.

IF G IS AN H-FREE GRAPH THEN $\exists$ ISG G' OF G WITH

- $|V(G')| \geq c\,|V(G)|$
- $\max\deg(G') \leq \varepsilon\,|V(G')|$

  OR

  $\min\deg(G') \geq (1-\varepsilon)\,|V(G')|$

WEIGHTED VERSION OF RÖDL'S THEOREM

$\forall H \; \forall \varepsilon \; \exists c$ S.T.

IF G IS AN H-FREE GRAPH AND $w : V(G) \to [0,1]$ WITH

$$\sum_{v \in V(G)} w(v) > 1$$

THEN $\exists$ ISG G' OF G AND $w' : V(G') \to [0,1]$ S.T.

- $w'(v) \leq w(v) \quad \forall v \in V(G')$
- $$\sum_{v \in V(G')} (v) \geq c$$
- $$\forall v \in V(G') \quad \sum_{u \in N_{G'}(v)} w'(u) \leq \varepsilon$$

OR

$$\forall v \in V(G') \quad \sum_{u \in N_{G'}(v)} w'(u) \geq 1 - \varepsilon$$

THM (BOUSQUET, LAGOUTTE, THOMASSÉ)

$\forall \varepsilon \ \exists c$ S.T.

IF G IS P_5^c - FREE

AND $w: V(G) \to [0,1]$ WITH

$$\sum_{v \in V(G)} w(v) > 1.$$

THEN $\exists$ ISG G' OF G AND

$w': V(G') \to [0,1]$ S.T.

- $w'(v) \le w(v) \quad \forall v \in V(G')$.
- $\sum_{v \in V(G')} (v) \ge c$
- $\forall v \in V(G') \quad \sum_{u \in N_{G'}(v)} w'(u) \le \varepsilon$

OR

- WEIGHTED STRONG EH-PROP

FROM NOW ON:

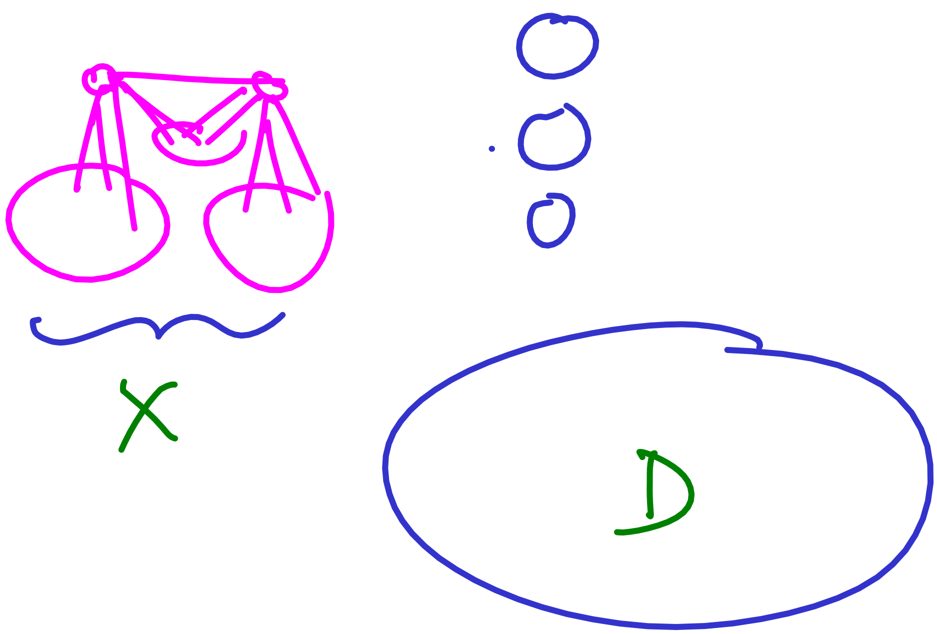
$f : V(G) \to [0,1]$ good

$$\sum_{v \in V(G)} f^{\alpha}(v) > c$$

$$\forall v \quad \sum_{u \in N(v)} f^{\alpha}(u) \leq \varepsilon$$

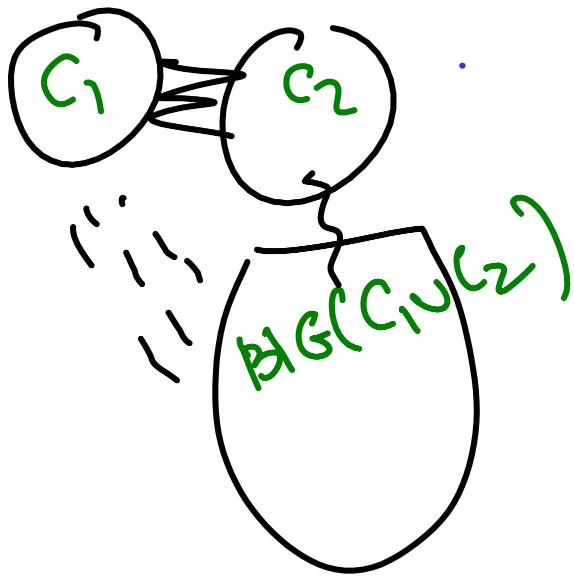
COR $\forall X \subseteq V(G)$ s.t. $\exists v_1, v_2$ with $X \subseteq N(v_1) \cup N(v_2)$

$\exists$ component D of $G \setminus X$ with $\sum_{v \in D} \varepsilon^{\alpha}(D) > 0.99c$.



$D = \mathrm{BIG}(X)$

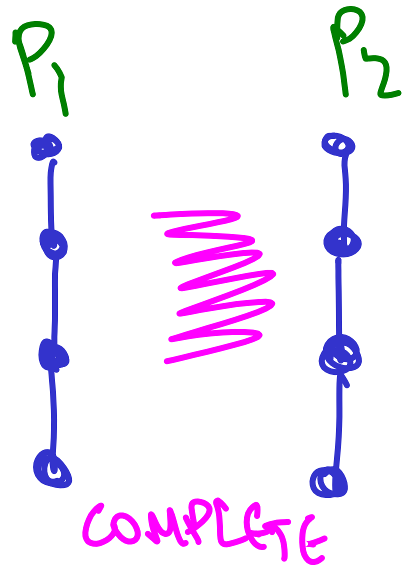
DEF A SPLIT

- 

- $G|C_1$ AND $G^c|C_1$ ARE CONNECTED

SPLIT $\Rightarrow$ SPLIT SEPARATION
CONTROLLED

A FORCER



THEOREM EVERY FORCER IS BROKEN BY A SPLIT

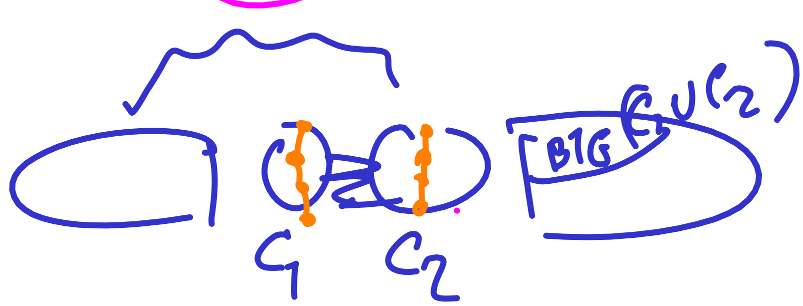
FORCER P_1 P_2

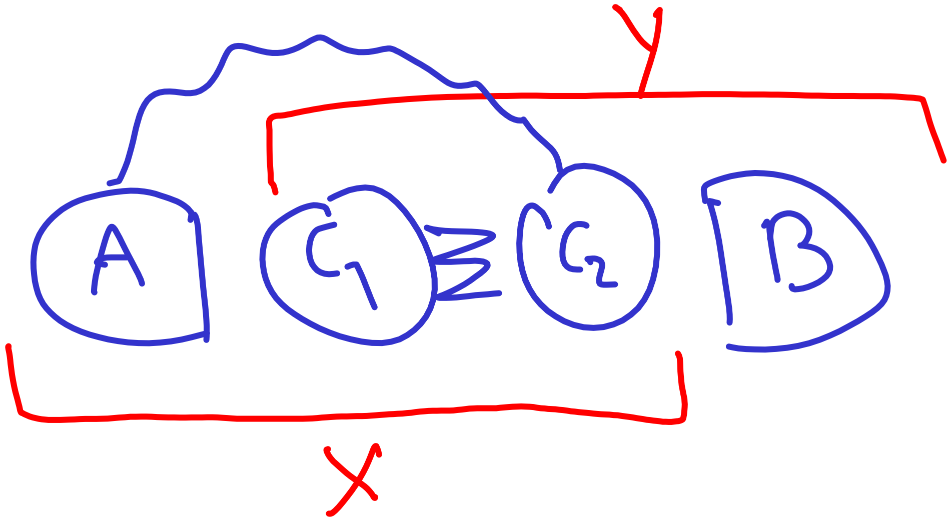
$\Downarrow$

SPLIT

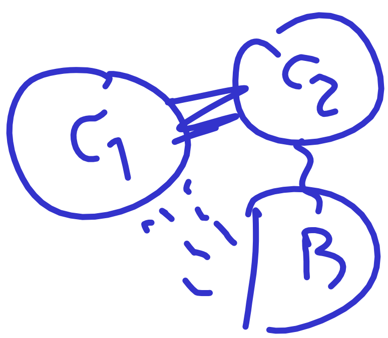
$\Downarrow$

SPLIT SEPARATION





FACT: C_1 IS A HOMOGENEOUS SET IN $G|Y$



"THEORY OF SEPARATIONS"

<u>DEF</u> (X,Y) AND (X',Y') ARE NON-CROSSING IF (UP TO SIMMETRY)

$$X \subseteq Y' \quad \& \quad X' \subseteq Y$$

THEOREM (ROBERTSON & SEYMOUR)

A SET $\mathcal{S}$ OF NON-CROSSING SEPARATIONS $\longrightarrow$ TREE DECOMP $(T, \mathcal{X})$ OF G WHERE THE SEPARATIONS GIVEN BY EDGES OF T ARE IN $\mathcal{S}$

IN THE PERFECT WORLD

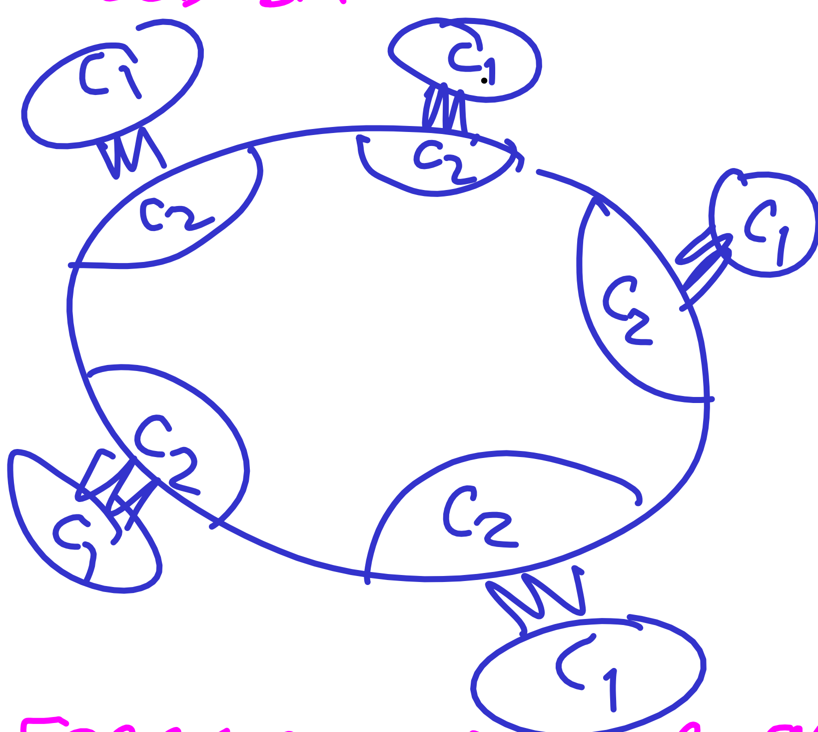
SPLIT SEPARATIONS ARE NON-CROSSING

$\rightsquigarrow$ TREE DECOMPOSITION

WEIGHTED STRONG FM-PROP

LARGE WEIGHT BAG WHERE ALL FORCERS ARE CONTROLLED

A CONTROLLED BAG:

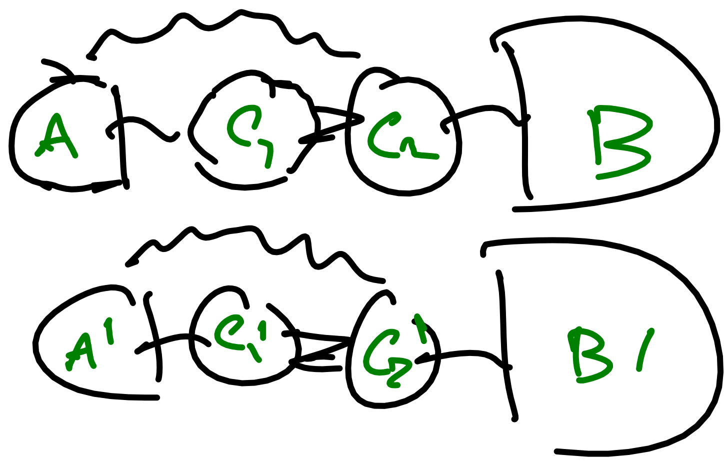


EVERY FORCER HAS ONE SIDE IN SOME "C_1"

IN THE REAL WORLD

THM SPLIT SEPARATIONS ARE NEAR-NON-CROSSING

GIVEN



EITHER

- EVERY COMP OF $A \cup A'$ IS A COMP OF A OR A COMP OF A'.

OR

- $BIG(C_1, C_2) = BIG(C_1', C_2')$

COR LET $\mathcal{S}$ BE THE SET OF ALL SPLIT-SEPARATIONS; LET

$$\mathcal{A} = \bigcup_{(A, C_1, C_2, B) \in \mathcal{S}} A .$$

THEN EVERY COMP OF $\mathcal{A}$ IS ANTICOMPLETE TO SOME BIG WEIGHT SUBSET OF $V(G)$.

COR $w(\mathcal{A}) \leq 0.01\, w(G)$.

PROOF: OTHERWISE WEIGHTED STRONG EH-PROP.

WRITE

$$\mathcal{X} = \bigcap (B \cup C_1 \cup C_2)$$

$$A - C_1 \equiv C_2 - B \in \mathcal{G}.$$

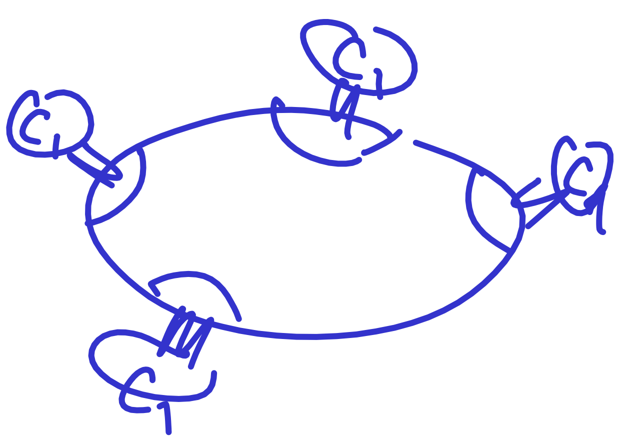
COR $w(\mathcal{X}) > 0.99\, w(G) > 0.99c$

NO WEIGHTED STRONG CH-PROD

$\Rightarrow \exists$ COMP $\mathcal{Y}$ OF $\mathcal{X}$

S.T. $w(\mathcal{Y}) > 0.98c$

FACT EVERY FORCER IN $G|\mathcal{Y}$ HAS ONE SIDE IN A HOMOGENEOUS SET OF $G|\mathcal{Y}$



LET H_0 BE OBTAINED FROM $G|\mathcal{Y}$ BY CONTRACTING ALL MAX'L HOMOGENEOUS SETS $X_1, \ldots, X_t$

WE HAVE :

- $G|\mathcal{Y}$ IS OBTAINED BY SUBSTITUTION FROM H_0, $G|X_1, \ldots, G|X_p$
- $f : \mathcal{Y} \to [0,1]$ GOOD
- $\sum_{v \in \mathcal{Y}} f^{\alpha}(v) > 0.98c$
- $\forall i \quad \sum_{v \in X_i} f^{\alpha}(v) \leq \varepsilon$
- $G|X_1, \ldots, G|X_p$ ARE α-NARROW
- $\boxed{H_0 \text{ IS } 30\text{-NARROW} \quad (\text{ALON, PACH, SOLYMOSY})}$ $\cancel{\ }$

THANK YOU!