G group, k field. A representation of G is a homomorphism

$$g: G \longrightarrow GL(V) \qquad \text{where } V/k \text{ is a vector space.}$$

... simple if only invariant subspaces are 0 and V.

It is often the case in mathematics that you need a mystery and a tool. Here the mystery is G and the tool is linear algebra.

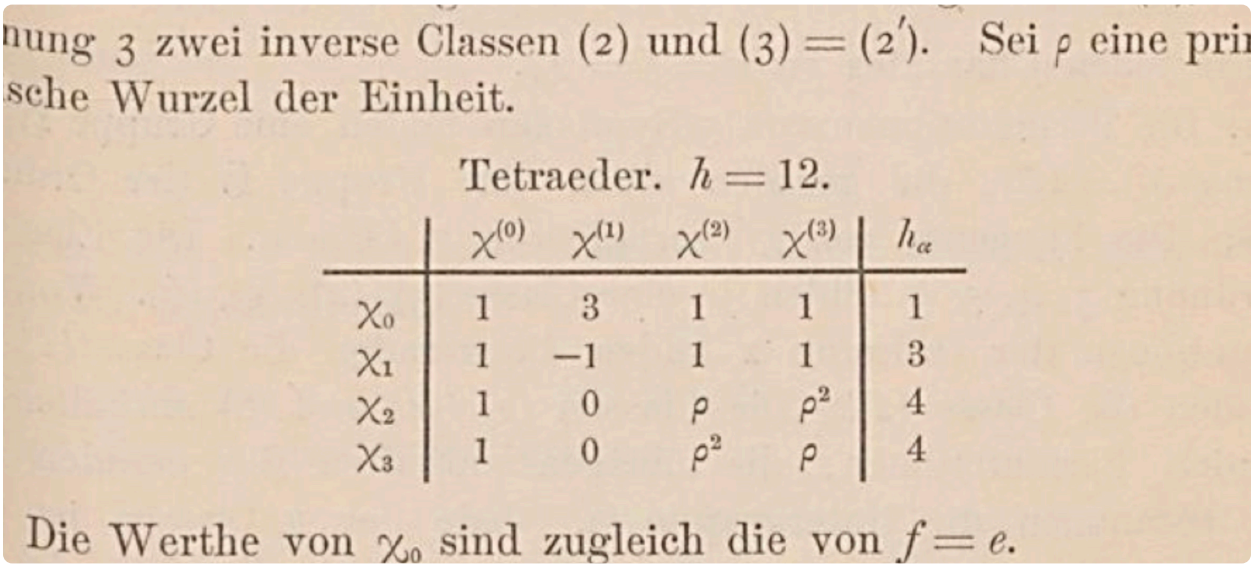
It is powerful because groups are ubiquitous and linear algebra is powerful. Often one linearizes to produce nice objects.

Example: $S^1 = \{z \in \mathbb{C}^\times \mid |z|=1\}$, simple continuous representations over $\mathbb{C}$ are $z \mapsto z^m$ for $m \in \mathbb{Z}$.

$$L^2(S^1, \mathbb{C}) = \widehat{\bigoplus_{m \in \mathbb{Z}}} \mathbb{C} z^m \qquad \text{(Fourier series).}$$

The importance of the group concept was realised by Ruffini and Galois in the early 19th century. Examples of abelian groups play an important role for Gauss, Dedekind and Dirichlet. It took mathematicians nearly 70 years to take representations seriously, and this was first done by Frobenius in 1896.

In the 20th century it was realised that representation theory lies at the heart of aspects of theoretical physics, and the Langlands program reveals that the concept of a representation is a unifying theme in number theory.



nung 3 zwei inverse Classen (2) und $(3) = (2')$. Sei ρ eine pri
sche Wurzel der Einheit.

Tetraeder. $h = 12$.

	$\chi^{(0)}$	$\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	h_α
χ_0	1	3	1	1	1
χ_1	1	−1	1	1	3
χ_2	1	0	ρ	ρ^2	4
χ_3	1	0	ρ^2	ρ	4

Die Werthe von χ_0 sind zugleich die von $f = e$.

Frobenius, *Über Gruppencharaktere*, S'ber. Akad. Wiss. Berlin, 1896.

Hermann Weyl: "The wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups."

REPRESENTATION THEORY
ITS RISE AND ITS ROLE IN NUMBER THEORY

ROBERT P. LANGLANDS

Introduction

By representation theory we understand the representation of a group by linear transformations of a vector space. Initially, the group is finite, as in the researches of Dedekind and Frobenius, two of the founders of the subject, or a compact Lie group, as in the theory of invariants and the researches of Hurwitz and Schur, and the vector space finite-dimensional,

Modern point of view

Representation theory: study of vector spaces w/ symmetries (endomorphisms)

Higher representation theory: study of (higher) categories w/ symmetries (endomorphisms).

Behind the scenes of this talk:

categories of representations ⟵ higher rep. theory ⟶ geometry (cohomology / constructible sheaves)

(This theory has been very powerful in representation theory and geometric Langlands. (and to a lesser extent algebraic geometry) We suspect it should have applications in other areas ...)

G finite group, k field

char k ∤ |G| (e.g. k = ℂ)

every representation ≅ ⊕ simple reps (semi-simplicity)

{simples / ≅} = # {conjugacy classes}

character table (non-abelian Fourier transform)

(reduces study to square matrix, character table of monster simple group was known before Monster was.)

(one can think of the theory over ℂ as being well known if it is easy, whereas the modular theory is rich and deep.)

semi-simplicity fails, "modular representation theory"

simples = # p-regular conjugacy classes.

many open problems.

Example: (McKay conjecture 1972)

simples of G w/ p ∤ dim G = # simples of $N_G(P)$ w/ p ∤ dim. ← Sylow p-subgroup

(Theorem for p=2 by Malle-Späth via classification, many other questions concerning reduction mod p of character table, derived equivalences etc.)

A fruitful analogy

G group

as k varies rep G is uniform except for a finite number of characteristics where interesting behaviour occurs

$X_1, \dots, X_n$ finite family of spaces of "finite type".

As k varies, $H^*(X,k)$ "looks the same" except for a finite number of characteristics.

There is another parallel in number theory where a lot of information is stored at "ramified" primes.

From now on we focus on S_n = symmetric group

Over $\mathbb{C}$: simple S_n representations $\xleftrightarrow{1:1}$ partitions λ of n.

$\uparrow$

$(\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ s.t. $\sum \lambda_i = n)$

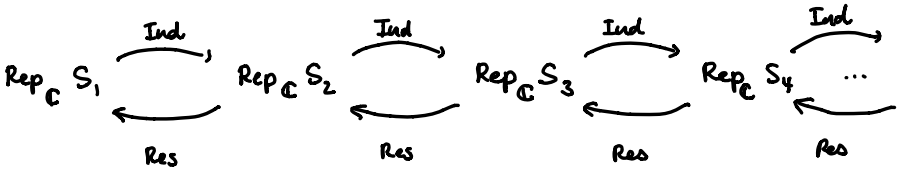
(Conjugacy classes are also labelled by partitions. Here the bijection is canonical.)

trivial $\hookleftarrow$

sign $\hookleftarrow$

(This was worked out by Frobenius in 1900, four years after his discovery of the character table.)

Okounkov-Vershik picture: $S_1 \subset S_2 \subset S_3 \subset \dots$

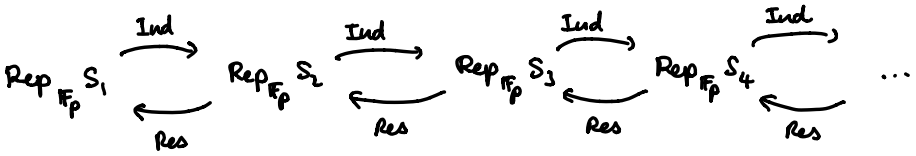


E.g.: Res ( ) =

(Passage to Grothendieck group yields a Fock space representation of the Heisenberg Lie algebra. With some work this picture can be use to explain all of the classical theory: standard tableaux, characters, …)

$\rightsquigarrow$ categorical Heisenberg Lie algebra action.

Modular setting:



$E = \mathrm{Res} = \bigoplus_{a \in \mathbb{F}_p} E_a$ $\qquad$ $F = \mathrm{Ind} = \bigoplus_{a \in \mathbb{F}_p} F_a$ $\qquad$ decompositions of functors.

$\rightsquigarrow$ categorical representation of $\widehat{\mathfrak{sl}}_p$ (affine Lie algebra).

James conjecture (1990) + Lascoux-Leclerc-Thibon conj. : Classes of simples agree with Lusztig's dual canonical basis if $p > \sqrt{n}$.

(Lusztig's canonical basis is defined via the quantum group of $\widehat{\mathfrak{sl}}_p$. It may be computed effectively.) ok for $n = 1, 2, \ldots, 22$.

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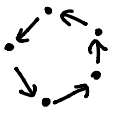
Connection to geometry: $Q =$  (p nodes)

$\mathcal{N}_{\mathbb{C}}(Q)$ = space of nilpotent representations of Q. (actually a stack)

Higher representation theory: $\mathrm{Sh}_{\mathbb{F}_p}(\mathcal{N}_{\mathbb{C}}(Q)) \twoheadrightarrow \bigoplus \mathrm{Rep}_{\mathbb{F}_p} S_n$.

(singularities of $\mathcal{N}ilp$ $\longleftrightarrow$ modular representations of S_n)

Thm: (W. 2013) James conjecture fails for large n, i.e. $n \geq 10^6$.

What are the simple representations of S_n in characteristic p?

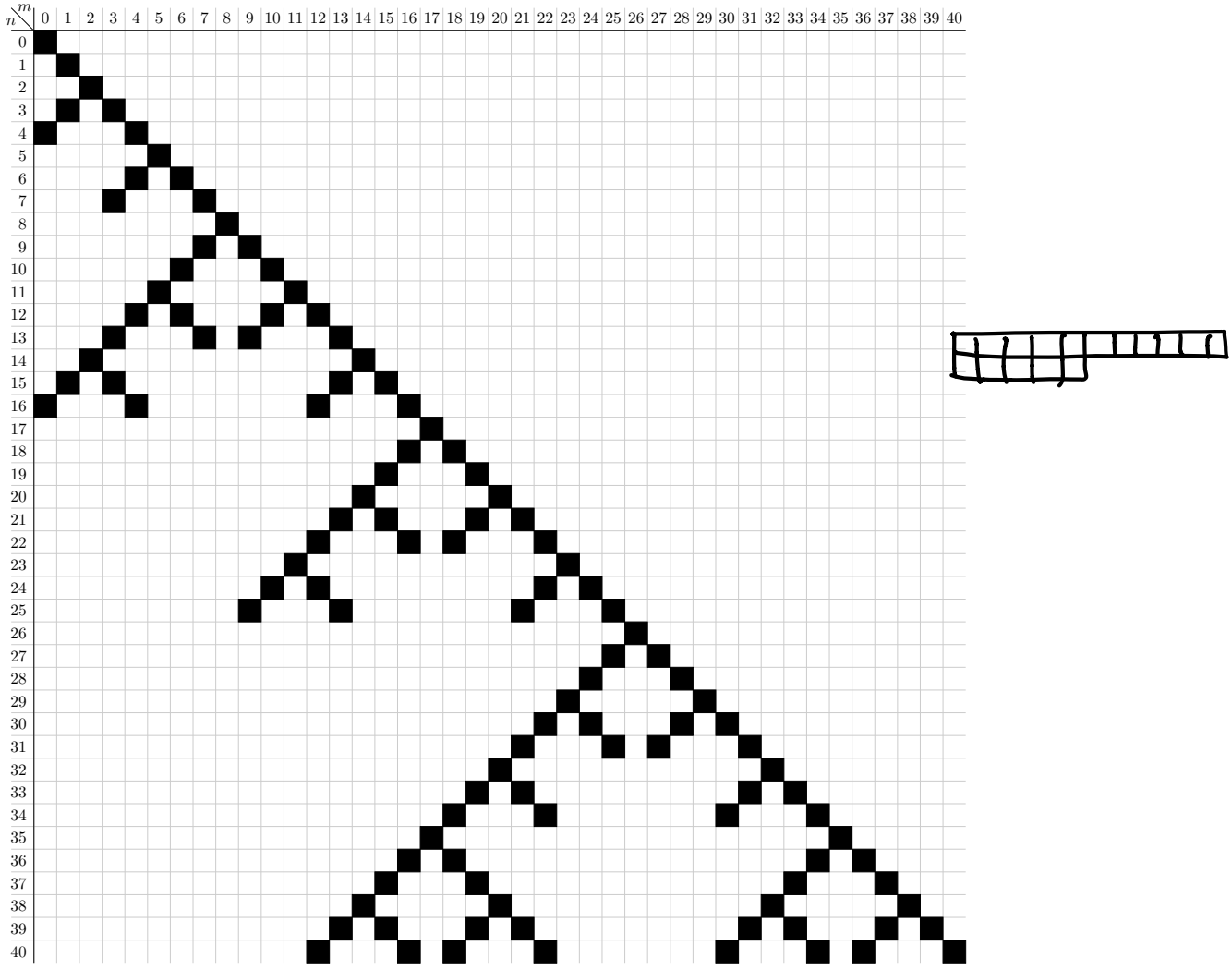
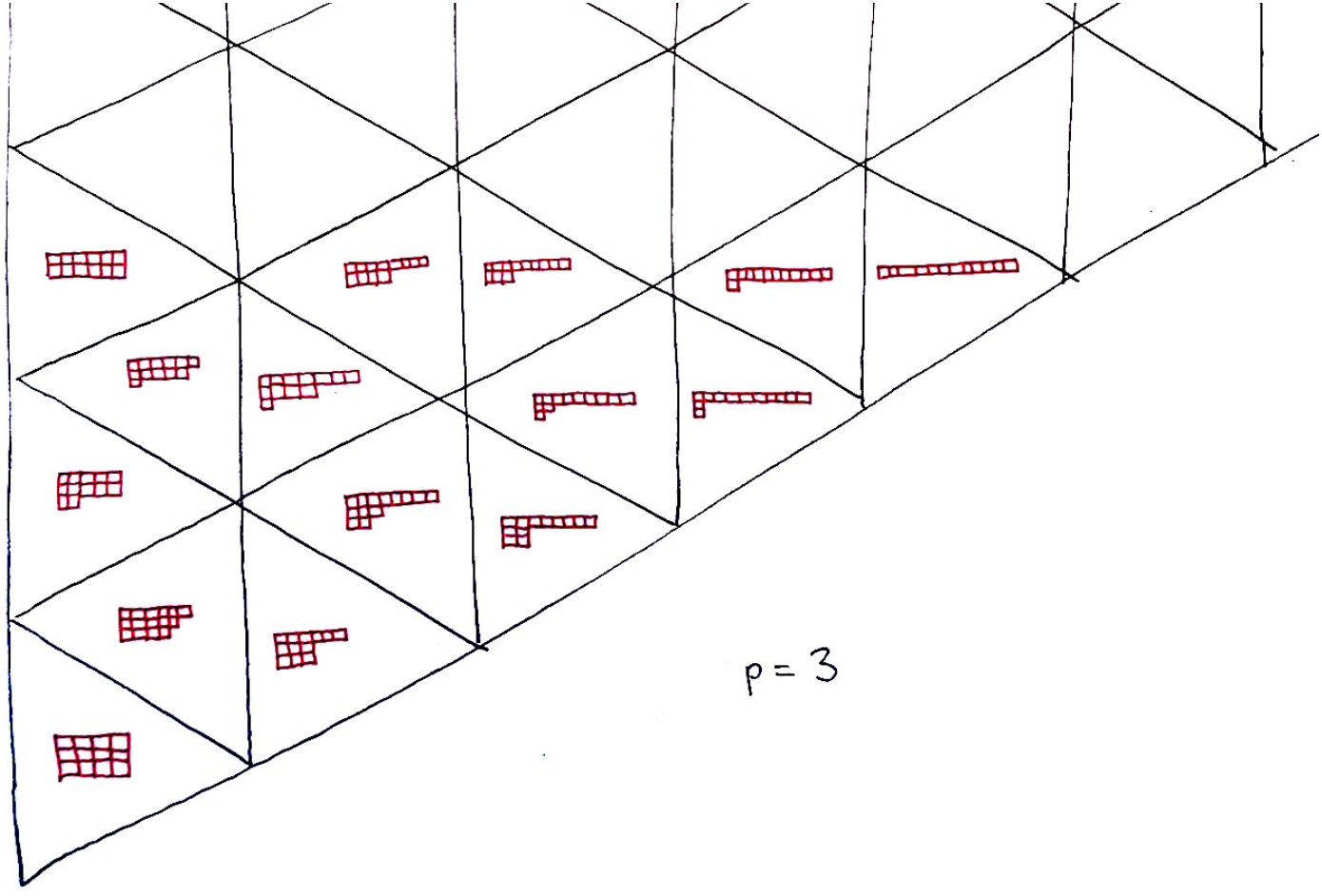
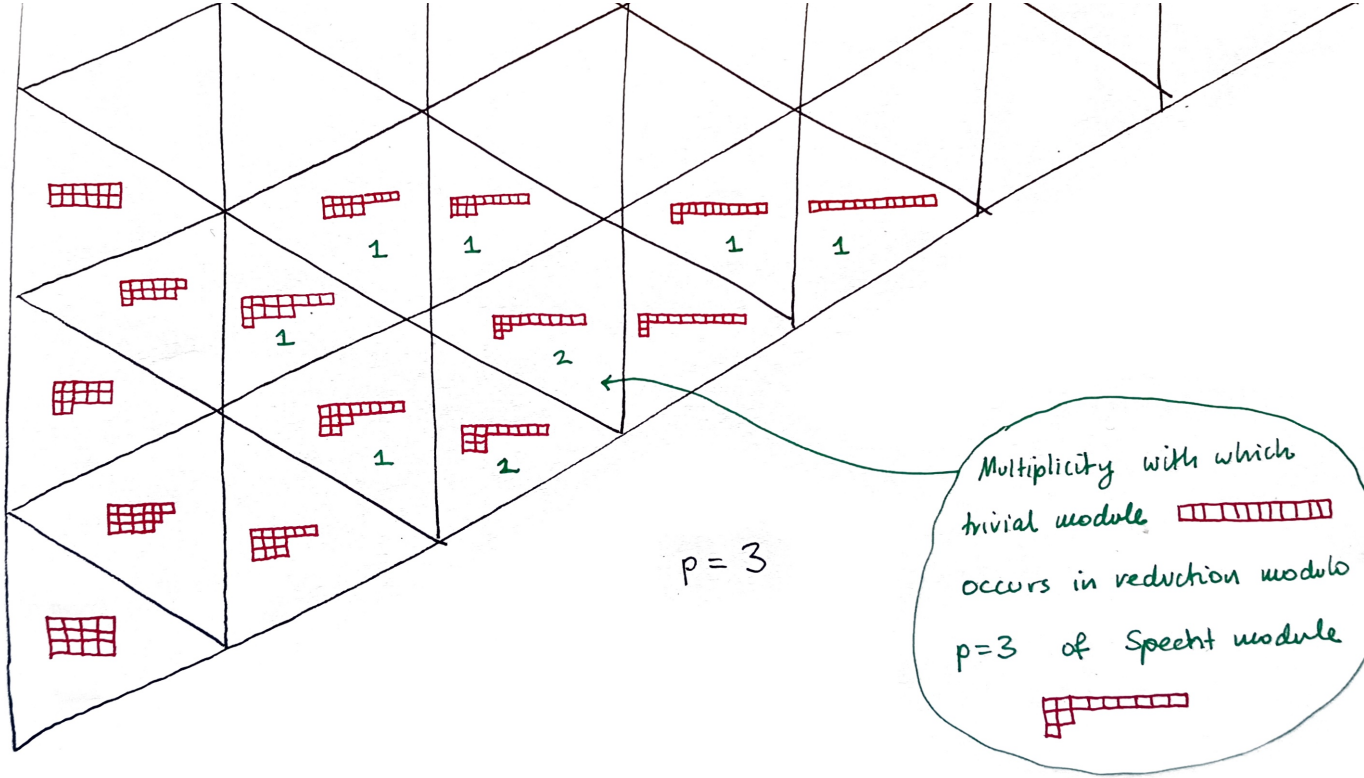


FIGURE 1. The multiplicities of $\Delta(m)$ in $T(n)$ for $p = 3$.


p = 3


1
1
1
1
1
1
2
1
1
p = 3
Multiplicity with which
trivial module
occurs in reduction modulo
p=3 of Specht module