Mock Modular Forms and Applications

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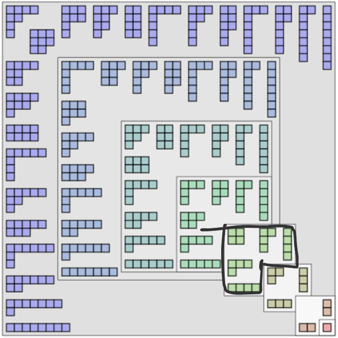
inf. q-series



eg. $n \in \mathbb{N}$

$$p(n) = \sum_{\substack{\text{part.}\\ \text{of } n}} 1$$

$p(4) = 5$

$p(5) = 7$

$\vdots$

Ramanujan's congruence

$p(4+5n) \equiv 0 \quad (5)$

$p(5+7n) \equiv 0 \quad (7)$

$\vdots$

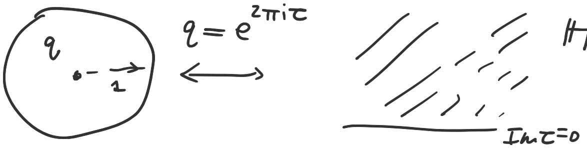
Ono et al

generating function.

$$q^{-\frac{1}{24}} \sum_{n=0}^{\infty} p(n)\, q^n = q^{-\frac{1}{24}}\left(1 + 1q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots\right)$$

$$\text{Euler} \quad = q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right) =: \frac{1}{\eta(\tau)}$$

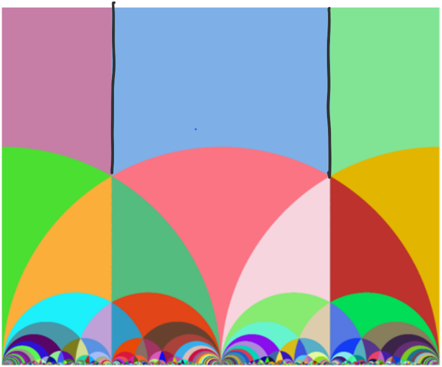
1)



2) $\eta : \mathbb{H} \to \mathbb{C}$

$$\eta(\tau+1) = \underbrace{e\left(\tfrac{1}{24}\right)}_{= e^{2\pi i/24}} \eta(\tau)$$

$$\eta(-1/\tau) = \sqrt{-i\tau}\; \eta(\tau)$$



$T: \tau \mapsto \tau + 1$

$S: \tau \mapsto -1/\tau$

generate

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \supset SL_2(\mathbb{Z})$

$$\tau \mapsto \gamma\tau := \frac{a\tau + b}{c\tau + d}$$

$k \in \frac{1}{2}\mathbb{Z}$, $\gamma \in SL_2(\mathbb{Z})$, $f : \mathbb{H} \to \mathbb{C}$

$$|_k \gamma : f(\tau) \longmapsto f|_k\gamma(\tau) := (c\tau+d)^{-k} f(\gamma\tau) \cdot \rho(\gamma)$$

Def (MF of wt k for $\Gamma < SL_2(\mathbb{Z})$)

$$f \begin{cases} \text{is holom,} \\ f = f|_k\gamma \quad \forall \gamma \in \Gamma \end{cases}$$

1) allow for exp. growth ~~$\{\infty\}$~~ $\mathbb{Q} \cup \{i\infty\}$

2) $f : \mathbb{H} \to \mathbb{C}^n$

3) $\rho : \Gamma \to GL(n, \mathbb{C})$

eg.1. $\eta^{24}(\tau) = q\prod(1-q^n)^{24} = (c\tau+d)^{-12} f(\gamma\tau)$

$= \sum \tau(n)\, q^n$, " $\tau(n)\tau(m) = \tau(nm)$ if $\gcd(n,m)=1$ "

eg 2. vertex operator alg. characters

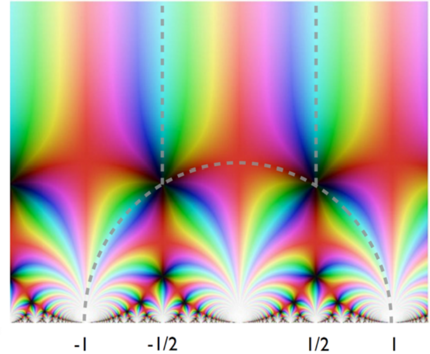
$$\frac{1}{\eta(\tau)} = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - 1/24}$$

$\mathcal{H}$ → Heisenberg alg.

eg 3. topology

$$q^{-1}\sum_{n=0}^{\infty} \chi\left(\mathrm{Hilb}^{[n]}(K3)\right) q^n = \frac{1}{\eta^{24}(\tau)}$$

eg 4. unique $J: \mathbb{H} \to \mathbb{C}$ holom.



$$J(\tau) = J(\tau+1) = J(-1/\tau)$$
$$= J\left(\frac{a\tau+b}{c\tau+d}\right)$$
$$= \underline{q^{-1} + 0} + \underline{\underline{O(q)}}$$
$$= \underline{1}\,q^{-1} + 1\underline{9688}4\,q + \underline{21493760}\,q^2 + \cdots$$
$$196884 = \underline{1 + \underline{\underline{196883}}}\left(+ 21296876\right)$$

dim. of the 3 smallest irreps of $\mathbb{M}$

26 sporadic simple fin. grps, largest $|\mathbb{M}| \sim 10^{54}$

Q: What does it naturally act on?

<u>Monstrous Moonshine</u>

There's a VOA $V^{\natural} = \bigoplus_{n=-1}^{\infty} V_n^{\natural}$ st

Modular Forms

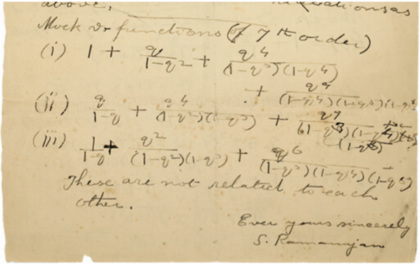
- combinatorics
- physics
- topology
- finite group rep

$\vdots$

Mock Modular Forms

What?

Where?



Mock θ-functions (of 7th order)

(i) $1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \frac{q^9}{(1-q^4)(1-q^5)(1-q^6)}$

(ii) $\frac{q}{1-q} + \frac{q^4}{(1-q^2)(1-q^3)} + \frac{q^9}{(1-q^3)(1-q^4)(1-q^5)}$

(iii) $\frac{1}{1-q} + \frac{q^2}{(1-q^2)(1-q^3)} + \frac{q^6}{(1-q^3)(1-q^4)(1-q^5)}$

These are not related to each other.

Ever yours sincerely
S. Ramanujan

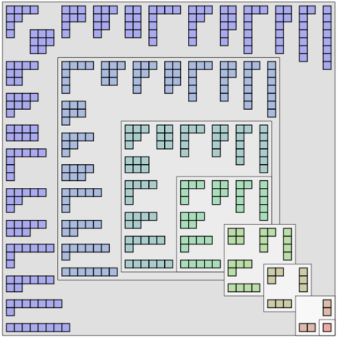
Letter to Hardy, 1920.

eg 1. q-hypergeom. series

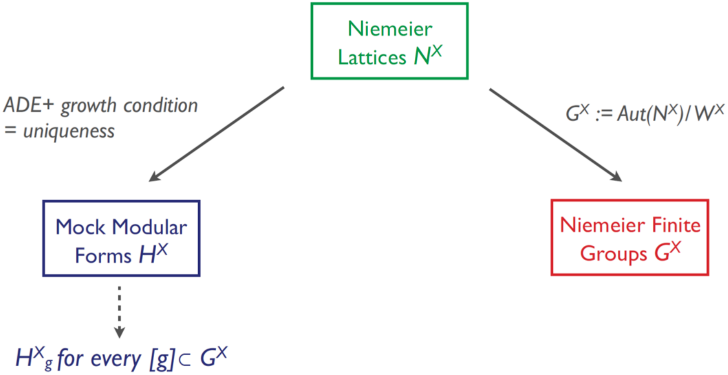
$$F_0(q) = 1 + \frac{q}{1-q^2} + \frac{q^4}{(1-q^3)(1-q^4)} + \cdots$$

$$= \sum \frac{q^{n^2}}{(1-q^{n+1})\cdots(1-q^{2n})} = 1 + q + q^3 + q^4 + q^5 + 2q^7 + \cdots$$

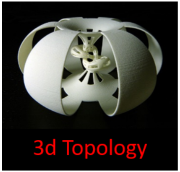
$$f(q) = \sum \frac{q^{n^2}}{((1+q)\cdots(1+q^n)}$$



Umbral Moonshine (Construction)



MMF and 3-Manifolds



[MC-Chun-Ferrari-Gukov-Harrison '18, MC-Ferrari-Sgroi '19, MC-Sgroi '20 (?)]

Questions and Motivations:

$\star$ What are the "good" topological invariants of 3-manifolds?

$\star$ We have the Witten–Reshetikhin–Turaev (WRT, $SU(2)$ Chern–Simons) invariants giving

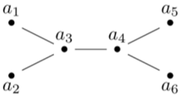
$$\mathbb{Z} \to \mathbb{C}, \quad k \mapsto CS(M_3; k).$$

Is there a "quantum" q-series invariants generalising the above in some way?

Such a quantum invariant, denoted $\widehat{Z}(M_3)$, has been proposed for specific families of 3-manifolds M_3, starting with [Gukov-Pei-Putrov-Vafa '17.]

M_3: Plumbed 3-manifold, determined by its **plumbing graph** Γ.

weighted graph $\Gamma := (V, E, a)$, $a : V \to \mathbb{Z}$.



glue (disk bdls over S^2 w. $\chi = a_i$)

$\parallel$

$M_4(\Gamma)$

plumbing graph Γ $\longleftrightarrow$ $M_4(\Gamma)$

$\updownarrow$

adjacency matrix M

$M_4(\Gamma) \big\downarrow$ take bdry

plumbed $M_{3,\Gamma}$

obtained with surgery along L

$$M = \begin{pmatrix} a_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 1 & 1 & a_3 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_4 & 1 & 1 \\ 0 & 0 & 0 & 1 & a_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & a_6 \end{pmatrix}$$

$H_1(M_{3,\Gamma}; \mathbb{Z}) \cong \mathbb{Z}^{|V|}/M\mathbb{Z}^{|V|}$ (Coker M)

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$\vdots$