KP integrability of triple Hodge integrals

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Quantum curves, integrability, and cluster algebras

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The **Kadomtsev–Petviashvili (KP) hierarchy** was introduced by Sato about 40 years ago. It can be represented in terms of **tau-function** $\tau(\mathbf{t}) \in \mathbb{C}[\![t_1, t_2, t_3, \dots]\!]$ by the **Hirota bilinear identity**

$$\oint_{\infty} e^{\sum_{k>0}(t_k - t'_k)z^k} \tau(\mathbf{t} - [z^{-1}])\tau(\mathbf{t}' + [z^{-1}])dz = 0,$$

which encodes all nonlinear equations of the KP hierarchy. Here we use the standard short-hand notation

$$\mathbf{t} \pm [z^{-1}] := \left\{t_1 \pm z^{-1}, t_2 \pm \frac{1}{2}z^{-2}, t_3 \pm \frac{1}{3}z^{-3}, \dots\right\}.$$

Infinite hierarchy of PDEs.

The first nontrivial term in the expansion of the l.h.s. of the Hirota bilinear identity gives the **KP equation**

$$\tau\tau_{1111} - 4\tau_1\tau_{111} + 3\left(\tau_{11}\right)^2 + 3\tau\tau_{22} - 3\left(\tau_2\right)^2 - 4\tau\tau_{13} + 4\tau_1\tau_3 = 0,$$

where $\tau_{i_1 i_2 \dots} = \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \dots \tau$. The second derivative of this equation with respect to t_1 gives the KP equation in its standard form

$$3u_{22} = \left(4u_3 - 12uu_1 - u_{111}\right)_1,$$

where $u = \frac{\partial^2}{\partial {t_1}^2} \log(\tau)$.

A KP tau-function, independent on even times $t_{2k}, k = 1, 2, \dots,$

$$\frac{\partial}{\partial t_{2k}} \tau = 0,$$

is a tau-function of the **KdV hierarchy**.

Sato Grassmannian

Let us consider the description of the space of solutions for the KP hierarchy, introduced by **[Sato '81]**. Let us consider the space $H = H_+ \oplus H_-$ with

$$H_- = z^{-1}\mathbb{C}[\![z^{-1}]\!], \quad H_+ = \mathbb{C}[z]$$

are generated by negative and nonnegative powers of z respectively. Then the Sato Grassmannian Gr consists of all closed linear spaces $\mathcal{W} \in H$, which are compatible with H_+. Namely, an orthogonal projection $\pi_+ : \mathcal{W} \to H_+$ should be a Fredholm operator, i.e. both the kernel $\ker \pi_+ \in \mathcal{W}$ and the cokernel $\mathrm{coker}\, \pi_+ \in H_+$ should be finite-dimensional vector spaces. The Grassmannian Gr consists of components $\mathrm{Gr}^{(k)}$, parametrized by an index of the operator π_+. We consider only the big cell $\mathrm{Gr}_+^{(0)}$ of the component $\mathrm{Gr}^{(0)}$, defined by the constraint $\ker \pi_+ = \mathrm{coker}\, \pi_+ = 0$. We call $\mathrm{Gr}_+^{(0)}$ the **Sato Grassmannian** for simplicity. There exists a bijection between the points of the Sato Grassmannian $\mathcal{W} \in \mathrm{Gr}_+^{(0)}$ and the tau-functions with $\tau(\mathbf{0}) = 1$.

A point of the Sato Grassmannian $\mathcal{W} \in \mathrm{Gr}_+^{(0)}$ can be described by an **admissible basis**

$$\mathcal{W} = \mathrm{span}\{\Phi_1^{\mathcal{W}}, \Phi_2^{\mathcal{W}}, \Phi_3^{\mathcal{W}}, \dots\}.$$

Convenient choice of the basis

$$\Phi_k(z) = z^{k-1}(1 + O(z^{-1})).$$

Kac–Schwarz operators

How to describe a given point of the Sato Grassmannian (=corresponding tau-function)?

Consider the ring of differential operators with coefficients formal Laurent series in the variable z^{-1}

$$\mathcal{D} := \mathbb{C}((z^{-1}))[\![\frac{\partial}{\partial z}]\!]$$

and its subrings $\mathcal{D}_{\pm} := H_{\pm}[\![\frac{\partial}{\partial z}]\!]$. Sometimes $\mathcal{D}$ is also called the $w_{1+\infty}$ algebra. A natural direct sum decomposition holds

$$\mathcal{D} = \mathcal{D}_{+} \oplus \mathcal{D}_{-}.$$

Definition

For any point of the Sato Grassmannian $\mathcal{W}$ the **Kac-Schwarz algebra**

$$\mathcal{A}_{\mathcal{W}} := \left\{ \mathtt{A} \in \mathcal{D} \,\middle|\, \mathtt{A} \cdot \mathcal{W} \subset \mathcal{W} \right\}$$

is the algebra of the differential operators which stabilize this point.

LINEAR CONSTRAINTS AND MIWA PARAMETRIZATION

To any Kac–Schwarz operator $\mathtt{A} \in \mathcal{A}_{\mathcal{W}}$ one can associate an operator $\widehat{A}$ acting of the space of functions of $\mathbf{t}$ variables such that the tau-function is its eigenfunction

$$\widehat{A} \cdot \tau_{\mathcal{W}} = c\tau_{\mathcal{W}}.$$

Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ be a diagonal matrix. For any function f, dependent on the infinite set of variables $\mathbf{t} = (t_1, t_2, t_3, \ldots)$, let

$$f\left([\Lambda^{-1}]\right) := f(\mathbf{t})\Big|_{t_k = \frac{1}{k}\operatorname{Tr}\Lambda^{-k}}$$

be the **Miwa parametrization**. For any admissible basis the tau-function of the KP hierarchy in the Miwa parametrization is equal to the ratio of the determinants

$$\tau_{\mathcal{W}}([\Lambda^{-1}]) = \frac{\det_{i,j=1}^{N} \Phi_i^{\mathcal{W}}(\lambda_j)}{\Delta(\lambda)},$$

where $\Delta(\lambda) := \prod_{i<j}(\lambda_j - \lambda_i)$ is the **Vandermonde determinant**. Moreover, if for some function $\tau_{\mathcal{W}}$ this equation holds for all $N \in \mathbb{Z}_{\geq 0}$, then $\tau_{\mathcal{W}}$ is a tau-function of the KP hierarchy.

Relation to matrix models.

Consider the space of the pairs of operators ($\mathcal{D}_- = z^{-1}\mathbb{C}[\![z^{-1}]\!][\![\frac{\partial}{\partial z}]\!]$)

$$\mathrm{Gr}_{\mathcal{D}} := \left\{ (\mathtt{P},\mathtt{Q}) \in \mathcal{D}^2 \,\middle|\, [\mathtt{P},\mathtt{Q}] = 1, \mathtt{P} - \frac{\partial}{\partial z} \in z^{-1}\mathcal{D}_-, \mathtt{Q} - z \in \mathcal{D}_- \right\}.$$

Theorem (A.A. '21)

There is a bijection between $\mathrm{Gr}_+^{(0)}$ *and* $\mathrm{Gr}_{\mathcal{D}}$

$$\mathrm{Gr}_+^{(0)} \ni \mathcal{W} \mapsto \rho(\mathcal{W}) = (\mathtt{P}_{\mathcal{W}}, \mathtt{Q}_{\mathcal{W}}) \in \mathrm{Gr}_{\mathcal{D}}.$$

Operator $\mathtt{P}_{\mathcal{W}}$, by construction, always annihilates the wave function

$$\mathtt{P}_{\mathcal{W}} \cdot \Phi_1 = 0,$$

and it can be considered as the **quantum spectral curve** operator. Operator $\mathtt{Q}_{\mathcal{W}}$ is the raising operator, which generates a basis for a point of the Sato Grassmannian

$$\mathcal{W} = \mathrm{span}\{\Psi, \mathtt{Q}_{\mathcal{W}} \cdot \Psi, \mathtt{Q}_{\mathcal{W}}^2 \cdot \Psi, \ldots\}.$$

Kac–Schwarz algebra is generated by $\mathtt{P}_{\mathcal{W}}$ and $\mathtt{Q}_{\mathcal{W}}$.

It is well known that a certain extension of the algebra $\mathfrak{gl}(\infty)$ and corresponding group $\mathrm{GL}(\infty)$ act on the space of KP tau-functions (Sato's Grassmannian). Let us consider the important Heisenberg–Virasoro subalgebra of $\mathfrak{gl}(\infty)$. The **Heisenberg–Virasoro subalgebra** of (the central extension of) $\mathfrak{gl}(\infty)$ is generated by the operators

$$\widehat{J}_k = \begin{cases} \dfrac{\partial}{\partial t_k} & \text{for} \quad k > 0, \\ 0 & \text{for} \quad k = 0, \\ |k| t_{|k|} & \text{for} \quad k < 0, \end{cases}$$

unit, and the Virasoro operators

$$\widehat{L}_m = \frac{1}{2} \sum_{a+b=-m} ab t_a t_b + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}.$$

These operators satisfy the commutation relations

$$\left[\widehat{J}_k, \widehat{J}_m\right] = k\delta_{k+m,0},$$

$$\left[\widehat{L}_k, \widehat{J}_m\right] = -m\widehat{J}_{k+m},$$

$$\left[\widehat{L}_k, \widehat{L}_m\right] = (k-m)\widehat{L}_{k+m} + \frac{1}{12}\delta_{k+m,0}(k^3 - k).$$

Operators $\widehat{J}_m$ and $\widehat{L}_m$ correspond to the operators

$$\mathtt{j}_m = z^m, \quad \mathtt{l}_m = -z^m \left(z\frac{\partial}{\partial z} + \frac{m+1}{2} \right)$$

acting on the Sato Grassmannian. The Heisenberg–Virasoro group $\mathcal{V}$ is generated by the operators $\widehat{J}_k$, $\widehat{L}_k$ and a unit,

$$\mathcal{V} := \{ C\, e^{\sum a_k \widehat{J}_k + b_k \widehat{L}_k} \,\big|\, a_k, b_k, C \in \mathbb{C} \}.$$

LEMMA

For a KP tau-function $\tau(\mathbf{t})$,

$$C\, e^{\sum a_k \widehat{J}_k + b_k \widehat{L}_k} \cdot \tau(\mathbf{t})$$

is also a tau-function of KP (if well-defined).

LEMMA

If $\sum a_k \mathtt{j}_k + b_k \mathtt{l}_k \in \mathcal{A}_{\mathcal{W}}$, *then* $\left(\sum a_k \widehat{J}_k + b_k \widehat{L}_k \right) \tau_{\mathcal{W}} = c \tau_{\mathcal{W}}$.

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Intersection numbers

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford compactification of the moduli space of stable complex curves of genus g with n distinct marked points. The moduli space $\overline{\mathcal{M}}_{g,n}$ is defined to be empty unless the **stability condition**

$$2g - 2 + n > 0$$

is satisfied. We consider **Hodge integrals**

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_k}\psi_1^{m_1}\psi_2^{m_2}\cdots\psi_n^{m_n} \in \mathbb{Q},$$

where $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ is the first Chern class of the line bundle corresponding to the cotangent space of the curve at the i-th marked point, and $\lambda_j \in H^{2j}(\overline{\mathcal{M}}_{g,n})$ is the j-th Chern class of the Hodge bundle $\mathbb{E}$. These integrals are trivial, unless the corresponding complex dimensions coincide

$$\sum_{l=1}^{k} j_l + \sum_{i=1}^{n} m_i = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n},$$

where $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$.

Kontsevich–Witten tau-function

In the early 90's Witten initiated new directions in the study of $\overline{\mathcal{M}}_{g,n}$. Consider the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\psi_2^{a_2}\cdots\psi_n^{a_n}.$$

Let T_j, $j \geq 0$, be formal variables and let

$$\tau_1 := \exp\left(\sum_{g=0}^{\infty}\sum_{n=0}^{\infty} \hbar^{2g-2+n} F^1_{g,n}\right),$$

where

$$F^1_{g,n} := \sum_{a_1,\dots,a_n\geq 0} \frac{\prod T_{a_i}}{n!}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\psi_2^{a_2}\cdots\psi_n^{a_n} \in \mathbb{Q}[T_1, T_3, T_5, \dots].$$

By Witten's conjecture the generating function τ_1 becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n+1)!!t_{2n+1}$, known as **times**.

Theorem (Kontsevich, Witten)

The generating function τ_1 is a tau-function of the KdV hierarchy.

Topological expansion

Instead of the **genus expansion** of the generating function

$$\exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\hbar^{2g-2}\mathcal{F}_{g,n}\right),$$

where g is the genus, n is the number of the marked points, we consider the **topological expansion**

$$\exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\hbar^{2g-2+n}\mathcal{F}_{g,n}\right).$$

The latter is more convenient for the KP hierarchy description. Two descriptions are related by a change of the normalization of the variables t_k by an additional factor of $\hbar$.

$$\log(\tau_1) = \hbar\left(\frac{1}{8}t_3 + \frac{1}{6}\,t_1^3\right) + \hbar^2\left(\frac{3}{16}t_3^2 + \frac{1}{2}\,t_3 t_1^3 + \frac{5}{8}t_1 t_5\right)$$
$$+\hbar^3\left(\frac{15}{4}t_1 t_3 t_5 + \frac{105}{128}t_9 + \frac{35}{16}t_7 t_1^2 + \frac{3}{2}\,t_1^3 t_3^2 + \frac{5}{8}\,t_1^4 t_5 + \frac{3}{8}t_3^3\right) + \dots$$

Norbury's classes and BGW tau-function

The most natural alternative to the Kontsevich–Witten tau-function is the **Brézin–Gross–Witten (BGW) tau-function**. This tau-function governs the intersection theory with the insertions of the fascinating **Norbury's Θ-classes**,

$$\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n}).$$

Consider the generating function of the intersection numbers of Θ-classes and ψ-classes

$$F^0_{g,n} = \sum_{a_1,\dots,a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n} \in \mathbb{Q}[T_1, T_3, T_5, \dots],$$

$$\tau_0 = \exp\left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F^0_{g,n} \right).$$

We have a direct analog of the Kontsevich–Witten tau-function:

Theorem (Norbury)

Generating function τ_0 becomes a tau-function of the KdV hierarchy after the change of variables $T_n = (2n+1)!!t_{2n+1}$.

These classes are related to the supersymmetric Riemann surfaces **[Norbury '20]**.

Kontsevich-type matrix integral

Generalized Kontsevich model

$$Z_V(\Lambda) = \mathcal{C}^{-1} \int [d\Phi] \exp\left(-\frac{1}{\hbar}\mathrm{Tr}\left(V(\Phi) - \Phi V'(\Lambda)\right)\right).$$

For $V = z^3/3!$ the asymptotic expansion of the $N \times N$ **Kontsevich matrix model** describes the Kontsevich–Witten tau-function in the **Miwa parametrization**

$$\begin{aligned}\tau_1\big|_{t_k=\frac{1}{k}\mathrm{Tr}\,\Lambda^{-k}} &= Z_{z^3/3!}(\Lambda)\\ &= \mathcal{C}^{-1}\int [d\Phi] \exp\left(-\frac{1}{\hbar}\mathrm{Tr}\left(\frac{\Phi^3}{3!} + \frac{\Lambda\Phi^2}{2}\right)\right).\end{aligned}$$

τ_0 is nothing but a tau-function of the BGW model **[Norbury '17]**. KdV integrability of the BGW model follows from the relation to the generalized Kontsevich model **[Mironov, Morozov, and Semenoff '96]**:

$$\tau_0\big|_{t_k=\frac{1}{k}\mathrm{Tr}\,\Lambda^{-k}} = \tilde{\mathcal{C}}^{-1}\int [d\Phi] \exp\left(\frac{1}{2\hbar}\mathrm{Tr}\,\left(\Lambda^2\Phi + \Phi^{-1} - 2\hbar N \log\Phi\right)\right).$$

Kac–Schwarz description of the Kontsevich–Witten tau-function

Admissible basis is given by the asymptotic expansion

$$\Phi_k^{KW}(z) = \sqrt{\frac{z}{2\pi\hbar}} \int_\gamma d\varphi \, (\varphi + z)^{k-1} e^{-\frac{1}{\hbar}\left(\frac{\varphi^3}{3!} + \frac{z\varphi^2}{2}\right)}.$$

Let us introduce

$$\mathtt{W} := \frac{1}{z}\frac{\partial}{\partial z} - \frac{1}{2z^2}.$$

Using integration by parts one can get the canonical pair of the KS operators

$$\mathtt{Q}_{KW} = z + \hbar\,\mathtt{W}, \qquad \mathtt{P}_{KW} = \frac{\partial}{\partial z} + \hbar\frac{\mathtt{W}^2}{2}.$$

The modified quantum spectral curve is given by the Airy equation

$$\left(\frac{\hat{y}^2}{2} - \hat{x}\right) \cdot \tilde{\Psi}_{KW} = 0,$$

where $\hat{x} = z^2/2$, $\hat{y} = \hbar\frac{\partial}{\partial x}$ and $\tilde{\Psi}_{KW} = \frac{e^{z^3/(3\hbar)}}{\sqrt{z}}\Phi_1^{KW}(z)$. The semi-classical limit yields the **Airy curve** $\frac{y^2}{2} = x$.

HEISENBERG–VIRASORO CONSTRAINTS

The Kontsevich–Witten ($\alpha = 1$) and Brézin–Gross–Witten ($\alpha = 0$) tau-functions τ_α satisfy the **Heisenberg–Virasoro constraints** **[Dijkgraaf, Verlinde, Verlinde '91; Gross, Newman '92]**

$$\frac{\partial}{\partial t_{2k}} \cdot \tau_\alpha = 0, \quad k > 0,$$
$$\widehat{L}_k^\alpha \cdot \tau_\alpha = 0, \quad k \geq -\alpha,$$

where the Virasoro operators are given by

$$\widehat{L}_k^\alpha = \frac{1}{2}\widehat{L}_{2k} - \frac{1}{2\hbar}\frac{\partial}{\partial t_{2k+1+2\alpha}} + \frac{\delta_{k,0}}{16}.$$

These operators satisfy the commutation relations

$$\left[\widehat{L}_k^\alpha, \widehat{L}_k^\alpha\right] = (k-m)\widehat{L}_{k+m}^\alpha, \qquad k, m \geq -\alpha.$$

The Virasoro constraints can be obtained from the Kac–Schwarz operators **[Kac, Schwarz '91]**. The Virasoro constraints completely specify the tau-functions τ_α, there are several ways to describe the solution.

Consider the **cut-and-join operators**

$$\widehat{W}_0 = \sum_{k,m\in\mathbb{Z}^+_{\mathrm{odd}}} \left(kmt_k t_m \frac{\partial}{\partial t_{k+m-1}} + \frac{1}{2}(k+m+1)t_{k+m+1}\frac{\partial^2}{\partial t_k \partial t_m}\right) + \frac{t_1}{8},$$

$$\widehat{W}_1 = \frac{1}{3}\sum_{k,m\in\mathbb{Z}^+_{\mathrm{odd}}} \left(kmt_k t_m \frac{\partial}{\partial t_{k+m-3}} + \frac{1}{2}(k+m+3)t_{k+m+3}\frac{\partial^2}{\partial t_k \partial t_m}\right) + \frac{t_1^3}{3!} + \frac{t_3}{8}.$$

THEOREM (A.A. '11;'18)

$$\tau_\alpha = \exp\left(\hbar \widehat{W}_\alpha\right)\cdot 1.$$

Topological expansion with $\tau_\alpha^{(0)} = 1$

$$\tau_\alpha = \sum_{k\geq 0} \hbar^k \tau_\alpha^{(k)}.$$

Algebraic topological recursion

$$\tau_\alpha^{(k)} = \frac{1}{k}\widehat{W}_\alpha \cdot \tau_\alpha^{(k-1)}.$$

Goal

Goal: generalize the Kontsevich–Witten/Brézin–Gross–Witten theory to other interesting enumerative geometry problems

- **Tau-function of the solitonic integrable hierarchy**
- **Virasoro constraints**
- **Cut-and-join description**
- **Matrix model**

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Cohomological field theories capture the formal universal properties of the Gromov-Witten theory **[Kontsevich, Manin '94]**. Givental's group acts on semi-simple cohomological field theories with unit **[Givental '01,Teleman '12]**.

We consider the Givental group for the simplest, **rank one case**. It is parametrized by the **R-matrix**, $R(z) \in 1 + z\mathbb{C}[\![z]\!]$, which satisfies the symplectic condition

$$R(z)R(-z) = 1$$

or, equivalently, $\log R(z)$ is odd function of z.

Let

$$\widehat{z^{2k-1}} = \widehat{W}_k, \quad k \in \mathbb{Z},$$

where for positive k

$$\widehat{W}_k = -\sum_m \tilde{T}_m \frac{\partial}{\partial T_{m+2k-1}} + \frac{1}{2}\sum_{m=0}^{2k-2}(-1)^m \frac{\partial^2}{\partial T_m \partial T_{2k-m-2}}.$$

Here the so-called **dilaton shift** is given by $\tilde{T}_k = T_k - \hbar^{-1}\delta_{k,1}$. These operators commute for $k > 0$. Quantization

$$\widehat{R} := \exp\left(\widehat{\log R(z)}\right).$$

Below we call the group of such operators the **Givental group**.

Observation: operators $\widehat{W}_k$ are similar to the Virasoro operators ($T_n = (2n+1)!!t_{2n+1}$)

$$\widehat{L}_m = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2}\sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}.$$

Factorization

Operators $\widehat{R}$ can be factorized. Let us introduce a formal series in two variables

$$V^R(z,w) := \frac{1 - R(-w)R(-z)}{w+z} \in \mathbb{C}[\![z,w]\!].$$

The matrix of its coefficients $V^R(z,w) = \sum_{k,l=0}^{\infty} V^R_{kl} w^k z^l$ is symmetric, $V^R_{kl} = V^R_{lk}$.
Let the linear change of the shifted variables from $\mathbf{T}$ to $\mathbf{T}^{\mathbf{R}}$ and associated with it change of the dilaton shift are given by

$$\sum_{k=0}^{\infty} \tilde{T}^R_k z^k := R(-z) \sum_{k=0}^{\infty} \tilde{T}_k z^k, \qquad \sum_{k=2}^{\infty} \delta_k z^k := z(1 - R(-z)).$$

Then, for any element of the Givental group and any series $Z(\mathbf{T})$ we have

Lemma (Givental)

$$\widehat{R} \cdot Z(\mathbf{T}) = e^{\frac{1}{2} \sum_{i,j=0}^{\infty} V^R_{ij} \frac{\partial^2}{\partial T_i \partial T_j}} \; e^{\hbar^{-1} \sum_{k=2}^{\infty} \delta_k \frac{\partial}{\partial T_k}} \; Z(\mathbf{T})\Big|_{T_k \mapsto T^R_k}.$$

Follows from the Baker–Campbell–Hausdorff formula.

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Idea: compare matrices v and V^R.

Compare the action of the Givental operator

$$\widehat{R}\cdot Z(\mathbf{T}) = e^{\frac{1}{2}\sum_{i,j=0}^{\infty} V^R_{ij}\frac{\partial^2}{\partial T_i \partial T_j}}\; e^{\hbar^{-1}\sum_{k=2}^{\infty}\delta_k \frac{\partial}{\partial T_k}}\; Z(\mathbf{T})\Big|_{T_k\mapsto T_k^R}$$

with the action of the Virasoro group element

$$\exp\left(\sum_{k=1}^{\infty} a_k \widehat{L}_k\right) = \exp\left(\sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} m t_m \frac{\partial}{\partial t_{k+m}}\right) \exp\left(\frac{1}{2}\sum_{k,m=1}^{\infty} v_{km}\frac{\partial^2}{\partial t_k \partial t_m}\right).$$

The coefficients v_{nm} are the Grunsky coefficients of $h(z) := e^{\sum_{k>0} a_k z^{m+1}\partial_z}\, z$,

$$\sum_{k,m=1}^{\infty} v_{km}\eta_1^k \eta_2^m = \log\left(\frac{h(\eta_1)-h(\eta_2)}{\eta_1-\eta_2}\right).$$

We identify $T_k \equiv (2k+1)!!t_{2k+1}$ and need to find $R(z)$ and $h(z)$ satisfying

$$V^R_{km} = (2k+1)!!(2m+1)!!v_{2k+12m+1},$$

$$\frac{1-R(-w)R(-z)}{w+z} = \frac{1}{2\pi(zw)^{3/2}}\int_{\gamma^2} d\eta_1 d\eta_2\, \eta_1\eta_2 e^{-\frac{\eta_1^2}{2z}-\frac{\eta_2^2}{2w}} \log\left(\frac{h(\eta_1)-h(\eta_2)}{\eta_1-\eta_2}\right).$$

Two-parametric family of Givental operators

Let us introduce two parameters, p and q, and

$$dx = f(z)df(z) = \frac{zdz}{(1+\sqrt{p+q}z)(1+qz/\sqrt{p+q})},$$

$$y_0 = \frac{1}{z},$$

$$y_1 = \int^z y_0(\eta)dx(\eta) = \frac{\sqrt{p+q}}{p}\left(\log\left(1+\sqrt{p+q}z\right) - \log\left(1+\frac{qz}{\sqrt{p+q}}\right)\right),$$

where we assume that $p+q \neq 0$. The functions

$$\phi_k(z) := \left(-\frac{\partial}{\partial x}\right)^k \cdot \frac{1}{z} \;\in\; \mathbb{C}[z^{-1}]$$

define a change of variables if we associate kt_k with z^{-k} and $T_k^{q,p}$ with $\phi_k(z)$. This change of variables can be described by the recursion

$$T_0^{q,p}(\mathbf{t}) = t_1,$$

$$T_k^{q,p}(\mathbf{t}) = \left(q\sum_{k=1}^{\infty} kt_k\frac{\partial}{\partial t_k} + \frac{2q+p}{\sqrt{p+q}}\sum_{k=1}^{\infty} kt_k\frac{\partial}{\partial t_{k-1}} + \sum_{k=1}^{\infty} kt_k\frac{\partial}{\partial t_{k-2}}\right)T_{k-1}^{q,p}(\mathbf{t}).$$

Let us consider a two-parametric family of functions satisfying symplectic condition

$$R_{q,p}(z)=\exp\left(-\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}\left(p^{2k-1}+q^{2k-1}-\left(\frac{pq}{p+q}\right)^{2k-1}\right)z^{2k-1}\right),$$

Here B_{2k} are the Bernoulli numbers. We introduce the coefficients v_k and a_k

$$v_k^{(\alpha)}:=[z^k]\int^z(f(\eta)^{2\alpha-1}-y_\alpha(\eta))dx(\eta),$$

$$e^{-\sum_{k>0}a_k z^{k+1}\frac{\partial}{\partial z}}z=f(z)=\sqrt{2\frac{p+q}{pq}\log\left(1+\frac{qz}{\sqrt{p+q}}\right)-\frac{2}{p}\log(1+\sqrt{p+q}z)}.$$

THEOREM

Rank one Givental operator coincides up to a linear change of variables with the element of the Heisenberg–Virasoro group of symmetries of the KP hierarchy iff $R(z)$ belongs to a family $R_{q,p}$ and then both act on any function of odd times $Z(\mathbf{T})\in\mathbb{C}[\![\mathbf{T}]\!]$

$$\widehat{R}_{q,p}\cdot Z(\mathbf{T})\Big|_{\mathbf{T}=\mathbf{T}^{q,p}(\mathbf{t})}=e^{\hbar^{-1}\sum_{k=4}v_k^{(1)}\frac{\partial}{\partial t_k}}e^{\sum_{k\in\mathbb{Z}_{>0}}a_k\widehat{L}_k}\cdot Z(\mathbf{T}).$$

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Let $\Lambda_g(u) = \sum_{i=0}^{g} u^i \lambda_i$ be the Chern polynomial of the Hodge bundle $\mathbb{E}$. Consider the m-parametric deformation of the Kontsevich–Witten tau-function

$$Z(\mathbf{T};\mathbf{u}) = \exp\left(\sum_{g,n=0}^{\infty} \hbar^{2g-2+n} \sum_{a_1,\dots,a_n} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_g(u_1)\dots\Lambda_g(u_m)\psi_1^{a_1}\cdots\psi_n^{a_n}\right).$$

In general this is not a KdV tau-function!

Insertion of the Hodge classes can be described by

$$Z(\mathbf{T};\mathbf{u}) = \widehat{R}(\mathbf{u}) \cdot \tau_1(\mathbf{T}),$$

where

$$\widehat{R}(\mathbf{u}) := \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k-1)} \sum_{j=1}^{m} u_j^{2k-1} \widehat{W}_k\right)$$

is an element of the Givental group **[Mumford '83; Faber, Pandharipande '00]**.

Triple Hodge integrals with Calabi–Yau condition

Let us consider triple Hodge integrals with an additional **Calabi–Yau condition**

$$\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} = 0.$$

It is convenient to use the parametrization

$$u_1 = -p, u_2 = -q, u_3 = \frac{pq}{p+q}.$$

For $\alpha \in \{0, 1\}$ consider the generating function

$$\mathcal{F}_{g,n}^{\alpha} = \sum_{a_1,\ldots,a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}^{1-\alpha} \Lambda_g(-q)\Lambda_g(-p)\Lambda_g\left(\frac{pq}{p+q}\right) \psi_1^{a_1}\psi_2^{a_2}\cdots\psi_n^{a_n},$$

$$Z_{q,p}^{\alpha}(\mathbf{T}) = \exp\left(\sum_{g=0}^{\infty}\sum_{n=0}^{\infty} \hbar^{2g-2+n}\mathcal{F}_{g,n}^{\alpha}\right).$$

Main theorem

Theorem

The generating functions of triple Hodge integrals, satisfying the Calabi–Yau condition, in the variables $\mathbf{T}^{q,p}$ are tau-functions of the KP hierarchy,

$$\tau^{(\alpha)}_{q,p}(\mathbf{t}) = Z^{\alpha}_{q,p}(\mathbf{T}^{q,p}(\mathbf{t})).$$

They are related to the Kontsevich–Witten ($\alpha = 1$) and Brézin–Gross–Witten ($\alpha = 0$) tau-functions by the elements of the Heisenberg–Virasoro group

$$\tau^{(\alpha)}_{q,p}(\mathbf{t}) = \exp\left(\hbar^{-1}\sum_{j=2+2\alpha}^{\infty} v^{(\alpha)}_j \frac{\partial}{\partial t_j}\right)\exp\left(\sum_{k=1}^{\infty} a_k \widehat{L}_k\right)\cdot \tau_{\alpha}(\mathbf{t}).$$

Linear Hodge integrals appear as a specification at $q = 0$ (or $p = 0$), because $\Lambda_g(0) = 1$. For this case with $\alpha = 1$ the KP integrability was proved by **[Kazarian '09]**.

Alternative proof of the integrability part of this theorem for $\alpha = 1$, based on the Mariño–Vafa formula, is given by **[Kramer '21]**.

We expect, that this is the most general family of the KP-integrable Hodge integrals.

HEISENBERG–VIRASORO CONSTRAINTS

To describe the Heisenberg–Virasoro constraints for the tau-functions $\tau_{q,p}^{(\alpha)}$ let us consider

$$\widehat{J}_k^{(q,p)} = \frac{1}{2}\sum_{m=2k}^{\infty} \rho[2k,m]\widehat{J}_m,$$

$$\widehat{L}_k^{\alpha,(q,p)} = \frac{1}{2}\sum_{m=2k}^{\infty} \sigma[2k,m]\widehat{L}_m - \frac{1}{2\hbar}\sum_{m=2k+1+2\alpha}^{\infty} \chi_\alpha[2k,m]\widehat{J}_m + \frac{\delta_{k,0}}{16} - \frac{\delta_{k,-1}}{48}\frac{p^2+pq+q^2}{p+q}.$$

Here the coefficients $\sigma[k,m]$, $\rho[k,m]$, and $\chi_\alpha[k,m]$ are given by

$$\rho[k,m] = [z^m]f(z)^k, \qquad \sigma[k,m] = [z^{m+1}]\frac{f(z)^{k+1}}{f'(z)}, \qquad \chi_\alpha[k,m] = [z^m]f(z)^{k+2}y_\alpha(z),$$

where

$$f(z) = \sqrt{2\frac{p+q}{pq}\log\left(1+\frac{qz}{\sqrt{p+q}}\right) - \frac{2}{p}\log(1+\sqrt{p+q}z)}.$$

Heisenberg–Virasoro constraints

Theorem

The tau-functions of triple Hodge integrals satisfy the Heisenberg–Virasoro constraints

$$\widehat{J}_k^{(q,p)} \cdot \tau_{q,p}^{(\alpha)} = 0, \qquad k > 0,$$
$$\widehat{L}_k^{\alpha,(q,p)} \cdot \tau_{q,p}^{(\alpha)} = 0, \qquad k \geq -\alpha.$$

These operators satisfy the commutation relations of the Heisenberg–Virasoro algebra

$$\left[\widehat{J}_k^{(q,p)}, \widehat{J}_m^{(q,p)}\right] = 0, \qquad k, m > 0,$$
$$\left[\widehat{L}_k^{\alpha,(q,p)}, \widehat{J}_m^{(q,p)}\right] = k \widehat{J}_{k+m}^{(q,p)}, \qquad k \geq -\alpha,\ m > 0,$$
$$\left[\widehat{L}_k^{\alpha,(q,p)}, \widehat{L}_m^{\alpha,(q,p)}\right] = (k-m) \widehat{L}_{k+m}^{\alpha,(q,p)}, \qquad k, m \geq -\alpha.$$

String equation $\widehat{L}_{-1}^{1,(q,p)} \cdot \tau_{q,p}^{(1)} = 0,$

$$\widehat{L}_{-1}^{1,(q,p)} = \frac{1}{2}\widehat{L}_{-2} + \frac{2q+p}{2\sqrt{p+q}}\widehat{L}_{-1} + \frac{q}{2}\widehat{L}_0 - \frac{1}{2\hbar}\sum_{m=1}^{\infty} \chi_1[-2,m]\widehat{J}_m - \frac{1}{48}\frac{p^2+pq+q^2}{p+q}.$$

Let

$$\tilde{h}(z) = \frac{p}{q\sqrt{p+q}} \frac{\exp\left(2\frac{q}{\sqrt{p+q}}z\right) - 1}{1 - \exp\left(-2\frac{p}{\sqrt{p+q}}z\right)} - \frac{1}{\sqrt{p+q}}.$$

and

$$f_0(z) = \tilde{f}(z), \qquad \frac{f_1(z)^3}{3} = \int_0^z \tilde{f}(\eta)dx(\eta).$$

where $\tilde{f}(z)$ is inverse to $\tilde{h}(z)$, $\tilde{h}(\tilde{f}(z)) = z$ so that $f_\alpha(z) \in z + z\mathbb{C}[\![z]\!]$. With coefficients

$$\rho_\alpha[k,m] = [z^m]f_\alpha(z)^k$$

we construct the operators

$$\widehat{J}_k^\alpha = \sum_{m=k}^{\infty} \rho_\alpha[k,m]\widehat{J}_m, \qquad k \in \mathbb{Z}.$$

$$\widehat{J}_k = \begin{cases} \dfrac{\partial}{\partial t_k} & \text{for} \quad k > 0, \\ |k|t_{|k|} & \text{for} \quad k < 0, \end{cases}$$

$$\widehat{W}_0^{q,p} = \sum_{k,m \in \mathbb{Z}^+_{\mathrm{odd}}}^{\infty} \left(\widehat{J}^0_{-k} \widehat{J}^0_{-m} \widehat{J}^0_{k+m-1} + \frac{1}{2} \widehat{J}^0_{-k-m-1} \widehat{J}^0_k \widehat{J}^0_m \right) + \frac{\widehat{J}^0_{-1}}{8},$$

$$\widehat{W}_1^{q,p} = \frac{1}{3} \sum_{k,m \in \mathbb{Z}^+_{\mathrm{odd}}}^{\infty} \left(\widehat{J}^1_{-k} \widehat{J}^1_{-m} \widehat{J}^1_{k+m-3} + \frac{1}{2} \widehat{J}^1_{-k-m-3} \widehat{J}^1_k \widehat{J}^1_m \right) + \frac{(\widehat{J}^1_{-1})^3}{3!} + \frac{\widehat{J}^1_{-3}}{24}.$$

THEOREM

$$\tau_{q,p}^{(\alpha)} = \exp\left(\hbar \widehat{W}_\alpha^{q,p}\right) \cdot 1$$

Algebraic topological recursion for topological expansion $\tau_{q,p}^{(\alpha)} = \sum_{k=0}^{\infty} \hbar^k \tau_{q,p}^{(\alpha,k)}$,

$$\tau_{q,p}^{(\alpha,k)} = \frac{1}{k} \widehat{W}_\alpha^{q,p} \cdot \tau_{q,p}^{(\alpha,k-1)}.$$

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It is also interesting to consider the specific values of the parameters q and p, for which the tau-function $\tau_{q,p}^{(\alpha)}$ describes k-reductions of the KP hierarchy.

COROLLARY

For $p = -2q$ the tau-functions $\tau_{q,p}^{(\alpha)}$ do not depend on even times,

$$\frac{\partial}{\partial t_{2k}} \tau_{q,-2q}^{(\alpha)}(\mathbf{t}) = 0, \qquad k \in \mathbb{Z}_{>0},$$

that is $\tau_{q,-2q}^{(\alpha)}(\mathbf{t})$ (with $\Lambda_g(-q)\Lambda_g(2q)\Lambda_g(2q)$) are tau-functions of the KdV hierarchy.

It follows from the form of the linear change of variables

$$T_k^{q,p}(\mathbf{t}) = \left(q \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + \frac{2q+p}{\sqrt{p+q}} \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k-2}} \right) T_{k-1}^{q,p}(\mathbf{t}).$$

In **[Dubrovin, Liu, Yang, Zhang, '20]** the relation between the generating function of the triple Hodge integrals for $\alpha = 1$ with $2q + p = 0$ and the **discrete** KdV was established. Our results shows that there is a simpler relation between $Z_{q,-2q}^{(\alpha)}$ and the ordinary KdV hierarchy given by a linear change of variables.

Mumford classes and translations

With the forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ we define the Mumford classes, $\kappa_k := \pi_* \psi_{n+1}^{k+1} \in H^{2k}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Let us consider the generating functions of the intersection numbers of ψ, κ, and Θ classes

$$F_{g,n}^{\alpha}(\mathbf{T}, \mathbf{s}) = \sum_{a_1,\ldots,a_n \geq 0} \frac{\prod T_{a_i}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n}^{1-\alpha} e^{\sum_{j=1}^{\infty} s_j \kappa_j} \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_n^{a_n},$$

$$\tau_\alpha(\mathbf{t}, \mathbf{s}) = \exp\left(\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \hbar^{2g-2+n} F_{g,n}^{\alpha}(\mathbf{T}, \mathbf{s}) \right)\Bigg|_{T_k = (2k+1)!! t_{2k+1}}.$$

We introduce the polynomials $q_j(\mathbf{s})$, defined by the generating function

$$1 - \exp\left(-\sum_{j=1}^{\infty} s_j z^j \right) = \sum_{j=1}^{\infty} q_j(\mathbf{s}) z^j,$$

Theorem (Manin, Zograf; Norbury)

$$\tau_\alpha(\mathbf{t}, \mathbf{s}) = \tau_\alpha\left(\left\{ t_{2k+1} + \frac{1}{\hbar} \frac{q_{k-\alpha}(\mathbf{s})}{(2k+1)!!} \right\} \right).$$

Identification of tau-functions

Let us put $q = -u^2$ with new parameter u, so that $p = 2u^2$. This parametrization corresponds to the insertion of the triple Hodge class of the form $\Lambda(-2u^2)\Lambda(-2u^2)\Lambda(u^2)$. Consider the parameters

$$\sum_{k=1}^{\infty} s_k^0 z^k = -\log\left(1 + \sum_{k=1}^{\infty}(2k-1)!!\left(zu^2\right)^k\right),$$

$$\sum_{k=1}^{\infty} s_k^1 z^k = -\log\left(1 + \sum_{k=1}^{\infty}(2k+1)!!\left(\sum_{j=0}^{k}\frac{1}{2j+1}\right)\left(zu^2\right)^k\right).$$

$s_1^0 = -u^2$, $s_2^0 = -\frac{5}{2}u^4$, $s_3^0 = -\frac{37}{3}u^6$; $s_1^1 = -4u^2$, $s_2^2 = -15u^4$, $s_3^1 = -\frac{316}{3}u^6$.

Theorem

The generating functions of the triple Hodge integrals satisfying Calabi–Yau condition for $p = -2q = 2u^2$ coincide with the generation functions of the intersection numbers containing κ classes,

$$\tau^{(\alpha)}_{-u^2,2u^2}(\mathbf{t}) = \tau_\alpha(\mathbf{t}, \mathbf{s}^\alpha).$$

Reduction to KdV: correlation functions

For $p = -2q = 2u^2$

$$\frac{\partial}{\partial x} = \frac{1 - u^2 z^2}{z} \frac{\partial}{\partial z}.$$

and

$$\Phi_k(z) := \left(-\frac{\partial}{\partial x}\right)^k \cdot \frac{1}{z}.$$

For $g, n \geq 0$ let us introduce the *correlation functions*

$$W^{\alpha}_{g,n}(z_1, \dots, z_n) = \sum_{a_1,\dots,a_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta^{1-\alpha}_{g,n} \Lambda(-2u^2)\Lambda(-2u^2)\Lambda(u^2) \prod_{j=1}^{n} \psi_j^{a_j} \Phi_{a_j}(z_j),$$

$$\tilde{W}^{\alpha}_{g,n}(z_1, \dots, z_n) = \sum_{a_1,\dots,a_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta^{1-\alpha}_{g,n} e^{\sum_{j=1}^{\infty} s_j^{\alpha} \kappa_j} \prod_{j=1}^{n} \psi_j^{a_j} \left(-\frac{1}{z_j} \frac{\partial}{\partial z_j}\right)^{a_j} z_j^{-1}.$$

Corollary

For $g, n \geq 0$

$$W^{\alpha}_{g,n}(z_1, \dots, z_n) = \tilde{W}^{\alpha}_{g,n}(z_1, \dots, z_n).$$

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How to generalize to the r-spin case?

Generalized Kontsevich model deformation!

Tau-function for linear Hodge case ($q = 0, p = w^2, \alpha = 1$) is given by the Kac–Schwarz operators

$$\mathtt{Q}_H = z + \hbar \left(\frac{1}{z} - w \right) \frac{\partial}{\partial z} - \hbar \frac{1}{2z^2},$$
$$\mathtt{X}_H = -\frac{z}{w} - \frac{1}{u^2} \log(1 - wz).$$

THEOREM

Tau-function for this case is given by the the generalized Kontsevich integral with the potential $V''(z) = \frac{z}{1-wz}$*:*

$$\tau_H(\mathbf{t}; w) = \tau_V(\mathbf{t}).$$

Subtlety: V - infinite series in z, matrix integral describes tau-function only at $N \to \infty$.

The modified wave function satisfies the quantum spectral curve equation

$$e^{w^2\hat{x}+w\hat{y}}\left(1-w\hat{y}-\hbar\frac{w^3}{2}\right)\cdot\tilde{\Psi}=\tilde{\Psi}.$$

The classical spectral curve

$$x=-w^{-2}\left(\log\left(1-wy\right)+wy\right)$$

reduces to Airy curve $x=y^2/2$ at $w=0$. It can be rewritten as

$$e^{-w^2x}=(1-wy)e^{wy}.$$

Kontsevich integral deformation: triple Hodge integrals

Tau-function for triple Hodge integral with Calabi–Yau constraint is described by the generalized Kontsevich model with the potential

$$V''(z) = \frac{z}{(1+\sqrt{p+q}z)(1+qz/\sqrt{p+q})}$$

and extra translation of times

$$\tilde{v}_k = [z^k] \int_0^z (\eta - y(\eta)) dx(\eta).$$

Theorem

Deformed generalized Kontsevich model is related to the generating function of cubic Hodge integrals satisfying the Calabi-Yau constraint by the translation of times

$$\tau^{(1)}_{q,p}(\mathbf{t}) = \tilde{C} e^{\sum \tilde{v}_k \frac{\partial}{\partial t_k}} \tau_V(\mathbf{t}).$$

Here $\tilde{C}$ does not depend on $\mathbf{t}$.

Translation of times – insertion of κ-classes.

Deformed Kontsevich matrix model

Cubic Hodge integrals with an additional Calabi-Yau condition are described by the potential

$$V''(z) = \frac{z}{(1+\sqrt{p+q}z)(1+qz/\sqrt{p+q})}$$

up to some explicit translation of times.

Shifted r-spin case

$$V''(z) = z^{r-1} + b_{r-2}z^{r-2} + \cdots + b_1 z$$

Deformation of the shifted r-spin case

$$V''(z) = \frac{z^{r-1} + b_{r-2}z^{r-2} + \cdots + b_1 z}{(1-w_1 z)(1-w_2 z)\dots(1-w_r z)}$$

Kac–Schwarz operators

$$\mathtt{Q} = z + \frac{\hbar}{V''}\frac{\partial}{\partial z} - \frac{\hbar V'''}{2(V'')^2}, \quad \mathtt{X} = V'(z).$$

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INTERSECTION OF THE SYMMETRY GROUPS

- Other combinations of the Givental groups and integrable hierarchies: (multicomponent) KP hierarchy, 2D Toda lattice, BKP, Drinfeld–Sokolov, Kac–Wakimoto...
- Relation between KP integrability and other integrable structures for Hodge integrals: Dubrovin–Zhang, Volterra, Hodge,...
- Geometrical interpretation of the matrix integral and KdV reduction
- Applications

CohFT $\cap$ Solitonic integrable hierarchies $\in$ Interesting geometry

$$\tau^{(0)}_{q,p}(\mathbf{t}) = \exp\left(\sum_{k=1}^{\infty} \hbar^k F^0_k(\mathbf{t})\right)$$

$$F_1^0 = \frac{1}{8} t_1,$$

$$F_2^0 = \frac{1}{16}\, {t_1}^2 - \frac{1}{128}\, \frac{p^2+pq+q^2}{p+q},$$

$$F_3^0 = \frac{9}{128}\, t_3 + \frac{1}{24}\, {t_1}^3 + \frac{3}{64}\, \frac{(p+2\,q)\, t_2}{\sqrt{p+q}} - \frac{1}{128}\, \frac{\left(2\,p^2 - pq - q^2\right) t_1}{p+q}$$

$$F_4^0 = \frac{27}{128}\, t_3 t_1 + \frac{1}{32}\, {t_1}^4 + \frac{9}{64}\, \frac{(p+2\,q)\, t_1 t_2}{\sqrt{p+q}} - \frac{3}{128}\, \frac{\left(p^2 - 2\,pq - 2\,q^2\right) {t_1}^2}{p+q} + \frac{1}{512}\, \frac{p^4 + 2\,p^3 q + 3\,p^2 q^2 + 2\,pq^3 + q^4}{(p+q)^2}$$

$$F_5^0 = \frac{225}{1024}\, t_5 + \frac{27}{64}\, t_3 {t_1}^2 + \frac{1}{40}\, {t_1}^5 + \frac{75}{256}\, \frac{(p+2\,q)\, t_4}{\sqrt{p+q}} + \frac{9}{32}\, \frac{(p+2\,q)\, t_2 {t_1}^2}{\sqrt{p+q}} + \frac{3}{512}\, \frac{t_3\left(4\,p^2 + 79\,pq + 79\,q^2\right)}{p+q} - \frac{1}{64}\, \frac{\left(2\,p^2 - 7\,pq - 7\,q^2\right) {t_1}^3}{p+q} - \frac{1}{512}\, \frac{\left(22\,p^3 + 21\,p^2 q - 69\,pq^2 - 46\,q^3\right) t_2}{(p+q)^{3/2}} + \frac{1}{1024}\, \frac{\left(8\,p^4 - 6\,p^3 q - 5\,p^2 q^2 + 2\,pq^3 + q^4\right) t_1}{(p+q)^2}$$

$$\tau^{(1)}_{q,p}(\mathbf{t}) = \exp\left(\sum_{k=1}^{\infty} \hbar^k F^1_k(\mathbf{t})\right)$$

$$F^1_1 = \frac{1}{8}t_3 + \frac{1}{6}{t_1}^3 + \frac{1}{12}\,\frac{(p+2\,q)\,t_2}{\sqrt{p+q}} - \frac{1}{24}\,\frac{p^2t_1}{p+q},$$

$$\begin{aligned}
F^1_2 = {} & \frac{1}{2}\,t_3{t_1}^3 + \frac{5}{8}\,t_5t_1 + \frac{3}{16}\,{t_3}^2 \\
& + \frac{5}{6}\,\frac{(p+2\,q)\,t_4t_1}{\sqrt{p+q}} + \frac{1}{4}\,\frac{(p+2\,q)\,t_3t_2}{\sqrt{p+q}} + \frac{1}{3}\,\frac{(p+2\,q)\,t_2{t_1}^3}{\sqrt{p+q}} \\
& + \frac{1}{8}\,\frac{\left(p^2+12\,pq+12\,q^2\right)t_3t_1}{p+q} + \frac{1}{12}\,\frac{\left(p^2+4\,pq+4\,q^2\right){t_2}^2}{p+q} + \frac{1}{6}\,q{t_1}^4 \\
& - \frac{1}{12}\,\frac{\left(p^3-p^2q-9\,pq^2-6\,q^3\right)t_1t_2}{(p+q)^{3/2}} \\
& - \frac{1}{48}\,\frac{q\left(2\,p^2-pq-q^2\right){t_1}^2}{p+q} \\
& - \frac{1}{5760}\,\frac{p^2q^2}{p+q},
\end{aligned}$$

$$\begin{aligned}
F^1_3 = {} & \frac{5}{8}\,t_5{t_1}^4 + \frac{3}{2}\,{t_3}^2{t_1}^3 + \frac{35}{16}\,t_7{t_1}^2 + \frac{15}{4}\,t_5t_3t_1 + \frac{3}{8}\,{t_3}^3 + \frac{105}{128}\,t_9 \\
& + \frac{35}{8}\,\frac{(p+2\,q)\,t_6{t_1}^2}{\sqrt{p+q}} + \frac{3}{4}\,\frac{(p+2\,q)\,{t_3}^2t_2}{\sqrt{p+q}} + \frac{35}{16}\,\frac{(p+2\,q)\,t_8}{\sqrt{p+q}} + \frac{5}{6}\,\frac{(p+2\,q)\,t_4{t_1}^4}{\sqrt{p+q}} \\
& + 2\,\frac{(p+2\,q)\,t_3t_2{t_1}^3}{\sqrt{p+q}} + \frac{5}{2}\,\frac{(p+2\,q)\,t_5t_2t_1}{\sqrt{p+q}} + 5\,\frac{(p+2\,q)\,t_4t_3t_1}{\sqrt{p+q}} + \ldots
\end{aligned}$$

Thank you!