Topological Recursion, Quantum Curves, and Atlantes Hurwitz Numbers

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The TR/QC correspondence – Motivation

$\boxed{\text{matrix models}}$

corr. functions

$$W_n = \left\langle \mathrm{Tr}\frac{1}{x_1 - M} \cdots \mathrm{Tr}\frac{1}{x_n - M} \right\rangle$$

- determined recursively (TR)

$$\Psi = \langle \det(x - M) \rangle$$

- determinental formulae (QC)

Relation?

" $\det = \exp \mathrm{Tr} \log$ "

$\rightsquigarrow$ Ψ in terms of W_n

$\rightsquigarrow$ TR/QC correspondence.

The TR/QC correspondence

Physical Mathematics: Use physics as a source of ideas for pure mathematics

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Physical Mathematics: Use physics as a source of ideas for pure mathematics

In [EO], Eynard and Orantin abstracted the topological recursion away from its physics origin.

Can the same be done for the TR/QC correspondence?

- The W_n often are generating functions for interesting enumerative invariants.
- Thus the TR/QC corrrespondence may lead to fascinating and unexpected properties for these invariants.

Outline

1. The CEO topological recursion
2. The TR/QC corrrespondence
3. The TR/QC correspondence for “transalgebraic spectral curves”
4. Enumerative interpretation: r-spin Hurwitz numbers vs Atlantes Hurwitz numbers

Algebraic spectral curves

Definition

An algebraic spectral curve is a quadruple $(C, x, \omega_{0,1}, \omega_{0,2})$ such that:

- C is a compact Riemann surface
- x is a meromorphic function on C
- $\omega_{0,1}$ is a meromorphic one-form on C
- $\omega_{0,2}$ is a symmetric bilinear differential on $C \times C$

$x : C \to \mathbb{P}^1$ finite degree branched covering.

Let $p \in C$, local normal form $x : z \mapsto z^m$, $m \geq 1$.

If $m \geq 2 \to p$ is a ramification point.

$R \subset C$

$\uparrow$ set of ramification points.

Algebraic spectral curves

Since C is a compact Riemann surface, and $\omega_{0,1}$ is a meromorphic one-form on C, we can write:

$$\omega_{0,1} = ydx,$$

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x and y are meromorphic functions on a compact Riemann surface C, and thus they identically satisfy an algebraic equation:

$$P(x, y) = 0.$$

This is why we call these spectral curves algebraic.

eg: $x = z^r$, $y = z$; $\quad P(x,y) = x - y^r = 0$.

The CEO topological recursion [CE,EO,BE]

$\omega_{g',n'+1}$ with $2g'-n'+1 < 2g-n+1$

$$\boxed{S = \text{spectral curve}} \xrightarrow{TR} \omega_{g,n+1}(z_0,\vec{z}) = \sum_{a\in R} \int_{\Gamma_a} K(z_0,z)\, \underbrace{R_{g,n+1}}, \quad g\geq 0,\ n\geq 0$$

$S \longrightarrow$ sequence $\{\omega_{g,n}\}_{g\geq 0, n\geq 1}$ of symmetric meromorphic differentials on C^n.

enum. invariants.

Wave-function

"$\det = \exp \operatorname{Tr} \log$" $\qquad \Psi \leftarrow W_n$.

$$S = (C, x, w_{0,1}, w_{0,2}) \xrightarrow{TR} w_{g,n}.$$

$$\Psi(z) = \exp\left[\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{2g-2+n}}{n!} \int_a^z \cdots \int_a^z \left(w_{g,n} - \delta_{g,0}\delta_{n,2} \frac{dx(z_1)\,dx(z_2)}{(x(z_1)-x(z_2))^2} \right) \right]$$

coord. on C

base point (pole of x)

Quantum curve and the TR/QC correspondence

Conjecture (the TR/QC correspondence)

There exists a quantization $\hat{P}(\hat{x}, \hat{y}; \hbar)$ of $P(x, y)$, known as quantum curve, such that

$$\hat{P}(\hat{x}, \hat{y}; \hbar)\Psi = 0,$$

where $\hat{x} = x$ and $\hat{y} = \hbar \frac{d}{dx}$.

$$\hat{P}\left(x, \hbar \frac{d}{dx}; \hbar\right) = \underbrace{P\left(x, \hbar \frac{d}{dx}\right)}_{\text{original spectral curve}} + \sum_{n \geq 1} \hbar^n P_n\left(x, \hbar \frac{d}{dx}\right)$$

(normal ordered)

Known results

- Many specific cases of TR/QC were proven case-by-case (all genus 0) ([GS, MSS,M,N,...])
- For higher genus spectral curves, the conjecture needs to be modified (the wave-function needs to be made "non-perturbative") ([BE,BCD,EGFMO])
- The conjecture was proven for a large class of genus 0 algebraic spectral curves in [BE]
- This was very recently extended to all genus algebraic spectral curves with simple ramification [EGF,EGFMO,MO] – We will hear more about these exciting results from Elba!

Transalgebraic spectral curves

Definition

A transalgebraic spectral curve is given by a quadruple $(C, x, \omega_{0,1}, \omega_{0,2})$ such that:

- C is a compact Riemann surface
- x is a transalgebraic function on C
- $\omega_{0,1}$ is a meromorphic one-form on C
- $\omega_{0,2}$ is a symmetric bilinear differential on $C \times C$

function potentially with exponential singularities.

e.g. $x = z e^{-z^n}$.

$x : C \to \mathbb{P}^1$ infinite degree

has finite ram. points and exponential singularities (infinite ram. points)

Transalgebraic spectral curves

Since x is a transalgebraic function on a compact Riemann surface C, $\mathrm{d}\log x$ is meromorphic one-form on C. Thus we can write:

$$\omega_{0,1} = y \; \mathrm{d}\log x,$$

which defines a meromorphic function y on C.

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which defines a meromorphic function y on C.

y is a meromorphic function on C and x is a transalgebraic function on C, and thus they identically satisfy a transalgebraic equation:

$$T(x, y) = 0.$$

e.g. $x = z e^{-z^r}, \quad y = z$

$$T(x,y) = y - x e^{y^r} = 0$$

The CEO topological recursion for transalgebraic spectral curves

The CEO topological recursion for transalgebraic spectral curves has been studied for many years. The standard formula is applied to the finite ramification points of a transalgebraic spectral curve, ignoring the exponential singularities.

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Take our example:

$$y - xe^{y^r} = 0.$$

It is know that the $\omega_{g,n}$ produced by topological recursion are generating functions for r-spin Hurwitz numbers (or "completed cycles Hurwitz numbers", studied for instance by Okounkov-Pandharipande in the context of relative Gromov-Witten theory of $\mathbb{P}^1$) [DBKPS].

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But...

Is this the most general definition of topological recursion for transalgebraic spectral curves? Should we really ignore the exponential singularities?

Transalgebraic spectral curves as limits of sequences of algebraic spectral curves

Take our example:

$$\left\{ x(z) = ze^{-z^r}, \qquad y = z. \right\} = S$$

Consider a sequence of alg. spectral curves:

$$\left\{ x_N(z) = z\left(1 - \frac{z^r}{N}\right)^N, \quad y = z \right\} = S^N$$

$$P^N(x_N, y) = x_N - y\left(1 - \frac{y^r}{N}\right)^N = 0$$

Clearly, $\boxed{S = \lim_{N \to \infty} S^N}$

The topological recursion and limits of sequences of algebraic spectral curves

Let $\omega_{g,n}[\mathcal{S}^N]$ be the correlation functions associated to the sequence of algebraic spectral curves $\mathcal{S}^N$, and $\omega_{g,n}\left[\lim_{N\to\infty} \mathcal{S}^N\right]$ be the correlation functions associated to the limiting transalgebraic spectral curves using the standard topological recursion, ignoring exponential singularities.

Question

Is it true that

$$\omega_{g,n}\left[\lim_{N\to\infty} \mathcal{S}^N\right] \stackrel{?}{=} \lim_{N\to\infty} \omega_{g,n}[\mathcal{S}^N]$$

No!

A "transalgebraic topological recursion" [BKW]

- We propose a transalgebraic topological recursion, which produces $\omega_{g,n}^{\text{TA}}\left[\lim_{N\to\infty}\mathcal{S}^N\right]$ for a transalgebraic spectral curve that satisfy:

$$\omega_{g,n}^{\text{TA}}\left[\lim_{N\to\infty}\mathcal{S}^N\right] = \lim_{N\to\infty}\omega_{g,n}[\mathcal{S}^N].$$

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$$\omega^{\mathrm{TA}}_{g,n}\left[\lim_{N\to\infty}\mathcal{S}^N\right] = \underbrace{\lim_{N\to\infty}\omega_{g,n}[\mathcal{S}^N]}_{\text{poles at exp. sing.}}.$$

- Our transalgebraic topological recursion includes contributions from the exponential singularities $\Rightarrow$ the $\omega^{\mathrm{TA}}_{g,n}$ have poles at the exponential singularities.

$$\omega_{g,n+1} = \sum_{a\in R}\oint_{\Gamma_a}(\ldots)$$

↑ includes exponential singularities

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- Our transalgebraic topological recursion includes contributions from the exponential singularities $\Rightarrow$ the $\omega_{g,n}^{\mathrm{TA}}$ have poles at the exponential singularities.

- In fact, for our r-spin example, only the $\omega_{g,1}^{\mathrm{TA}}$ have poles at the exponential singularity, and the poles have a very specific form.

The TR/QC correspondence for transalgebraic spectral curves

The TR/QC follows directly in our construction, if we use the transalgebraic topological recursion.

The TR/QC correspondence for transalgebraic spectral curves [BKW]

Let $\omega_{g,n}^{\mathrm{TA}}$ be the correlation functions associated to a transalgebraic spectral curve by the transalgebraic TR, and construct the associated wave-function Ψ^{TA}. Then

$$\Psi^{\mathrm{TA}}\left[\lim_{N\to\infty}\mathcal{S}^N\right]=\lim_{N\to\infty}\Psi[\mathcal{S}^N].$$

If there exists quantum curves for the sequence of algebraic spectral curves:

$$\hat{P}^N\left(\Psi[\mathcal{S}^N]\right)=0,$$

we obtain the quantum curve for the transalgebraic spectral curve as:

$$\hat{P}\left(\Psi^{\mathrm{TA}}\left[\lim_{N\to\infty}\mathcal{S}^N\right]\right)=0,\qquad \text{with}\qquad \hat{P}=\lim_{N\to\infty}\hat{P}^N.$$

Our favourite example

Consider our favourite example:

$$y - xe^{y^r} = 0.$$

sequence $\quad x_N - y\left(1 - \frac{y^r}{N}\right)^N = 0 \quad \leftarrow$ quantum curves $\hat{P}_N$ exist [BE]

Our favourite example

Consider our favourite example:

$$y - xe^{y^r} = 0.$$

Theorem [BKW]

The wave-function Ψ^{TA} constructed by the transalgebraic TR is annihilated by the quantum curve:

$$\hat{P} = \hat{y} - \hat{x}e^{\hat{y}^r},$$

with $\hat{x} = x$ and $\hat{y} = \hbar x \frac{d}{dx}$.

r-spin Hurwitz numbers

Consider again:

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The usual TR that ignores exponential singularities calculates $\omega_{g,n}$ that generate r-spin Hurwitz numbers. It was shown in [MSS] (using different methods, i.e. the semi-infinite wedge formalism for Hurwitz numbers) that the associated wave-function Ψ is also annihilated by a quantum curve, which takes a different form:

$$\hat{P} = \hat{y} - \hat{x}^{3/2} e^{\frac{1}{r+1}\sum_{i=0}^{r} \frac{1}{\hat{x}} \hat{y}^i \hat{x} \hat{y}^{r-i}} \hat{x}^{-1/2},$$

which is also a quantization of the spectral curve with a different (and less natural) ordering.

Not the same QC!

$\omega_{g,n} \neq \omega_{g,n}^{TA}$

But then... what is the transalgebraic TA calculating?

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Theorem [BKW]

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Theorem [BKW]

The $\omega_{g,n}^{\mathrm{TA}}$ are generating functions for Atlantes Hurwitz numbers.

- In genus 0, Atlantes Hurwitz numbers and r-spin Hurwitz numbers are the same, as expected. In particular, they share the same transalgebraic spectral curve.
- Using the semi-infinite wedge formalism, it was in shown in [ALS] that Atlantes Hurwitz numbers do satisfy the quantum curve that we also obtain from transalgebraic TR.

From [ALS]:

"We have an example where the dequantization of the quantum curve doesn't give a spectral curve suitable for the corresponding topological recursion."

"We can conclude that the dequantization of $\hat{y}^r - \hat{x}e^{\hat{y}^r}$ cannot be the spectral curve for the atlantes Hurwitz numbers, suitable for the construction of the topological recursion."

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Indeed, but that's because we're using the wrong TR! It is the spectral curve for atlantes Hurwitz numbers, if one uses the transalgebraic TR that takes into account the exponential singularities of the spectral curve.

$$\left.\begin{array}{rcl} \text{old school TR} & \leftrightarrow & r\text{-spin Hurwitz numbers} \\ \text{transalgebraic TR} & \leftrightarrow & \text{atlantes Hurwitz numbers} \end{array}\right\} \text{same spectral curve}$$

Conclusion

- We propose a new definition of TR for transalgebraic spectral curves that take into account the exponential singularities.
- We claim that it is more "natural", in the sense that it commutes with limits of sequences of algebraic spectral curves.
- We prove the TR/QC correspondence for transalgebraic spectral curves using transalgebraic TR.
- In the example of $y - xe^{y^r} = 0$, we show that transalgebraic TR computes atlantes Hurwitz numbers, and recovers the quantum curve of [ALS].

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A few open questions:

- Enumerative interpretation more generally?
- ELSV-type formula for atlantes Hurwitz numbers?
- More general class of curves? (Gromov-Witten theory)